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The construction of P-expansive maps of regular continua: A geometric approach

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Abstract

In this paper, we prove the following: Let G be a graph, $f:G \to G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Then there exist a regular continuum Z, a continuous map $g:Z \to Z$ and a semi-conjugacy $\pi:G \to Z$ such that

(1) g is $\pi(P)$-expansive, and

(2) if $p, q \in P$ and Q is a subset of P with $A \cap Q \neq \emptyset$ for any arc A in G between p and q, then $A' \cap \pi(Q) \neq \emptyset$ for any arc A' in Z between $\pi(p)$ and $\pi(q)$.

In addition, f is point-wise P-expansive if and only if $\pi|_P$ is one-to-one.

In this paper we are especially interested in the geometrical structure of Z. Actually we can see the complicated construction of Z. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

In recent years, there has been a growing interest in the study of the dynamical behavior of continuous maps of a graph. Especially, one of the central questions in the theory of dynamical systems is how to recognize "chaos". The theme of this paper is how to describe visually the chaoticity of continuous maps of a graph. To do this, for each graph G and continuous map f of G, we shall construct a new subspace Z of the Euclidean 3-dimensional space and a continuous map g of Z which (G, f) is semi-conjugate to. And we shall use the notion of P-expansiveness in order to investigate how complicated

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the dynamical behavior of f is. The fractal and complicated structure of the new space Z implies the chaoticity of f.

In [2] and [1, Theorem 4.1] the following result has been shown. Let D be a dendrite, $f: D \to D$ a continuous map and P a finite subset of D such that $f(P) \subset P$. Then there exist a dendrite E, a map $g: E \to E$ and a semi-conjugacy $\pi: D \to E$ (i.e., $\pi \circ f = g \circ \pi$) such that

(1) g is $\pi(P)$-expansive, and

(2) if $x, y, z \in P$ and $y \in [x, z]$ then $\pi(y) \in [\pi(x), \pi(z)]$.

If, in addition, the Markov graph of P has no basic intervals of order 0 and no loops of order 1, then $\pi|_P$ is one-to-one.

In this paper we expand the above result to a graph. Our main theorem is as follows:

Theorem 3.4. *Let G be a graph, $f: G \to G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Then there exist a regular continuum Z, a continuous map $g: Z \to Z$ and a semi-conjugacy $\pi: G \to Z$ such that*

(1) g is $\pi(P)$-expansive, and

(2) if $p, q \in P$ and Q is a subset of P with $A \cap Q \neq \emptyset$ for any arc A in G between p and q, then $A' \cap \pi(Q) \neq \emptyset$ for any arc A' in Z between $\pi(p)$ and $\pi(q)$.

In addition, f is point-wise P-expansive if and only if $\pi|_P$ is one-to-one.

We will show this by using a more geometrical method than that of Baldwin. Our interest is in what structure Z has. We can see visually that Z is a subset of a 3-dimensional space which has a fractal structure.

A *continuum* is a nonempty connected compact metric space. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. A *dendrite* is a locally connected, uniquely arcwise connected continuum. A continuum X is said to be *regular at* $x \in X$ if for any neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and the boundary of V has finite cardinality. And X is said to be *regular* provided that X is regular at each of its points.

Let X_i be a compact metric space and $f_i: X_i \to X_i$ a continuous map for $i = 1, 2$. We say that (X_1, f_1) is *semi-conjugate* (or *conjugate*, respectively) to (X_2, f_2) if there exists a continuous map (or homeomorphism) π from X_1 onto X_2 such that $\pi \circ f_1 = f_2 \circ \pi$. We call π a *semi-conjugacy* (or *conjugacy*).

Let G be a graph, $f: G \to G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Put

$$S(G, P) = P \cup \{C \mid C \text{ is a component of } G \setminus P\}.$$

Given $x \in G$, the *itinerary* of x with respect to P and f, written $I_{P,f}(x)$ (or just $I(x)$ if P and f are obvious from context), is defined to be the unique infinite sequence $(C_n)_{n \geqslant 0}$ from $S(G, P)$ given by the rule $f^n(x) \in C_n$ for all $n \geqslant 0$. If no two points of G have the same itinerary, then f will be called *P-expansive.* And f is *point-wise P-expansive* if for each $p, q \in P$, there exists some non-negative integer m such that

$$A \cap \left(P \setminus \left\{f^m(p), f^m(q)\right\}\right) \neq \emptyset$$

for each arc A in G between $f^m(p)$ and $f^m(q)$.

Let K be a continuum and P a finite subset of K. Then we say that *P graph-separates* K if and only if there exists a finite set $S(K, P)$ of subsets of K such that

(1) the element of $S(K, P)$ partition K, i.e., every point of K is in exactly one member of $S(K, P)$,
(2) for each $p \in P$, $\{p\} \in S(K, P)$,
(3) for each $A \in S(K, P)$, the closure of A in K is arc-wise connected, and
(4) if $A, B \in S(K, P)$, then the closure of A and B either have empty intersection or intersect in only elements of P.

Note that we can also define P-expansive for a graph-separated continuum in a similar way.

2. Constructions of $X_{\rightarrow}$ and $X_{\rightharpoonup}$

Let G be a graph, $f : G \to G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. We will construct new spaces $X_{\rightarrow}$ and $X_{\rightharpoonup}$ from P and f.

