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# Morasses, square and forcing axioms

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**Abstract**

The paper discusses various relationships between the concepts mentioned in the title. In Section 1 Todorcevic functions are shown to arise from both morasses and square. In Section 2 the theme is of supplements to morasses which have some of the flavour of square. Distinctions are drawn between differing concepts. In Section 3 forcing axioms related to the ideas in Section 2 are discussed.

## Introduction

One sign of vibrancy and maturation in an area of mathematics is that its practitioners are increasingly discriminatory in the choice of hypotheses for their theorems. This discrimination manifests itself formally and, equally importantly, informally: in distinguishing those hypotheses from which theorems can be proven from those for which there are counterexamples, but also in improved understanding of which techniques are likely to be adequate for a particular task. An example of this in infinite combinatorics and the subjects on which it impinges is the growing delicacy with which prediction principles ($\diamondsuit$, CH, weak $\diamondsuit$, and others) were used in the 1970s.

A second example in the same field has recently become apparent. Since 1980 gap-one construction devices, for constructing "large" objects (of size $\kappa^+$, where $\kappa$ is some infinite cardinal) from "small" pieces, have been employed with burgeoning facility. Necessarily, many people have been involved in this development, albeit that while carrying out their work they may not have thought in such terms. The following is a brief, and partial, listing of some significant trends and papers connected with them.

Morasses, have been considerably simplified, particularly by Velleman [23]. Enhancements of simplified morasses have been examined and used by Velleman, Donder and others [24, 6, 17]. Various trees have been used for stepping-up purposes, particularly innovatively by Todorcevic (Kurepa trees [18, 4], Aronszajn trees [19, 20]). $\square$ has also been used to this end [20, 22, 21, 8]. (Of course the title construction of Todorcevic's amazing [20] is also a gap-one construction.) Forcing notions have been introduced


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and axiomatised by Shelah, Shelah and Stanley, and Velleman, [2, 14–16, 23, 24]. Gap-one constructions also occur in [1, 7, 9], and many other places.

This paper tries to provide insights into the relationships between some of these devices, with an emphasis on those concerning on morasses. In Section 1 the theme is Todorcevic functions, in Section 2 several augmentations of morasses, and in Section 3 forcing axioms. Each section is logically, but not culturally, independent of the others.

Most of the terminology is fairly standard, and hopefully any which is not will be readily understandable. The cardinality of $x$ is $\bar{\bar{x}}$, while its ordertype and cofinality are $\mathrm{otp}(x)$ and $\mathrm{cf}(x)$, respectively. $\kappa$ will always be a *regular* infinite cardinal. $\mathscr{M} = \langle\langle\theta_\alpha \mid \alpha\leqslant\kappa\rangle, \langle\mathscr{F}_{\alpha\beta} \mid \alpha\leqslant\beta\leqslant\kappa\rangle\rangle$ will be a $(\kappa,1)$-simplified morass. Consequently each $\theta_\alpha$ is an ordinal and each $\mathscr{F}_{\alpha\beta}$ is a set of order preserving maps from $\theta_\alpha$ to $\theta_\beta$. Recall that the axioms for a $(\kappa,1)$-simplified morass are as follows:

