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Exact Solutions to the Double Sinh-Gordon Equation by the Tanh Method and a Variable Separated ODE Method

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Abstract—New exact travelling wave solutions for the double sinh-Gordon equation and its generalized form are formally derived by using the tanh method and the variable separated ODE method. The Painlevé property $v = e^u$ is employed to support the tanh method in deriving exact solutions. The work emphasizes the power of the methods in providing distinct solutions of different physical structures.

Keywords—Traveling wave solutions, Tanh method, Variable separated ODE method, Painlevé property, Double sinh-Gordon equation.

1. INTRODUCTION

The sinh-Gordon equation,

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{1}$$

appears in integrable quantum field theory, kink dynamics, and fluid dynamics [1–8]. The sinh-Gordon equation is completely integrable because it possesses similarity reductions to third Painlevé equation. There is a growing interest in the study of the sinh-Gordon equation, the double sinh-Gordon, and the triple sinh-Gordon equations [1–8]. It is well known that searching for explicit solutions for nonlinear evolution equation, by using different methods, is the goal for many researchers. Many powerful methods, such as Bäcklund transformation, inverse scattering method, Hirota bilinear forms, pseudo spectral method, the tanh-sech method [9–13], the sine-cosine method [14], and many others were successfully used to investigate these types of equations. However, some of these methods are not easy to use and require a thorough knowledge. Practically, there is no unified method that can be used to handle all types of nonlinearity.

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In our previous work [12,13], the sine-Gordon equation, the sinh-Gordon equation, and the famous sinh-Gordon equation given by

$$u_{tt} - u_{xx} + \sin u = 0, \tag{2}$$

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{3}$$

and

$$u_{xt} + \sinh u = 0, \tag{4}$$

respectively, were investigated by using the standard tanh method [9–13].

In this work, we aim to extend our previous work in [12–13] to make further progress for other forms of the sinh-Gordon equation. In particular, we will investigate the double sinh-Gordon equation and its generalized form given by

$$u_{tt} - ku_{xx} + 2\alpha \sinh u + \beta \sinh(2u) = 0 \tag{5}$$

and

$$u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0, \qquad n \geq 1, \tag{6}$$

respectively. The last form was introduced and studied in [8] by using specific approaches.

Two reliable methods will be employed to formally derive distinct sets of solutions. The first method is the tanh method developed by Malfliet [9–11]. The Painlevé property will be employed to support the tanh method. The second method, depends on a variable separated ODE, developed by Sirendaoreji *et al.* [1], used by Fu *et al.* [2] and by Wazwaz [15,16]. Sirendaoreji *et al.* [1] have fully described the variable separated ODE method that has established scientific value and reliability. The variable separated ODE method works effectively if the equation involves sine, cosine, hyperbolic sine, and hyperbolic cosine functions.

In what follows, we highlight the main features of the two methods as introduced in [9–11] and in [1,2,15,16], where more details and examples can be found there.

2. THE TANH METHOD

We first unite the independent variables x and t [9–13] into one wave variable $\xi = x - ct$ to carry out a PDE in two independent variables,

$$P\left(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots\right) = 0, \tag{7}$$

into an ODE,

$$Q\left(u, u', u'', u''', \ldots\right) = 0, \tag{8}$$

that can be integrated as long as all terms contain derivatives. Usually the integration constants are considered to be zeros in view of the localized solutions. The tanh technique is based on the *a priori* assumption that the traveling wave solutions can be expressed in terms of the tanh function. Then, we introduce a new independent variable,

$$Y = \tanh\left(\mu\xi\right), \tag{9}$$

that leads to the change of derivatives,

$$\begin{aligned} \frac{d}{d\xi} &= \mu\left(1 - Y^2\right)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= \mu^2\left(1 - Y^2\right)\left(-2Y\frac{d}{dY} + \left(1 - Y^2\right)\frac{d^2}{dY^2}\right). \end{aligned} \tag{10}$$

The solutions can be proposed as a finite power series in Y in the form,

$$u\left(\mu\xi\right) = S\left(Y\right) = \sum_{k=0}^{M} a_k Y^k, \tag{11}$$

limiting them to solitary and shock wave profiles. The parameter M is a positive integer, mostly 1 or 2, that will be determined by balancing Y^M in the highest derivative against its counterpart within the nonlinear terms. Consequently, the other parameters can be determined as well.