First we want to define an equivalence relation $\sim_1$ on P. Let $p, q \in P$. If for any non-negative integer i, there exists an arc A_i in G between $f^i(p)$ and $f^i(q)$ such that $A_i \cap P = \{f^i(p), f^i(q)\}$, then we put $p \sim_1' q$, where A_i may now consist of a single point. Now, if for $p, q \in P$, there exist some points $p_1, p_2, \ldots, p_k$ of P such that

$$p \sim_1' p_1 \sim_1' p_2 \sim_1' \cdots \sim_1' p_k \sim_1' q,$$

then we set $p \sim_1 q$. This relation $\sim_1$ is an equivalence relation on P. Let $[p]_1$ be the equivalence class of p, $P_1 = \{[p]_1 \mid p \in P\}$ and $G_1 = G/{\sim_1}$ the space obtained from G by identifying each equivalence class of P. Then we define a continuous map $f_1 : G_1 \to G_1$ such that $f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_1([p]_1) = [f(p)]_1$ for $[p]_1 \in P_1$. Similarly, if for any $p, q \in P_1$ and non-negative integer i, there exists an arc A_i in G_1 between $f_1^i(p)$ and $f_1^i(q)$ such that $A_i \cap P_1 = \{f_1^i(p), f_1^i(q)\}$, then we put $p \sim_2' q$. And if there exist some points $p_1, p_2, \ldots, p_k$ of P_1 such that $p \sim_2' p_1 \sim_2' p_2 \sim_2' \cdots \sim_2' p_k \sim_2' q$, then we set $p \sim_2 q$. This relation $\sim_2$ is also an equivalence relation on P_1. Let $[p]_2 = \{q \mid p \sim_2 q$ and $p, q \in P_1\}$, $P_2 = \{[p]_2 \mid p \in P_1\}$ and $G_2 = G_1/{\sim_2}$ the space obtained from G_1 by identifying each equivalence class of P_1. Then we define a continuous map $f_2 : G_2 \to G_2$ such that $f_2|_{G_2 \setminus P_2} = f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_2([p]_2) = [f_1(p)]_2$ for $[p]_2 \in P_2$. In the same way, we can obtain the space G_ℓ and a continuous map $f_\ell : G_\ell \to G_\ell$ for $\ell \geqslant 1$. Since P is finite, there is some natural number m such that $f_m : G_m \to G_m$ is point-wise P-expansive. There exists a semi-conjugacy π_i between (G_{i-1}, f_{i-1}) and (G_i, f_i) for $i = 1, 2, \ldots, m$, where $(G_0, f_0) = (G, f)$. We will construct Z and π' in Theorem 3.4 by the use of the point-wise P_m-expansiveness of f_m.

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\pi_1"'] & G \arrow[d, "\pi_1"] \\
G_1 \arrow[r, "f_1"'] \arrow[d, "\pi_2"'] & G_1 \arrow[d, "\pi_2"] \\
G_2 \arrow[r, "f_2"'] \arrow[d, "\pi_3"'] & G_2 \arrow[d, "\pi_3"] \\
\vdots \arrow[d, "\pi_m"'] & \vdots \arrow[d, "\pi_m"] \\
G_m \arrow[r, "f_m"'] \arrow[d, "\pi'"'] & G_m \arrow[d, "\pi'"] \\
Z \arrow[r, "g"'] & Z
\end{tikzcd}$$

By the argument above, we may proceed with our construction, under the assumption that f *is point-wise P-expansive*, in the rest part of this section.

Let $S(G,P)\setminus P=\{C_1,C_2,\ldots,C_n\}$ and $P=\{p_1,p_2,\ldots,p_k\}$. We will express the relation of elements of $S(G,P)$ as follows: If $p,q\in P$ and $f(p)=q$, then $p\to q$. This arrow $\to$ defines the Markov graph $P_\to$ on P (see Section 4). If $C_i,C_j\in S(G,P)\setminus P$ and $C_j\subset f(C_i)$, then $C_i\to C_j$. If $f(C_i)\cap C_j\neq\emptyset$, then $C_i\rightharpoonup C_j$. These arrows $\to$ and $\rightharpoonup$ define the Markov graphs $M_\to$ and $M_\rightharpoonup$ of elements of $S(G,P)\setminus P$, respectively. Note that $\to$ implies $\rightharpoonup$.

Now we will construct a new space $X_\rightharpoonup$ by using the Markov graphs $M_\rightharpoonup$ and $P_\to$. First we will construct a subspace X which is the union of 3-dimensional balls $B_1,B_2,\ldots,B_n$ in the Euclidean 3-dimensional space $\mathbb{E}^3$ by regarding elements $C_1,C_2,\ldots,C_n$ of $S(G,P)\setminus P$ as 3-dimensional balls $B_1,B_2,\ldots,B_n$ of $\mathbb{E}^3$. That is to say, $X=\bigcup_{i=1}^n B_i$, where the relationship of B_i and B_j is decided as follows: If $\mathrm{cl}(C_i)\cap\mathrm{cl}(C_j)=\emptyset$ for $C_i,C_j\in S(G,P)\setminus P$, then $B_i\cap B_j=\emptyset$. And if $\mathrm{cl}(C_i)\cap\mathrm{cl}(C_j)=\{q_1,q_2,\ldots,q_\ell\}\subset P$, then $B_i\cap B_j=\mathrm{Bd}(B_i)\cap\mathrm{Bd}(B_j)=\{q_1',q_2',\ldots,q_\ell'\}$, where $\mathrm{Bd}(B)$ is the boundary of B. Without confusion, we can express elements of $\mathrm{cl}(C_i)\cap\mathrm{cl}(C_j)$ and $B_i\cap B_j$ in a similar way. And for each $p\in(P\cap\mathrm{cl}(C_i))\setminus\bigcup\{\mathrm{cl}(C_j)\cap\mathrm{cl}(C_{j'})\mid j\neq j' \text{ and } 1\leqslant j,j'\leqslant n\}$, we take a corresponding point $p'\in\mathrm{Bd}(B_i)\setminus\bigcup\{B_j\cap B_{j'}\mid j\neq j' \text{ and } 1\leqslant j,j'\leqslant n\}$. For simplicity, we set $p'=p\in P$ (see Fig. 1).