- $\forall\alpha<\beta<\kappa\,(\theta_\alpha<\theta_\beta<\kappa)$ and $\theta_\kappa=\kappa^+$.
- $\forall\alpha\leqslant\beta<\kappa\,(\overline{\overline{\mathscr{F}_{\alpha\beta}}}<\kappa)$.
- $\forall\alpha\leqslant\beta\leqslant\gamma\leqslant\kappa\,(\mathscr{F}_{\alpha\gamma}=\{f\cdot g \mid f\in\mathscr{F}_{\beta\gamma}\ \&\ g\in\mathscr{F}_{\alpha\beta}\})$.
- $\forall\alpha\leqslant\kappa\,(\mathscr{F}_{\alpha\alpha}=\{\mathrm{id}\}$ and $\forall\alpha<\kappa$ either $\mathscr{F}_{\alpha\alpha+1}$ is a singleton or a pair $\{f, g\}$ such that for some ordinal $\sigma_\alpha$ (called the *splitting point* of $\mathscr{F}_{\alpha\alpha+1}$) $f\mid\sigma_\alpha = g\mid\sigma_\alpha$ and $\mathrm{rge}(f)\subseteq g(\sigma_\alpha)$.
- $\forall\lim(\alpha)\leqslant\kappa\,\forall\beta_0,\beta_1<\alpha\forall f_0\in\mathscr{F}_{\beta_0\alpha}\,\forall f_1\in\mathscr{F}_{\beta_1\alpha}\,\exists\gamma\,(\beta_0,\beta_1<\gamma<\alpha$ and $\exists f_0'\in\mathscr{F}_{\beta_0\gamma}$ $\exists f_1'\in\mathscr{F}_{\beta_1\gamma}\,\exists g\in\mathscr{F}_{\gamma\alpha}\,(f_0=g\cdot f_0'$ and $f_1=g\cdot f_1'))$.
- $\kappa^+=\bigcup\{f''\theta_\alpha \mid \alpha<\kappa\ \&\ f\in\mathscr{F}_{\alpha\kappa}\}$.

(This is an 'expanded simplified morass' in the terminology of [23]. It has become usual, see for example [26], to refer to these structures as simplified morasses, although it is still sometimes useful in applications to think about the set $\{f''\theta_\alpha \mid \alpha<\kappa\ \&\ f\in\mathscr{F}_{\alpha\kappa}\}$ – the meaning of 'simplified morass' in [23].)

Facts about simplified morasses will continually be used implicitly, particularly the fact that if $f, g\in\mathscr{F}_{\alpha\beta}$ and $\sup(f''\nu_1)=\sup(g''\nu_2)$ then $\nu_1=\nu_2$ and $f\mid\nu_1=g\mid\nu_2$, and consequently the relation $\lhd$, defined by $(\alpha,\nu)\lhd(\beta,\tau)$ if $f(\nu)=\tau$ for some $f\in\mathscr{F}_{\alpha\beta}$, is a tree relation. Finally recall that a $(\kappa,1)$-simplified morass is *neat* if $\theta_0=1$, and $\theta_\alpha=\bigcup\{f''\theta_\beta \mid f\in\mathscr{F}_{\beta\alpha}\}$ for each $\beta<\alpha\leqslant\kappa$, and it is *slow* if $\theta_0=1$,

- $\forall\alpha<\kappa$ either $\mathscr{F}_{\alpha\alpha+1}$ is *almost trivial*: $\mathscr{F}_{\alpha\alpha+1}=\{\mathrm{id}\}$ and $\theta_{\alpha+1}=\theta_\alpha+1$, or $\mathscr{F}_{\alpha\alpha+1}$ is *neat*: $\mathscr{F}_{\alpha\alpha+1}=\{\mathrm{id}, h_\alpha\}$ and $\theta_{\alpha+1}=\theta_\alpha\cup h_\alpha''\theta_\alpha$, and in the latter case $\alpha$ and $\sigma_\alpha$ are limit ordinals, and
- $\forall\lim(\alpha)\leqslant\kappa\,(\theta_\alpha=\bigcup\{f''\theta_\beta \mid \beta<\alpha\ \&\ f\in\mathscr{F}_{\beta\alpha}\})$.

The reader can consult the papers mentioned above for more on simplified morasses per se.

## 1. Morasses and Todorcevic functions

In the first two sections of [20] Todorcevic showed that $\square$ is a much more powerful construction principle in its own right than had previously been imagined. From it he extracted several stepping-up functions and so solved problems whose solutions were envisaged in more primitive times to require the use of morasses (see [20], Sections 1, 2, [21]). In this section functions with the same properties as Todorcevic's function $\rho$ are derived from simplified morasses. Consequently these functions form part of the non-trivial common core of simplified morasses and $\Box$'s. Work uncovering more of this core would be of great interest. The results proven below can be seen as confirming that the above-mentioned problems do have morass-theoretic solutions. (Note that, for $\kappa > \omega_1$, the existence of a $(\kappa, 1)$-simplified morass and $\Box_\kappa$ holding are mutually independent, relative to large cardinals–for one direction see [24], for the other the remarks at the end of this section.)