3. THE VARIABLE SEPARATED ODE METHOD

As stated before, this method was developed by Sirendaoreij *et al.* [1]. We first unite the independent variables x and t into one wave variable $\xi = x - ct$ to carry out a PDE into an equivalent ODE. The method depends mainly on assuming that $u(\xi)$ satisfies an additional variable separated ODE given by

$$u' = \frac{du}{d\xi} = G(u), \tag{12}$$

where $G(u)$ is a suitable function of sine, cosine, hyperbolic sine, hyperbolic cosine. Substituting (12) into the given equation yields a system of algebraic equations that can be solved to determine the unknown parameters. It is worth noting that the variable separated ODE (12) can be solved easily by using the method of separation of variables.

4. USING THE TANH METHOD

In what follows, we will apply the tanh method to the double sinh-Gordon equation and to a generalized form of this equation.

4.1. The Double Sinh-Gordon Equation

We first examine the double sinh-Gordon equation,

$$u_{tt} - ku_{xx} + 2\alpha \sinh u + \beta \sinh(2u) = 0, \tag{13}$$

that, by using the wave variable $\xi = x - ct$, can be converted to the ODE,

$$\left(c^2 - k\right) u'' + 2\alpha \sinh u + \beta \sinh(2u) = 0. \tag{14}$$

We next use the Painlevé transformation,

$$v = e^u, \tag{15}$$

or equivalently,

$$u = \ln v, \tag{16}$$

from which we find

$$\begin{aligned} u' &= \frac{1}{v} v', \\ u'' &= \frac{1}{v} v'' - \frac{1}{v^2} \left(v'\right)^2. \end{aligned} \tag{17}$$

The transformation (15) also gives

$$\sinh u = \frac{v - v^{-1}}{2}, \qquad \sinh(2u) = \frac{v^2 - v^{-2}}{2}, \qquad \cosh u = \frac{v + v^{-1}}{2}, \tag{18}$$

that also gives

$$u = \operatorname{arccosh}\left[\frac{v + v^{-1}}{2}\right]. \tag{19}$$

Substituting the transformations introduced above into the double sinh-Gordon equation (14) gives the ODE,

$$\beta v^4 + 2\alpha v^3 - 2\alpha v - \beta + 2\left(c^2 - k\right) vv'' - 2\left(c^2 - k\right)\left(v'\right)^2 = 0. \tag{20}$$

Applying the balancing process of v^4 with vv'' gives

$$M = 1. \tag{21}$$

The tanh method admits the use of the finite expansion,

$$v(x,t) = S(Y) = a_0 + a_1 Y, Y = \tanh(\mu\xi). \tag{22}$$

Substituting (22) into (20), collecting the coefficients of each power of Y, and using any symbolic computation program such as MATHEMATICA, we obtain

$$\begin{aligned} a_0 &= -\frac{\alpha}{\beta}, \\ a_1 &= \frac{\sqrt{\alpha^2 - \beta^2}}{\beta}, \\ \mu &= \sqrt{\frac{\alpha^2 - \beta^2}{2\beta(k - c^2)}}, \qquad \alpha > \beta, \quad k > c^2, \end{aligned} \tag{23}$$

where c is left as a free parameter. This gives

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \tanh\left[\mu (x - ct)\right], \qquad \alpha > \beta, \tag{24}$$

and

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \coth\left[\mu (x - ct)\right], \qquad \alpha > \beta. \tag{25}$$

Recall that

$$u(x,t) = \operatorname{arccosh}\left[\frac{v + v^{-1}}{2}\right], \tag{26}$$

from (19), therefore, we obtain the solutions, for $\alpha > \beta$, $k > c^2$,

$$\begin{aligned} u(x,t) = \operatorname{arccosh}\Bigg\{\frac{1}{2}\Bigg(&-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh\left[\mu(x-ct)\right] \\ &+ \frac{1}{-\alpha/\beta + \left(\sqrt{\alpha^2-\beta^2}/\beta\right)\tanh\left[\mu(x-ct)\right]}\Bigg)\Bigg\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} u(x,t) = \operatorname{arccosh}\Bigg\{\frac{1}{2}\Bigg(&-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\coth\left[\mu(x-ct)\right] \\ &+ \frac{1}{-\alpha/\beta + \left(\sqrt{\alpha^2-\beta^2}/\beta\right)\coth\left[\mu(x-ct)\right]}\Bigg)\Bigg\}. \end{aligned} \tag{28}$$