Put $X_0=X$. We will construct a subspace X_1 contained in X_0 by using the Markov graph $M_\rightharpoonup$ and $P_\to$. For each $i=1,2,\ldots,n$, we have an embedding $h_i:X\hookrightarrow B_i$ such that

(1) $h_i(X)\cap\mathrm{Bd}(B_i)\subset P$, and

(2) for each $p,q\in P$ with $p\in\mathrm{Bd}(B_i)$ and $p\to q$, $h_i(q)=p\in\mathrm{Bd}(B_i)$.

If $C_i\rightharpoonup C_j$ ($C_i,C_j\in S(G,P)\setminus P$) in the Markov graph $M_\rightharpoonup$, then let $B_{i,j}=h_i(B_j)$ which is a copy of B_j. If $C_i\not\rightharpoonup C_j$, then $B_{i,j}=\emptyset$. Let

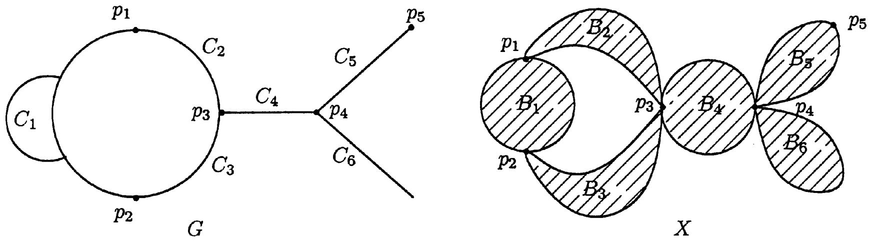

Fig. 1.

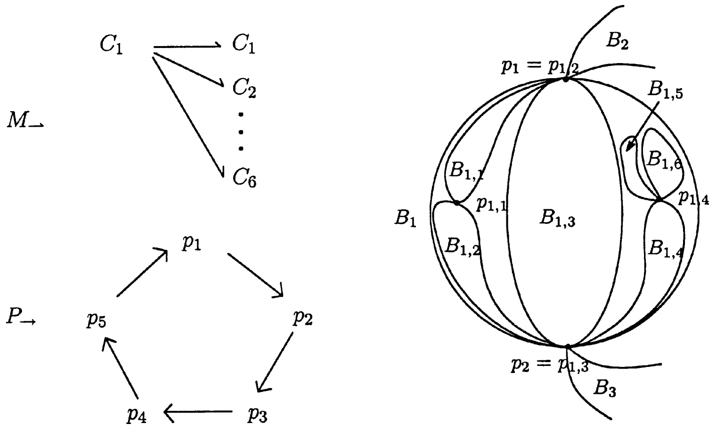

Fig. 2.

$$Y_i = \bigcup_{j=1}^{n} B_{i,j}, \quad \boldsymbol{B}_i = \{B_j \mid C_i \rightharpoonup C_j\} \quad \text{and}$$

$$\Bigl(\bigcup \boldsymbol{B}_i\Bigr) \cap P = \{p_{t(i:1)}, p_{t(i:2)}, \ldots, p_{t(i:k(i))}\},$$

where $t(i:\ell)$ and $k(i)$ are natural numbers with $1 \leqslant t(i:\ell),\ k(i) \leqslant k$ $(1 \leqslant \ell \leqslant k(i))$. And put $h_i(p_{t(i:\ell)}) = p_{i,t(i:\ell)}$. Then we obtain a connected subset $X_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ (see Fig. 2).

Similarly, we will construct a subspace X_2 in X_1. Let $h_{i_0,i_1} : X \hookrightarrow B_{i_0,i_1}$ be an embedding such that

(1) $h_{i_0,i_1}(X) \cap \mathrm{Bd}(B_{i_0,i_1}) \subset h_{i_0}(P)$, and

(2) for each $p_{i_0,j} \in \mathrm{Bd}(B_{i_0,i_1}) \cap h_{i_0}(P)$ and $q \in P$ with $p_j \to q$, $h_{i_0,i_1}(q) = p_{i_0,j} \in \mathrm{Bd}(B_{i_0,i_1})$.

If $C_{i_1} \rightharpoonup C_j$ in the Markov graph $M_{\rightharpoonup}$, then let $B_{i_0,i_1,j} = h_{i_0,i_1}(B_j)$. And if $C_{i_1} \not\rightharpoonup C_j$, then $B_{i_0,i_1,j} = \emptyset$. Let $Y_{i_0,i_1} = \bigcup_{j=1}^{n} B_{i_0,i_1,j}$, $\boldsymbol{B}_{i_1} = \{B_j \mid C_{i_1} \rightharpoonup C_j\}$ and $(\bigcup \boldsymbol{B}_{i_1}) \cap P = \{p_{t(i_0,i_1:1)}, p_{t(i_0,i_1:2)}, \ldots, p_{t(i_0,i_1:k(i_0,i_1))}\}$. Put $h_{i_0,i_1}(p_{t(i_0,i_1:j)}) = p_{i_0,i_1,t(i_0,i_1:j)}$ $(1 \leqslant j \leqslant t(i_0,i_1:k(i_0,i_1)))$. Then we obtain $X_2 = \bigcup\{Y_{i_0,i_1} \mid 1 \leqslant i_0, i_1 \leqslant n\}$ (see Fig. 3).

When this operation is repeated inductively, we obtain $X_0 \supset X_1 \supset X_2 \supset \cdots$ and a subspace $X_{\rightharpoonup} = \bigcap_{i=0}^{\infty} X_i$ of $\mathbb{E}^3$. Note that $X_{\rightharpoonup}$ is connected.

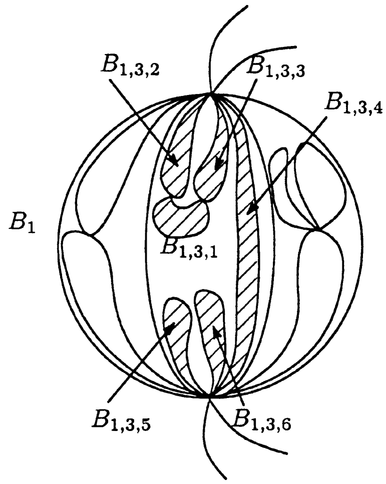

Fig. 3.