For the following definition let $\kappa$ be *any* infinite cardinal.

**Definition 1.** A *Todorcevic* function ($T$-function) is a function $f:[\kappa^+]^2 \to \kappa$ such that

(a) $\overline{\overline{\{\xi \leqslant \zeta \mid f(\xi,\zeta) < \nu\}}} < \kappa$, for all $\zeta < \kappa^+$and $\nu < \kappa$,

(b) $f(\xi,\nu) \leqslant f(\xi,\zeta)$ or $f(\zeta,\nu)$, for all $\xi < \zeta < \nu < \kappa^+$,

(c) $f(\xi,\zeta) \leqslant f(\xi,\nu)$ or $f(\zeta,\nu)$, for all $\xi < \zeta < \nu < \kappa^+$,

(d) for all $\lambda = \bigcup\lambda < \tau < \kappa^+$ there is some $\zeta(\lambda,\tau)$ such that $f(\xi,\lambda) \leqslant f(\xi,\tau)$, for all $\xi \in [\zeta(\lambda,\tau),\lambda)$.

In Todorcevic's original definition [20], Section 2.3) the bound in (a) is sharpened to $\nu^+$: the first infinite cardinal greater than $\nu$. This bound has not been used in any application of these functions considered so far, but let functions obeying (b)–(d) and the strengthened form of (a), say (a$^+$), be called $T^+$-functions.

Assume now that $\kappa$ is a regular cardinal. Let $\mathcal{M} = \langle\langle\theta_\alpha \mid \alpha \leqslant \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leqslant \beta \leqslant \kappa\rangle\rangle$ be a $(\kappa, 1)$-simplified morass. The following four functions can be defined immediately from $\mathcal{M}$.

**Definition 2.** (a) $d_\nu : \theta_\nu \to \nu + 1$, the "date-of-birth" function (at $\nu$), is given by

$$d_\nu(\xi) = \text{the least } \alpha \text{ such that } \exists f \in \mathcal{F}_{\alpha\nu}\,(\xi \in \mathrm{rge}(f)).$$

(b) $c_\nu : [\theta_\nu]^2 \to \nu + 1$, the "coupling-point" function (at $\nu$), is given by

$$c_\nu(\xi,\zeta) = \text{the least } \alpha \text{ such that } \exists f \in \mathcal{F}_{\alpha\nu}\,(\xi,\zeta \in \mathrm{rge}(f)).$$

(c) $a_\nu : [\theta_\nu]^2 \to [\nu + 1]^{<\omega}$, the "archive (of coupling points)" function, is defined as follows. Let $\xi,\zeta < \theta_\nu$ and for $\alpha < \nu + 1$ let $\xi_\alpha, \zeta_\alpha$ be such that $(\alpha,\xi_\alpha) \lhd (\nu,\xi)$ and $(\alpha,\zeta_\alpha) \lhd (\nu,\zeta)$. Let $\alpha_0 = c_\nu(\xi,\zeta)$. Inductively let $\beta_i$ be the immediate ordinal predecessor of $\alpha_i$, if there is one, and let $\alpha_{i+1} = c_{\beta_i}(\xi_{\beta_i},\zeta_{\beta_i})$ if $\xi_{\beta_i}$ and $\zeta_{\beta_i}$ exist and are distinct, and be undefined otherwise. Plainly $\alpha_{i+1} < \alpha_i$, so there is some $n$ such that $\alpha_n$ exist but $\alpha_{n+1}$ does not.