On the other hand, for $\alpha < \beta$, we find

$$u(x,t) = \operatorname{arccosh}\left\{\frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\tan\left[\bar{\mu}(x-ct)\right] - \frac{1}{\alpha/\beta + \left(\sqrt{\beta^2-\alpha^2}/\beta\right)\tan\left[\bar{\mu}(x-ct)\right]}\right)\right\}, \tag{29}$$

and

$$u\left(x,t\right) = \operatorname{arccosh}\left\{\frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\cot\left[\bar{\mu}(x-ct)\right] - \frac{1}{(\alpha/\beta) + \left(\sqrt{\beta^2-\alpha^2}/\beta\right)\cot\left[\bar{\mu}\left(x-ct\right)\right]}\right)\right\}, \tag{30}$$

where

$$\bar{\mu} = \sqrt{\frac{\beta^2-\alpha^2}{2\beta\left(k-c^2\right)}}. \tag{31}$$

4.2. Generalized Form of the Double Sinh-Gordon Equation

We next examine a generalized form of the double sinh-Gordon equation [8] given by

$$u_{tt} - ku_{xx} + 2\alpha\sinh\left(nu\right) + \beta\sinh\left(2nu\right) = 0, \qquad n \geq 1, \tag{32}$$

that can be converted to the ODE,

$$\left(c^2 - k\right)u'' + 2\alpha\sinh\left(nu\right) + \beta\sinh\left(2nu\right) = 0, \tag{33}$$

upon using the wave variable $\xi = x - ct$. Proceeding as before we use the transformation,

$$v = e^{nu}, \tag{34}$$

or equivalently,

$$u = \frac{1}{n}\ln v, \tag{35}$$

from which we find

$$\sinh\left(nu\right) = \frac{v - v^{-1}}{2}, \qquad \sinh\left(2nu\right) = \frac{v^2 - v^{-2}}{2}, \qquad \cosh\left(nu\right) = \frac{v + v^{-1}}{2}, \tag{36}$$

that also gives

$$u = \frac{1}{n}\operatorname{arccosh}\left[\frac{v + v^{-1}}{2}\right]. \tag{37}$$

Substituting these transformations introduced into (33) yields the ODE,

$$\beta nv^4 + 2\alpha nv^3 - 2\alpha nv - \beta n + 2\left(c^2-k\right)vv'' - 2\left(c^2-k\right)\left(v'\right)^2 = 0. \tag{38}$$

Balancing v^4 with vv'' gives

$$M = 1. \tag{39}$$

Consequently, we substitute the finite expansion,

$$v(x,t) = S(Y) = a_0 + a_1 Y, \qquad Y = \tanh(\mu\xi), \tag{40}$$

into (38), collecting the coefficients of each power of Y, and using any symbolic computation program such as Mathematica, we obtain

$$\begin{aligned} a_0 &= -\frac{\alpha}{\beta}, \\ a_1 &= \frac{\sqrt{\alpha^2-\beta^2}}{\beta}, \qquad \alpha > \beta, \\ \mu &= \sqrt{\frac{n(\alpha^2-\beta^2)}{2\beta(k-c^2)}}, \qquad \alpha > \beta, \quad k > c^2, \end{aligned} \tag{41}$$

where c is left as a free parameter. This in turn gives

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh[\mu(x-ct)], \qquad \alpha > \beta, \tag{42}$$

and

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\coth[\mu(x-ct)], \qquad \alpha > \beta. \tag{43}$$

Recall that

$$u(x,t) = \frac{1}{n}\operatorname{arccosh}\left[\frac{v+v^{-1}}{2}\right], \tag{44}$$

from (37), we therefore obtain the solutions, for $\alpha > \beta$,

$$\begin{aligned} u(x,t) = \frac{1}{n}\operatorname{arccosh}\Bigg\{\frac{1}{2}\Bigg(&-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh[\mu(x-ct)] \\ &+ \frac{1}{-(\alpha/\beta) + \left(\sqrt{\alpha^2-\beta^2}/\beta\right)\tanh[\mu(x-ct)]}\Bigg)\Bigg\}, \end{aligned} \tag{45}$$

and

$$\begin{aligned} u(x,t) = \frac{1}{n}\operatorname{arccosh}\Bigg\{\frac{1}{2}\Bigg(&-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\coth[\mu(x-ct)] \\ &+ \frac{1}{-(\alpha/\beta) + \left(\sqrt{\alpha^2-\beta^2}/\beta\right)\coth[\mu(x-ct)]}\Bigg)\Bigg\}. \end{aligned} \tag{46}$$