Next let $X'_1, X'_2, \ldots$ be subspaces constructed in a similar way on basis of the Markov graph $M_{\rightarrow}$. Then we obtain a subspace $X_{\rightarrow} = \bigcap_{i=1}^{\infty} X'_i$ of $\mathbb{E}^3$. Note that $X_{\rightarrow}$ is not always connected.

By using the construction of $X_{\rightharpoondown}$, we show that $\lim_{m\to\infty} \operatorname{diam}(B_{i_0,i_1,\ldots,i_m}) = 0$. Since we have assumed that f is point-wise P-expansive, for any distinct points p, q of P there exists a non-negative integer $N_{p,q}$ such that $A \cap (P \setminus \{f^{N_{p,q}}(p), f^{N_{p,q}}(q)\}) \neq \emptyset$ for any arc A in G between $f^{N_{p,q}}(p)$ and $f^{N_{p,q}}(q)$. Let $N = \max\{N_{p,q} \mid p, q \in P \text{ and } p \neq q\}$. Let m be a natural number and $B_{i_0,i_1,\ldots,i_m}$ a 3-dimensional ball from the construction of X_m. For any natural numbers $j_{m+1}, j_{m+2}, \ldots, j_{m+N}$ $(1 \leqslant j_{m+1}, j_{m+2}, \ldots, j_{m+N} \leqslant n)$, where $n = \mathit{Card}(S(G,P) \setminus P)$, the 3-dimensional ball $B_{i_0,i_1,\ldots,i_m,j_{m+1},j_{m+2},\ldots,j_{m+N}}$ from the constructing of X_{m+N} cannot contain two or more points of $\bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} \mid 1 \leqslant \ell_0, \ell_1, \ldots, \ell_m \leqslant n \text{ and } (\ell_0,\ell_1,\ldots,\ell_m) \neq (i_0,i_1,\ldots,i_m)\}$. Suppose that there exist distinct points x, y of $\bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} \mid 1 \leqslant \ell_0,\ell_1,\ldots,\ell_m \leqslant n$ and $(\ell_0,\ell_1,\ldots,\ell_m) \neq (i_0,i_1,\ldots,i_m)\}$ such that $x, y \in B_{i_0,i_1,\ldots,i_m,j_{m+1},\ldots,j_{m+N}}$. Put $x = p_{i_0,i_1,\ldots,i_{m-1},s} = h_{i_0,i_1,\ldots,i_{m-1}}(p_s)$ and $y = p_{i_0,i_1,\ldots,i_{m-1},t} = h_{i_0,i_1,\ldots,i_{m-1}}(p_t)$. Then $p_s, p_t \in P \cap B_{i_m}$. By the construction, for each $i = 0, 1, \ldots, N$, there exists an arc A_i in G between $f^i(p_s)$ and $f^i(p_t)$ such that $A_i \cap P = \{f^i(p_s), f^i(p_t)\}$. This contradicts the definition of N. Thus any two points $x, y \in \bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} \mid 1 \leqslant \ell_0,\ell_1,\ldots,\ell_m \leqslant n$ and $(\ell_0,\ell_1,\ldots,\ell_m) \neq (i_0,i_1,\ldots,i_m)\}$ are connected by the union of two or more 3-dimensional balls $B_{i_0,i_1,\ldots,i_m,j_{m+1},\ldots,j_{m+N}}$ $(1 \leqslant j_{m+1},\ldots,j_{m+N} \leqslant n)$. Hence we may assume that

$$\operatorname{diam}(B_{i_0,i_1,\ldots,i_m,j_{m+N},\ldots,j_{m+N}}) \leqslant \frac{1}{2} \operatorname{diam}(B_{i_0,i_1,\ldots,i_m}).$$

Thus we can suppose that

$$\lim_{n\to\infty} \operatorname{diam}(B_{i_0,i_1,\ldots,i_n}) = 0$$

(see Fig. 4).

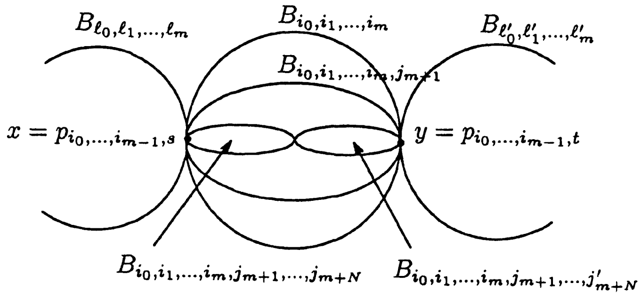

Fig. 4.

3. Construction of Z

Let G be a graph, $f:G\to G$ a continuous map, $P=\{p_1,p_2,\ldots,p_k\}$ a finite subset of G such that $f(P)\subset P$ and $S(G,P)\setminus P=\{C_1,C_2,\ldots,C_n\}$. We may also assume that f is point-wise P-expansive in this section from the argument in Section 2. And let $X_{\rightarrow}$, $X_{\rightharpoondown}$ be the above spaces constructed by the Markov graphs $(M_{\rightarrow},P_{\rightarrow})$, $(M_{\rightharpoondown},P_{\rightharpoondown})$ on $S(G,P)$, respectively.