$$a_\nu(\xi,\zeta) = \langle\alpha_n, \alpha_{n-1}, \ldots, \alpha_0\rangle.$$

(d) $b_\nu : [\theta_\nu]^2 \to \nu + 1$, the "branching-point" function is given by

$$b_\nu(\xi,\zeta) = \alpha_n, \text{ where } \alpha_n \text{is defined from } \xi,\zeta \text{ as in (c).}$$

Write $f$ for $f_\kappa$ in each case. Note that if $\nu = \kappa$ the range of each function is $\kappa$ (or $[\kappa]^{<\omega}$) rather than $\kappa + 1$ (or $[\kappa + 1]^{<\omega}$).

It is perhaps helpful to elaborate somewhat on the previous definition. Firstly, if $\mathscr{M}$ is neat $\forall \xi\, d(\xi) = 0$, but if $\mathscr{M}$ is slow there will be points that are born further up the morass (for example, $\theta_{\alpha+1}+n+1$ for any $\alpha < \kappa$ and $n < \omega$). Secondly, $b_\nu(\xi, \zeta)$ is simply the point below $(\nu, \xi)$ and $(\nu, \zeta)$ where $\lhd$ branches if $(\nu, \xi)$ and $(\nu, \zeta)$ are in the same component of $\lhd$, and if $\mathscr{M}$ is neat (and $\theta_0 = 1$), $\lhd$ is connected. If, on the other hand, $(\nu, \xi)$ and $(\nu, \zeta)$ are not in the same component of $\lhd$ then $b_\nu(\xi, \zeta) = \max\{d(\xi), d(\zeta)\}$. Lastly, the note to the definition is explained by observing that if $\lim(\alpha) \leqslant \nu$, $f \in \mathscr{F}_{\alpha\nu}$, $f(\xi_\alpha) = \xi$ and $f(\zeta_\alpha) = \zeta$ and if neither $d(\xi_\alpha)$ nor $d(\zeta_\alpha) = \alpha$, then the directedness of $\mathscr{M}$ at $\alpha$ ensures $c_\nu(\xi, \zeta) < \alpha$.

Variants of $c_\nu$, $a_\nu$ and $b_\nu$ whose domains consist of longer sequences of ordinals can be considered. The function $a'_\nu$ given by $a'_\nu(\xi, \zeta) = \langle \alpha'_n, \ldots, \alpha'_0 \rangle$, where $a_\nu(\xi, \zeta) = \langle \alpha_n, \ldots, \alpha_0 \rangle$, $\alpha'_n = \alpha_n$ and $\alpha'_i = \mathrm{otp}(\alpha_i \setminus \alpha_{i+1})$ for $i < n$, is also of interest.

Recall that the partition relation $\kappa^+ \not\to [\lambda]^2_{\kappa,<\lambda}$ means that there is some $f : [\kappa^+]^2 \to \kappa$ such that $\forall A \in [\kappa^+]^\lambda\, (\overline{\overline{f''[A]^2}} = \lambda)$. It is immediate that both $b$ and $c$ demonstrate the relation $\kappa^+ \not\to [\kappa]^2_{\kappa,<\kappa}$, as would the branching function of any Kurepa tree.

In fact, $b$ shows the relation $\kappa^+ \not\to [\lambda]^2_{\kappa,<\lambda}$ for each $\lambda \leqslant \kappa$ if $\mathscr{M}$ has simple limits (see Definition 2.2(b)).

**Proposition 3.** *If $\mathscr{M}$ is a neat simplified morass $c\,(= c_\kappa)$ is a T-function.*

**Proof.** First of all here is some notation which is also useful elsewhere.

**Notation 4.** If $r \lhd s$ write $\psi_{rs}$ for the map from $r_1 + 1$ to $s_1 + 1$ which is the common part of the simplified morass maps witnessing $r \lhd s$. If $\mu < \kappa^+$ and $d(\mu) < \alpha < \kappa$ write $\mu_\alpha$ for the ordinal (less than $\theta_\alpha$) such that $(\alpha, \mu_\alpha) \lhd (\kappa, \mu)$ and $\psi^\alpha_\mu$ for $\psi_{(\alpha,\mu_\alpha)(\kappa,\mu)}$.