On the other hand, for $\alpha < \beta$, we find

$$\begin{aligned} u(x,t) = \frac{1}{n}\operatorname{arccosh}\Bigg\{\frac{1}{2}\Bigg(&-\frac{\alpha}{\beta} - \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\tan[\bar{\mu}(x-ct)] \\ &- \frac{1}{\alpha/\beta + \left(\sqrt{\beta^2-\alpha^2}/\beta\right)\tan[\bar{\mu}(x-ct)]}\Bigg)\Bigg\}, \end{aligned} \tag{47}$$

and

$$u(x,t) = \frac{1}{n}\operatorname{arccosh}\left\{\frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\cot\left[\bar{\mu}(x-ct)\right] - \frac{1}{\alpha/\beta + \left(\sqrt{\beta^2-\alpha^2}/\beta\right)\cot\left[\bar{\mu}(x-ct)\right]}\right)\right\}, \tag{48}$$

where

$$\bar{\mu} = \sqrt{\frac{n\left(\beta^2-\alpha^2\right)}{2\beta\left(k-c^2\right)}} \tag{49}$$

5. USING THE VARIABLE SEPARATED ODE METHOD

In what follows, we will employ the variable separated ODE method [1] to formally derive new distinct travelling wave solutions to the two equations examined before. The aim here is to obtain explicit solutions with distinct physical structures compared to the solutions obtained by using the tanh method.

5.1. The Double Sinh-Gordon Equation

The double sinh-Gordon equation,

$$u_{tt} - ku_{xx} + 2\alpha\sinh u + \beta\sinh(2u) = 0, \tag{50}$$

can be converted to the ODE,

$$\left(c^2-k\right)u'' + 2\alpha\sinh u + \beta\sinh(2u) = 0, \tag{51}$$

or equivalently,

$$u'' + \frac{2\alpha}{\left(c^2-k\right)}\sinh u + \frac{\beta}{\left(c^2-k\right)}\sinh(2u) = 0. \tag{52}$$

We next assume that $u(\xi)$ satisfies the variable separated ODE given by

$$u'(\xi) = \frac{du}{d\xi} = a + b\cosh u, \tag{53}$$

where a and b are parameters that will be determined. Differentiating (53) with respect to ξ gives

$$u''(\xi) - ab\sinh u - \frac{b^2}{2}\sinh(2u) = 0. \tag{54}$$

Comparing (52) with (54), we obtain

$$\begin{aligned} ab &= \frac{2\alpha}{k-c^2}, \\ b^2 &= \frac{2\beta}{k-c^2}, \end{aligned} \tag{55}$$

so that

$$\begin{aligned} a &= \frac{2\alpha}{\sqrt{2\beta(k-c^2)}}, \qquad k > c^2, \\ b &= \frac{2\beta}{\sqrt{2\beta(k-c^2)}}, \qquad k > c^2. \end{aligned} \tag{56}$$

Equation (53) is separable, hence, we set

$$\frac{1}{a+b\cosh u}\,du = d\xi, \tag{57}$$

where by integrating both sides, we find the following solutions,

$$\frac{2}{\sqrt{a^2-b^2}}\text{arctanh}\left(\sqrt{\frac{a-b}{a+b}}\tanh\frac{u}{2}\right) = \xi+\xi_0, \qquad a>b, \tag{58}$$

$$\frac{2}{\sqrt{a^2-b^2}}\text{arccoth}\left(\sqrt{\frac{a-b}{a+b}}\tanh\frac{u}{2}\right) = \xi+\xi_0, \qquad a>b, \tag{59}$$

$$\frac{2}{\sqrt{b^2-a^2}}\arctan\left(\sqrt{\frac{b-a}{b+a}}\tanh\frac{u}{2}\right) = \xi+\xi_0, \qquad a<b, \tag{60}$$