Theorem 3.1. *The subspace $X_{\rightharpoondown}$ of $\mathbb{E}^3$ is a regular continuum.*

Proof. Let $\varepsilon>0$ and $x\in X_{\rightharpoondown}$. As $\lim_{m\to\infty}\operatorname{diam}(B_{i_0,i_1,\ldots,i_m})=0$, for an ε-neighborhood $U_\varepsilon(x)$ of x in $X_{\rightharpoondown}$ there exists a non-negative integer ℓ such that $B_{i_0,i_1,\ldots,i_\ell}\cap X_{\rightharpoondown}\subset U_\varepsilon(x)$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ containing x. Let

$$B=\bigcup\bigl\{B_{i_0,i_1,\ldots,i_\ell}\mid x\in B_{i_0,i_1,\ldots,i_\ell}\cap X_{\rightharpoondown}\subset U_\varepsilon(x)\bigr\}.$$

Then $B\cap X_{\rightharpoondown}$ is a neighborhood of x in $X_{\rightharpoondown}$ such that $B\cap X_{\rightharpoondown}\subset U_\varepsilon(x)$. By the construction of $X_{\rightharpoondown}$, the boundary of B has finite cardinality. Thus $X_{\rightharpoondown}$ is a regular continuum. $\square$

We define a map $\pi:G\to X_{\rightharpoondown}$ as follows: Given $x\in G$, if $f^\ell(x)\in\operatorname{cl}(C_{i_\ell})$ for any $\ell=0,1,2,\ldots$, then $\pi(x)=\bigcap_{\ell=0}^{\infty}B_{i_0,i_1,i_2,\ldots,i_\ell}$. We will investigate the uniqueness of $\pi(x)$ for each $x\in G$. Let $\{i_\ell\}_{\ell\geqslant 0},\{j_\ell\}_{\ell\geqslant 0}$ be sequences of natural numbers such that $\{i_\ell\}_{\ell\geqslant 0}\neq\{j_\ell\}_{\ell\geqslant 0}$, $f^\ell(x)\in\operatorname{cl}(C_{i_\ell})\cap\operatorname{cl}(C_{j_\ell})$ for each $\ell\geqslant 0$ and $1\leqslant i_\ell,j_\ell\leqslant n$. We will show that

$$\bigcap_{\ell=0}^{\infty}B_{i_0,i_1,\ldots,i_\ell}=\bigcap_{\ell=0}^{\infty}B_{j_0,j_1,\ldots,j_\ell}.$$

Let

$$m=\min\{\ell\mid C_{i_\ell}\neq C_{j_\ell}\},$$

then $f^m(x)\in P$. We put $x'=p_{i_0,i_1,\ldots,i_{m-1},t}$, where $p_{i_0,i_1,\ldots,i_{m-1},t}\in B_{i_0,i_1,\ldots,i_{m-1},i_m}\cap B_{j_0,j_1,\ldots,j_{m-1},j_m}$ and $p_t\in P$. Since $f(p_t)\in\operatorname{cl}(C_{i_{m+1}})\cap\operatorname{cl}(C_{j_{m+1}})$, $p_{i_0,i_1,\ldots,i_{m-1},t}\in$

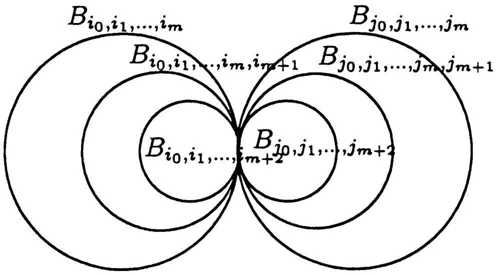

Fig. 5.

$B_{i_0,i_1,\dots,i_{m-1},i_m,i_{m+1}} \cap B_{j_0,j_1,\dots,j_{m-1},j_m,j_{m+1}}$ by the construction of X_{m+1}. Similarly, since $f^2(p_t) \in \mathrm{cl}(C_{i_{m+2}}) \cap \mathrm{cl}(C_{j_{m+2}})$, $p_{i_0,i_1,\dots,i_{m-1},t} \in B_{i_0,i_1,\dots,i_{m+2}} \cap B_{j_0,j_1,\dots,j_{m+2}}$. Inductively, for each $\ell \geqslant 0$, $p_{i_0,i_1,\dots,i_{m-1},t} \in B_{i_0,i_1,\dots,i_{m+\ell}} \cap B_{j_0,j_1,\dots,j_{m+\ell}}$. As $\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\dots,i_{m+\ell}}$ and $\bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\dots,j_{m+\ell}}$ are degenerate,

$$\{p_{i_0,i_1,\dots,i_{m-1},t}\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\dots,i_{m+\ell}} = \bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\dots,j_{m+\ell}}.$$

Thus we can define

$$\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\dots,i_{m+\ell}} = \bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\dots,j_{m+\ell}} = x'$$

(see Fig. 5).

Lemma 3.2. *$\pi : G \to X_{\rightharpoonup}$ is continuous.*

Proof. Let $x \in G$ and V be a neighborhood of $\pi(x)$ in $X_{\rightharpoonup}$.

Case 1. Assume that $f^{\ell}(x) \notin P$ for any $\ell = 0, 1, 2, \dots$ and $\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\dots,i_\ell}$. There exists a non-negative integer ℓ such that $B_{i_0,i_1,\dots,i_\ell} \cap X_{\rightharpoonup} \subset V$, where $B_{i_0,i_1,\dots,i_\ell}$ is a 3-dimensional ball containing $\pi(x)$. Since $C_{i_0,i_1,\dots,i_\ell} = \{x \in G \mid x \in C_{i_0}, f(x) \in C_{i_1}, \dots, f^{\ell}(x) \in C_{i_\ell}\}$ is an open set containing x and $\pi(C_{i_0,i_1,\dots,i_\ell}) \subset B_{i_0,i_1,\dots,i_\ell} \cap X_{\rightharpoonup}$, π is continuous at x.