(a) Let $\zeta < \kappa^+$ and $\nu < \kappa$. If $\nu \leqslant d(\zeta)$ then $S = \{\xi \leqslant \zeta \mid c(\xi, \zeta) < \nu\}$ is empty. Otherwise let $(\nu, \zeta_\nu) \lhd (\kappa, \zeta)$. If $c(\xi, \zeta) < \nu$ then $\exists \bar{\xi} (= \xi_\nu) \leqslant \zeta_\nu (\psi^\nu_\zeta(\bar{\xi}) = \xi)$. Thus $\overline{\overline{S}} \leqslant \zeta_\nu + 1 \leqslant \theta_\nu < \kappa$.

(Clearly, $c$ also satisfies ($\mathrm{a}^+$) if for each $\alpha < \kappa$ $\theta_\alpha < \alpha^+$ . This occurs automatically if $\kappa < \omega_2$, and if $\kappa$ is not a limit cardinal it is easy to convert $\mathscr{M}$ into a $(\kappa, 1)$-simplified morass with this property. Further information about which limit $\kappa$ have $(\kappa, 1)$-simplified morass with small limits is given in Section 2.)

(b), (c) Let $\xi < \zeta < \nu < \kappa^+$. Set $\alpha = c(\xi, \zeta)$. Suppose $c(\zeta, \nu) \leqslant c(\xi, \zeta)$. As $\psi^\alpha_\nu \restriction (\zeta_\alpha + 1) = \psi^\alpha_\zeta$, the supposition gives $\psi^\alpha_\nu(\xi_\alpha) = \psi^\alpha_\zeta(\xi_\alpha) = \xi$, and thus $c(\xi, \nu) \leqslant c(\xi, \zeta)$. Moreover, if $\max\{c(\xi, \nu), c(\zeta, \nu)\} \leqslant \beta \leqslant \alpha$ the maps witnessing $c(\xi, \nu) \leqslant \beta$ and $c(\zeta, \nu) \leqslant \beta$ are both equal to $\psi^\beta_\nu$ on $\nu_\beta$, and so agree at $\zeta_\beta$. Thus $c(\xi, \zeta) \leqslant \beta$ and so $\beta = \alpha$. Whence either $c(\xi, \zeta) = c(\zeta, \nu)$, which implies $d(\xi)$, $d(\zeta)$ or $d(\nu) = \alpha$, or $c(\xi, \zeta) = c(\xi, \nu)$. Similarly if $c(\zeta, \nu) \leqslant c(\xi, \nu)$ then $c(\xi, \zeta) = c(\xi, \nu)$ or $c(\zeta, \nu) = c(\xi, \nu)$. Thus if $c(\zeta, \nu) < c(\xi, \zeta)$ or $c(\xi, \nu)$ then $c(\xi, \zeta) = c(\xi, \nu)$. (b) and (c) follow immediately.

(d) Let $\lambda < \nu < \kappa^+$ and $\lim(\lambda)$. Let $S = \{\xi < \lambda \mid c(\xi, \nu) < c(\xi, \lambda)\}$. The aim is to find some bound $\zeta < \lambda$ for $S$. If $S = \emptyset$ it suffices to set $\zeta = 0$, so suppose $S \neq \emptyset$. It is at this point (only) that the assumption of neatness comes in. It is needed locally, i.e. what is actually needed is that $d(\lambda) \leqslant d(v)$. Of course, this holds if $\mathscr{M}$ is neat, since that assumption is equivalent to assuming that $d(\gamma) = 0$ for all $\gamma < \kappa^+$.