or

$$\frac{2}{\sqrt{b^2-a^2}}\text{arccot}\left(\sqrt{\frac{b-a}{b+a}}\tanh\frac{u}{2}\right) = \xi+\xi_0, \qquad a<b, \tag{61}$$

where ξ_0 is constant of integration. Using (56) gives the exact solutions,

$$u(x,t) = 2\text{arctanh}\left\{\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\tanh\left(\sqrt{\frac{\alpha^2-\beta^2}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right\}, \qquad \alpha>\beta, \tag{62}$$

$$u(x,t) = 2\text{arctanh}\left\{\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\coth\left(\sqrt{\frac{\alpha^2-\beta^2}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right\}, \qquad \alpha>\beta, \tag{63}$$

$$u(x,t) = 2\text{arctanh}\left\{\sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\tan\left(\sqrt{\frac{\beta^2-\alpha^2}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right\}, \qquad \alpha<\beta. \tag{64}$$

and

$$u(x,t) = 2\text{arctanh}\left\{\sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\cot\left(\sqrt{\frac{\beta^2-\alpha^2}{2\beta(c^2-k)}}\left((x-ct)+\xi_0\right)\right)\right\}, \qquad \alpha<\beta, \tag{65}$$

However, for the case where $a=b$, equation (57) becomes

$$\frac{1}{1+\cosh u}\,du = a\,d\xi, \tag{66}$$

where by integration, we obtain

$$\tanh\frac{u}{2} = a\left(\xi+\xi_0\right), \tag{67}$$

therefore, we find

$$u(x,t) = 2\,\text{arctanh}\left[a\left(\xi+\xi_0\right)\right], \tag{68}$$

or equivalently,

$$u(x,t) = 2\,\text{arctanh}\left(\sqrt{\frac{2\alpha}{(k-c^2)}}\left((x-ct)+\xi_0\right)\right). \tag{69}$$

Concerning the case where $b = -a$, equation (57) becomes

$$\frac{1}{1-\cosh(u)}\,du = a\,d\xi, \tag{70}$$

where by integration, we obtain

$$\coth\frac{u}{2} = a(\xi+\xi_0), \tag{71}$$

therefore, we find

$$u(x,t) = 2\,\mathrm{arccoth}\left[a(\xi+\xi_0)\right], \tag{72}$$

or equivalently,

$$u(x,t) = 2\,\mathrm{arccoth}\left(\sqrt{\frac{2\alpha}{c^2-k}}\left((x-ct)+\xi_0\right)\right). \tag{73}$$

5.2. A Generalized Form of the Double Sinh-Gordon Equation

We finally consider a generalized form of the double sinh-Gordon equation,

$$u_{tt} - ku_{xx} + 2\alpha\sinh(nu) + \beta\sinh(2nu) = 0, \tag{74}$$

that can be converted to the ODE,

$$\left(c^2-k\right)u'' + 2\alpha\sinh(nu) + \beta\sinh(2nu) = 0, \tag{75}$$

or equivalently,

$$u'' + \frac{2\alpha}{(c^2-k)}\sinh(nu) + \frac{\beta}{(c^2-k)}\sinh(2nu) = 0. \tag{76}$$

We next assume that $u(\xi)$ satisfies a variable separated ODE given by

$$u'(\xi) = \frac{du}{d\xi} = a + b\cosh(nu), \tag{77}$$

where a and b are parameters that will be determined. Differentiating (77) with respect to ξ gives

$$u''(\xi) - abn\sinh(nu) - \frac{b^2n}{2}\sinh(2nu) = 0. \tag{78}$$

Comparing (76) with (78), we obtain

$$\begin{aligned} abn &= \frac{2\alpha}{k-c^2},\\ b^2n &= \frac{2\beta}{k-c^2}, \end{aligned} \tag{79}$$

so that

$$\begin{aligned} a &= \frac{2\alpha}{\sqrt{2n\beta\,(k-c^2)}},\\ b &= \frac{2\beta}{\sqrt{2n\beta\,(k-c^2)}}. \end{aligned} \tag{80}$$