Case 2. Assume that there exists $m = \min\{\ell \mid f^{\ell}(x) \in P\} < \infty$. There exists $\ell > m$ such that $B_{i_0,i_1,\dots,i_\ell} \cap X_{\rightharpoonup} \subset V$ for each $B_{i_0,i_1,\dots,i_\ell}$ containing $\pi(x)$. Let $\boldsymbol{C}_\ell = \{C_{i_0,i_1,\dots,i_\ell} \mid 1 \leqslant i_0, i_1, \dots, i_\ell \leqslant \mathit{Card}(S(G,P))\}$ and $U = \bigcup\{C \in \boldsymbol{C}_\ell \mid x \in \mathrm{cl}(C)\}$. Since f is continuous, U is a neighborhood of x such that $\pi(U) \subset V$. Thus π is continuous. $\square$

Now we will put $Z = \pi(G)$. Then $X_{\to} \subset Z \subset X_{\rightharpoonup}$. In general it is difficult to recognize the precise structure of Z, but by the above relation $X_{\to} \subset Z \subset X_{\rightharpoonup}$, we can realize the approximate structure of Z. Since $X_{\rightharpoonup}$ is regular, Z is also regular.

Note that by the construction, if for any element $C \in S(G,P) \setminus P$, there exist finitely many elements $C_1, C_2, \dots, C_m$ of $S(G,P)$ such that $f(C) = \bigcup_{i=1}^{m} C_i$, then $X_{\to} = Z = X_{\rightharpoonup}$.

Define a map $g : X_{\rightharpoondown} \to X_{\rightharpoondown}$ as follows: If $\{x\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}$, then $\{g(x)\} = g(\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}) = \bigcap_{\ell=1}^{\infty} B_{i_1,i_2,\ldots,i_\ell}$. We can investigate the uniqueness of g as we did that of π. Note that $g(Z) \subset Z$.

Lemma 3.3. *$g : X_{\rightharpoondown} \to X_{\rightharpoondown}$ is continuous.*

Proof. Let $x \in X_{\rightharpoondown}$ and V be a neighborhood of $g(x)$ in $X_{\rightharpoondown}$. Then there exists a non-negative integer ℓ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_{\rightharpoondown} \subset V$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ containing $g(x)$. Let $B = \bigcup\{B_{j_0,j_1,\ldots,j_{\ell+1}} \mid x \in B_{j_0,j_1,\ldots,j_{\ell+1}}\}$. Then B is a neighborhood of x and $g(B \cap X_{\rightharpoondown}) \subset V$. Thus g is continuous. $\square$

The following is the main theorem in this paper.

Theorem 3.4. *Let G be a graph, $f : G \to G$ a continuous map and P a finite subset of G such that $f(P) \subset P$. Then there exist a regular continuum Z, a continuous map $g : Z \to Z$ and a semi-conjugacy $\pi : G \to Z$ such that*

(1) *g is $\pi(P)$-expansive, and*
(2) *if $p, q \in P$ and Q is a subset of P with $A \cap Q \neq \emptyset$ for any arc A in G between p and q, then $A' \cap \pi(Q) \neq \emptyset$ for any arc A' in Z between $\pi(p)$ and $\pi(q)$.*

In addition, f is point-wise P-expansive if and only if $\pi|_P$ is one-to-one.

Proof. Let π and g be the above maps. Let $x \in G$ with $f^\ell(x) \in \mathrm{cl}(C_{i_\ell})$ for $\ell = 0, 1, 2, \ldots$. Then

$$\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell} \quad \text{and} \quad \{g \circ \pi(x)\} = \bigcap_{\ell=1}^{\infty} B_{i_1,i_2,\ldots,i_\ell} = \{\pi \circ f(x)\}.$$

Thus π is a semi-conjugacy between (G, f) and (Z, g).

We will show that (1) g is $\pi(P)$-expansive. Let x, y be distinct points of Z. There exists a 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ such that $x, y \in B_{i_0,i_1,\ldots,i_{\ell-1}}$, $x \in B_{i_0,i_1,\ldots,i_\ell}$ and $y \notin B_{i_0,i_1,\ldots,i_\ell}$. Then $g^\ell(x) \in B_{i_\ell}$ and $g^\ell(x) \notin B_{i_\ell}$. Thus $I_{\pi(P),g}(x) \neq I_{\pi(P),g}(y)$.

By the construction of $X_{\rightharpoondown}$, we can easily check (2). $\square$

Proposition 3.5. *Let G be a graph, $f : G \to G$ a continuous map and P the set of vertices of G with $f(P) \subset P$. If f is point-wise P-expansive and $f|_{[p,q]}$ is one-to-one for each edge $[p, q]$ between p and q, then Z is homeomorphic to G.*

Proof. Let $p, q \in P$ and $[p, q]$ be the edge between p and q. Since $f|_{[p,q]}$ is one-to-one, $f([p, q])$ is an arc between $f(p)$ and $f(q)$. Let $\{C_{m_1}, C_{m_2}, \ldots, C_{m_\ell}\}$ be the set of elements of $S(G, P) \setminus P$ which is contained in $f([p, q])$. As f is point-wise P-expansive, by the construction of X the 3-dimensional balls $B_{m_1}, B_{m_2}, \ldots, B_{m_\ell}$ corresponding to $C_{m_1}, C_{m_2}, \ldots, C_{m_\ell}$ form a chain between $\pi(p)$ and $\pi(q)$, i.e., $B_{m_i} \cap B_{m_j} \neq \emptyset$ if and only if $|i - j| \leqslant 1$. Similarly, by the construction of X_1, finitely many smaller balls form a chain in each ball B_{m_i} $(i = 1, 2, \ldots, \ell)$, too. When we repeat this operation, $\pi([p, q])$ is an arc between $\pi(p)$ and $\pi(q)$. Thus Z is homeomorphic to G. $\square$

Remark. In Theorem 3.4, we can obtain the same result by using a graph-separated continuum instead of a graph.

4. Examples

In this section, a few concrete examples will be given to clarify the explanation given so far.