Let $a(\lambda, v) = \langle \alpha_n, \ldots, \alpha_0 \rangle$. Note that each $\alpha_i$ is a successor ordinal, so setting $\beta_i + 1 = \alpha_i$ for $i \leqslant n$ is a good definition. Let $k \leqslant n$ be the least $j$ such that $\sigma_{\beta_j} < \lambda_{\beta_j}$ and, for $i < j$, $\sigma_{\beta_i} = \lambda_{\beta_i} (< v_{\beta_i})$ if there is any such $j$, and be undefined otherwise. Note that $\lambda_\delta \leqslant v_\delta$ for all $\delta \leqslant \kappa$ if $k$ is undefined, and for all $\delta \in [\alpha_k, \kappa]$ if $k$ is defined.

Suppose $\xi \in S$. Temporarily setting $\gamma = \delta + 1 = c(\xi, \lambda)$, one has $\sigma_\delta \leqslant \xi_\delta < v_\delta = v_\gamma < \theta_\delta \leqslant \lambda_\gamma$. Thus $k$ is defined, $c(\xi, \lambda) \leqslant \beta_k$, the latter shows that $\xi_{\beta_k} < \sigma_{\beta_k}$. As $\xi \in S$ was arbitrary it clearly suffices to set $\zeta = \psi_\lambda^{\beta_k}(\sigma_{\beta_k})$.

Notice, the assumption that $\lambda$ is a limit ordinal has not been used, and so can be dropped.

The definition of Todorcevic's $\rho$ can seem rather mysterious. The function $c$, on the other hand, can be thought of as being built up by a familiar morass construction if the reader so desires, because for each

$$\alpha \leqslant \kappa, \{(\xi, \zeta, \gamma) \mid c_\alpha(\xi, \zeta) = \gamma\} = \bigcup_{\varepsilon < \alpha} \{(f(\bar{\xi}), f(\bar{\zeta}), \bar{\gamma}) \mid c_\varepsilon(\bar{\xi}, \bar{\zeta}) = \bar{\gamma} \;\&\; f \in \mathscr{F}_{\varepsilon\alpha}\}$$

$$(\cup \{(j_\beta(\xi), h_\beta(\zeta), \alpha) \mid \xi, \zeta \in [\sigma_\beta, \theta_\beta)\}$$

$$\cup \{(\xi_0, \xi_1, \alpha) \mid \exists i < 2\; d(\xi_i) = \alpha\} \text{ if } \alpha = \beta + 1).$$

In an earlier version of this paper it was shown that a certain function extracted from an $(\omega, 1)$-simplified morass witnessed Todorcevic's theorem $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$ (see [20], Section 4). The existence of such a morass is a theorem of ZFC (see [23]). Despite the use of morass technology, which to some extent simplified the combinatorics of the proof, the definition of the function was rather complicated. The specific definition is not given here, since after reading the earlier version Velleman came up with a very elementary proof of Todorcevic's result. (The reader should consult [27].)

Returning to $T$-functions, Velickovic showed that when $\square_\kappa$ holds Todorcevic's $T$-function $\rho$ also has the following properties.

**Fact 5** (Velickovic, Lemmas 4.3–4.5).

(a) $\forall X, Y \in [\kappa^+]^\kappa (\alpha = \sup(X) \leqslant \beta = \sup(Y)$

$$\rightarrow \exists X' \in [X]^\kappa\, \exists Y' \in [Y]^\kappa\, \forall \xi \in X' \forall \eta \in Y' [\xi < \eta \rightarrow \rho(\xi, \alpha) \leqslant \rho(\xi, \eta)]).$$

(b) $\forall X, Y \in [\kappa^+]^\kappa \forall \gamma < \kappa^+ \exists X' \in [X]^\kappa\, \exists Y' \in [Y]^\kappa\, \forall \xi \in X'\, \forall \eta \in Y'\, (\rho(\xi, \eta)$

$$\geqslant \rho(\xi, \gamma) \textit{ or } \rho(\eta, \gamma)).$$

(c) $\forall F \in [[\omega_2]^{<\omega}]^{\omega_1} \forall \mu < \omega_1 \exists x, y \in F\, \forall \alpha \in (x \setminus y) \forall \beta \in (y \setminus x) \forall \gamma \in (x \cap y)$

$$(\rho(\alpha, \beta) \geqslant \mu \text{ and at least one of } \rho(\alpha, \gamma) \text{ and } \rho(\beta, \gamma)). \quad (\textit{When } \kappa = \omega_1.)$$