Equation (77) is separable, hence, we set

$$\frac{1}{a+b\cosh(nu)}\,du = d\xi, \tag{81}$$

where by integrating both sides, we find the following

$$\frac{2}{n\sqrt{a^2-b^2}}\operatorname{arctanh}\left(\sqrt{\frac{a-b}{a+b}}\tanh\left(\frac{nu}{2}\right)\right) = \xi+\xi_0, \qquad a>b, \tag{82}$$

$$\frac{2}{n\sqrt{a^2-b^2}}\operatorname{arccoth}\left(\sqrt{\frac{a-b}{a+b}}\tanh\left(\frac{nu}{2}\right)\right) = \xi+\xi_0, \qquad a>b, \tag{83}$$

$$\frac{2}{n\sqrt{b^2-a^2}}\arctan\left(\sqrt{\frac{b-a}{b+a}}\tanh\left(\frac{nu}{2}\right)\right) = \xi+\xi_0, \qquad a<b, \tag{84}$$

or

$$\frac{2}{n\sqrt{b^2-a^2}}\operatorname{arccot}\left(\sqrt{\frac{b-a}{b+a}}\tanh\left(\frac{nu}{2}\right)\right) = \xi+\xi_0, \qquad a<b, \tag{85}$$

where ξ_0 is constant of integration. This in turn gives the exact solution,

$$u(x,t) = \frac{2}{n}\operatorname{arctanh}\left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\tanh\left(\sqrt{\frac{n(\alpha^2-\beta^2)}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right), \qquad \alpha>\beta, \tag{86}$$

$$u(x,t) = \frac{2}{n}\operatorname{arctanh}\left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\coth\left(\sqrt{\frac{n(\alpha^2-\beta^2)}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right), \qquad \alpha<\beta. \tag{87}$$

$$u(x,t) = \frac{2}{n}\operatorname{arctanh}\left(\sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\tan\left(\sqrt{\frac{n(\beta^2-\alpha^2)}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right), \qquad \alpha<\beta, \tag{88}$$

and

$$u(x,t) = \frac{2}{n}\operatorname{arctanh}\left(\sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\cot\left(\sqrt{\frac{n(\beta^2-\alpha^2)}{2\beta(k-c^2)}}\left((x-ct)+\xi_0\right)\right)\right), \qquad \alpha<\beta. \tag{89}$$

However, for the case where $a=b$, equation (81) becomes

$$\frac{1}{1+\cosh(nu)}\,du = a\,d\xi, \tag{90}$$

where by integration we obtain

$$\frac{1}{n}\tanh\left(\frac{nu}{2}\right) = a(\xi+\xi_0), \tag{91}$$

therefore, we find

$$u(x,t) = \frac{2}{n}\operatorname{arctanh} a(\xi+\xi_0), \tag{92}$$

or equivalently

$$u(x,t) = \frac{2}{n}\,\text{arctanh}\left(\sqrt{\frac{2n\alpha}{k-c^2}}\,((x-ct)+\xi_0)\right). \tag{93}$$

Concerning the case where $b=-a$, equation (81) becomes

$$\frac{1}{1-\cosh(nu)}\,du = a\,d\xi, \tag{94}$$

where by integration, we obtain

$$\frac{1}{n}\coth\left(\frac{nu}{2}\right) = a(\xi+\xi_0), \tag{95}$$

therefore, we find

$$u(x,t) = \frac{2}{n}\,\text{arccoth}\,(a(\xi+\xi_0)), \tag{96}$$

or equivalently,

$$u(x,t) = \frac{2}{n}\,\text{arccoth}\left(\sqrt{\frac{2n\alpha}{\sqrt{k-c^2}}}\,((x-ct)+\xi_0)\right). \tag{97}$$

6. DISCUSSION

The double sinh-Gordon and its generalized form were investigated by using the tanh method and a variable separated ODE method. Several exact distinct travelling wave solutions were formally derived by using the two methods. The obtained results clearly demonstrate the efficiency of the methods used in this work. Moreover, the methods are capable of greatly minimizing the size of computational work compared to other existing techniques. Unlike what was thought before that the sinh-Gordon and the sine-Gordon equations cannot be solved by the tanh method, the introduced analysis confirms that the tanh method can be effectively used for these equations.

The variable separable ODE method is useful in that it provides new distinct solutions other than the solutions obtained by the tanh method. As indicated in [1], this method changes the problem from solving nonlinear partial differential equations to solving a separable ordinary differential equations.

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