Example 4.1. Let $G = [0, 1]$ be the unit interval and $P = \{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$. We will define a continuous map f of G such that $f(x) = 2x$ (if $0 \leqslant x \leqslant \frac{1}{2}$) and $f(x) = -2x + 2$ (if $\frac{1}{2} \leqslant x \leqslant 1$). This map f is point-wise P-expansive. Then $S(G, P) = \{[0, \frac{2}{7}), \{\frac{2}{7}\}, (\frac{2}{7}, \frac{4}{7}), \{\frac{4}{7}\}, (\frac{4}{7}, \frac{6}{7}), \{\frac{6}{7}\}, (\frac{6}{7}, 1]\}$, where put $C_1 = [0, \frac{2}{7})$, $C_2 = (\frac{2}{7}, \frac{4}{7})$, $C_3 = (\frac{4}{7}, \frac{6}{7})$, $C_4 = (\frac{6}{7}, 1]$, $p_1 = \frac{2}{7}$, $p_2 = \frac{4}{7}$ and $p_3 = \frac{6}{7}$. The Markov graph of $S(G, P) \setminus P$ and P is as in Fig. 6.

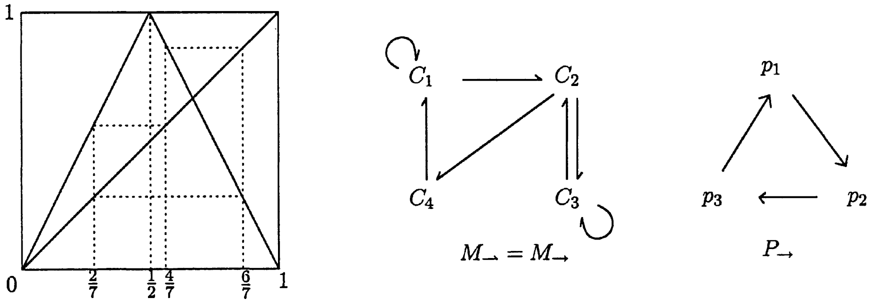

Fig. 6.

The above Markov graphs $(M_{\rightharpoondown}, P_{\rightarrow})$ will give information useful in constructing the space Z. Let $X = B_1 \cup B_2 \cup B_3 \cup B_4$ and $h_i : X \hookrightarrow B_i$ $(i = 1, 2, 3, 4)$ be an embedding such that (1) $h_i(X) \cap \mathrm{Bd}(B_i) \subset P$, and (2) for each $p, q \in P$ with $p \in \mathrm{Bd}(B_i)$ and $p \to q$, $h_i(q) = p \in \mathrm{Bd}(B_i)$. Then we describe the union Y_i of finitely many balls in each ball B_i. For example, when $i = 2$, $Y_2 = B_{2,3} \cup B_{2,4} \subset h_2(X)$, $p_1 = h_2(p_2)$ and $p_2 = h_2(p_3)$, since $C_2 \to C_3$, $C_2 \to C_4$, $p_1 \to p_2$ and $p_2 \to p_3$. In this way, we obtain a subspace $X_1 = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ of $\mathbb{E}^3$ (see Fig. 7).

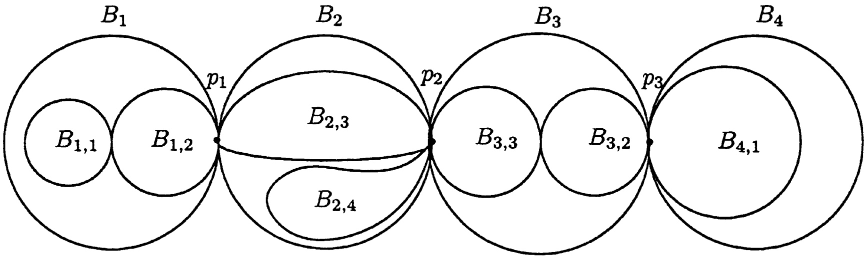

Fig. 7.

Next we describe finitely many 3-dimensional balls in X_1. Let $h_{i,j} : X \hookrightarrow B_{i,j}$ be an embedding such that (1) $h_{i,j}(X) \cap \mathrm{Bd}(B_{i,j}) \subset h_i(P)$, and (2) for each $p, q \in P$ with $h_i(p) \in \mathrm{Bd}(B_{i,j})$ and $f^2(p) = q$, $h_{i,j}(q) = h_i(p)$. Then we describe the union $Y_{i,j}$ of finitely many balls in $B_{i,j}$. For example, when $i = 2$ and $j = 3$, $h_{2,3}(X) = B_{2,3,2} \cup B_{2,3,3} = Y_{2,3}$, $h_2(p_2) = h_{2,3}(p_1)$ and $h_2(p_3) = h_{2,3}(p_2)$, since $C_3 \to C_2$, $C_3 \to C_3$, $f^2(p_2) = p_1$ and $f^2(p_3) = p_2$. Put

$$X_2 = \bigcup_{i,j=1}^{4} Y_{i,j}$$

(see Fig. 8).

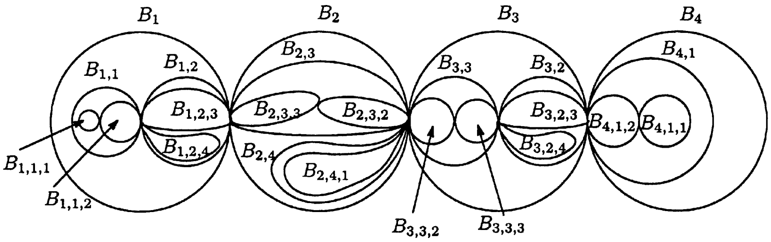

Fig. 8.

Similarly, we can describe X_i $(i = 3, 4, \ldots)$. Finally the space $Z = \bigcap_{i=1}^{\infty} X_i$ is the universal dendrite (see Fig. 9).

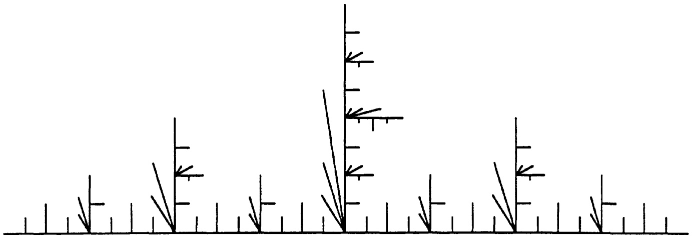

Fig. 9.

Example 4.2. Let $G = [0, 1]$ be the unit interval, $P = \{0, \frac{1}{2}, 1\}$ and f the same continuous map of G as in Example 4.1. Then $X_{\rightarrow} = Z = X_{\rightharpoondown}$ and Z is homeomorphic to $G = [0, 1]$. This implies that the structure of Z depends on the way of selecting the points of P (see Proposition 3.5).

Example 4.3. Let $G = [0, 1]$ be the unit interval and $P = \{\frac{1}{2}, \frac{3}{4}, 1\}$. We will define a continuous map f of G as follows: $f(x) = 4x\,(0 \leqslant x \leqslant \frac{1}{4})$, $f(x) = -2x + \frac{3}{2}(\frac{1}{4} \leqslant x \leqslant \frac{1}{2})$, $f(x) = 2x - \frac{1}{2}(\frac{1}{2} \leqslant x \leqslant \frac{3}{4})$ and $f(x) = -2x + \frac{5}{2}(\frac{3}{4} \leqslant x \leqslant 1)$. Then

$$S(G, P) = \left\{\left[0, \tfrac{1}{2}\right), \left\{\tfrac{1}{2}\right\}, \left(\tfrac{1}{2}, \tfrac{3}{4}\right), \left\{\tfrac{3}{4}\right\}, \left(\tfrac{3}{4}, 1\right), \{1\}\right\},$$

where put $C_1 = [0, \frac{1}{2})$, $C_2 = (\frac{1}{2}, \frac{3}{4})$, $C_3 = (\frac{3}{4}, 1)$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$ and $p_3 = 1$. The Markov graph of $S(G, P) \setminus P$ and P is as in Fig. 10.

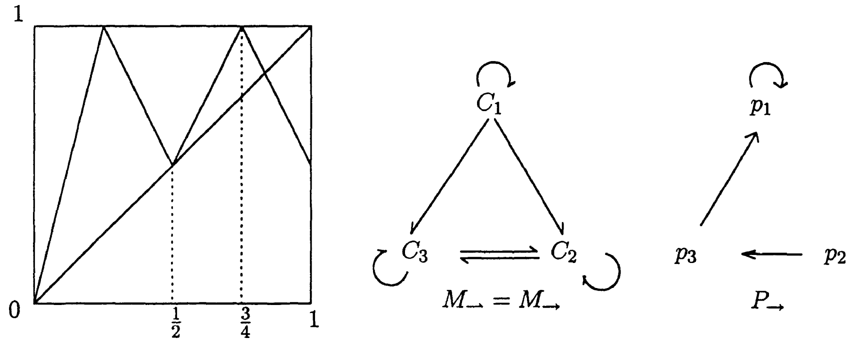

Fig. 10.

From the above Markov graph of $S(G, P) \setminus P$, we know that $B_{2,1} = \emptyset$ and $B_{3,1} = \emptyset$. Furthermore, the Markov graph of P suggests the way of connection of each ball $B_{i,j}$, where $i, j = 1, 2, 3$ (see Fig. 11).

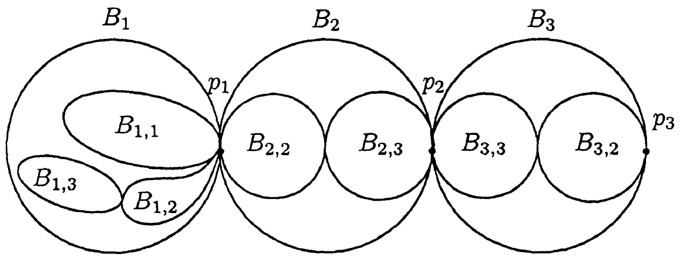

Fig. 11.

Finally, Z is the following dendrite (see Fig. 12).

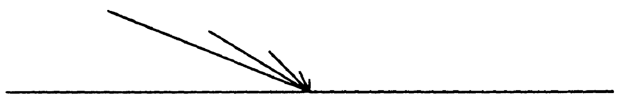

Fig. 12.

Example 4.4. Let G be the following graph, $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ a finite subset of G and $f : G \to G$ a continuous map. And assume that $f(\mathrm{cl}(C)) = G$ for any $C \subset S(G, P) \setminus P$, $f(p_1) = p_1 = f(p_4)$, $f(p_3) = p_2 = f(p_6)$ and $f(p_2) = p_3 = f(p_5)$. Note that f is point-wise P-expansive (see Fig. 13).

Then Z is the triangular Sierpinski curve (see Fig. 14).

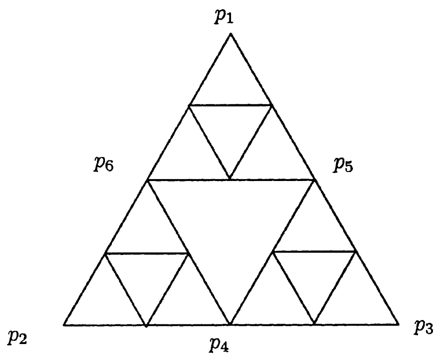

Fig.13.

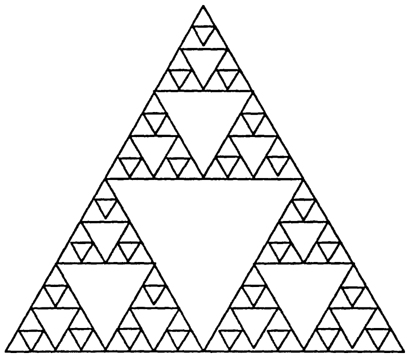

Fig. 14.

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