**Proposition 6.** *If $\mathscr{M}$ is neat, $c$ satisfies the conditions* 5(a)–(c).

**Proof.** Velickovic showed that (b) and (c) follow from (a), so it suffices to prove this alone. The structure of this proof is similar to Velickovic's, although, of course, the details are different. Let $X$, $Y$, $\alpha$ and $\beta$ be as in the statement of (5(a)). For $\varepsilon < \kappa$ let $\eta^\varepsilon$ be the least $\eta$ such that $\operatorname{dom} \psi_\alpha^\varepsilon \setminus \{\alpha\} \subseteq \eta$. (Hence $(\varepsilon, \alpha_\varepsilon) \lhd (\kappa, \eta^\varepsilon)$.) Then $C = \{\eta^\varepsilon \mid \varepsilon < \kappa\}$ is closed and unbounded in $\alpha$. If $\eta \in C$ and $\gamma < \eta$ then either $c(\eta, \alpha) < c(\gamma, \alpha) = c(\gamma, \eta)$ or $c(\eta, \alpha) \geqslant c(\gamma, \alpha)$, $c(\gamma, \eta)$. In the latter case clearly $c(\gamma, \alpha) = c(\gamma, \eta)$ again as $\alpha_\varepsilon = \eta_\varepsilon^\varepsilon$. For each $\eta \in C$ pick some distinct $\delta_\eta \in Y \setminus \eta$ and use property (1.1d) to pick $\zeta_\eta < \eta$ such that $c(\xi, \delta_\eta) \geqslant c(\xi, \eta) = c(\xi, \alpha)$ for all $\xi \in [\zeta_\eta, \eta)$. By Fodor's Lemma fix $S \in [C]^\kappa$ and $\gamma < \alpha$ such that $\forall \eta \in S\ \zeta_\eta < \gamma$. Let $\bar{X} = X \setminus \gamma$. If $\alpha = \beta$, let $X' \in [\bar{X}]^\kappa$, $S' \in [S]^\kappa$ and $Y' \in [\{\delta_\eta \mid \eta \in S'\}]^\kappa$ be such that for all $\eta$ in $S'$, $[\eta, \delta_\eta) \cap X' = \emptyset$. If $\alpha < \beta$, let $X' \in [\bar{X}]^\kappa$, $S' \in [S]^\kappa$ and $Y' \in [\{\delta_\eta \mid \eta \in S'\}]^\kappa$ be such that $c(\xi, \alpha) > c(\alpha, \delta_\eta)$ for all $\xi \in X'$ and $\eta$ in $S'$ with $\xi > \eta$. For then $c(\xi, \alpha) \leqslant c(\xi, \delta_\eta)$ or $c(\alpha, \delta_\eta)$ and $c(\xi, \delta_\eta) \geqslant c(\xi, \alpha)$.

Todorcevic and Velickovic have found many uses for $T$-functions, and $T$-functions satisfying 5(a)–(c) in stepping up arguments. These are arguments whereby some proposition about $\kappa$ is transferred to a similar, but usually slightly weaker one about $\kappa^+$ (although often the weakened result is known to be best possible). The principle use of $T$-functions in [22] is to show that a certain forcing is ccc. Baumgartner and Shelah have also introduced a variety of function designed to help show certain forcing notions have ccc. They call their functions "functions with the property $\Delta$".

**Definition 7** ([2], *Section 8*). A function $f : [\omega_2]^2 \to [\omega_2]^{\leqslant \omega}$ has property $\Delta$ if

(i) $\quad \forall \xi < \zeta < \omega_2\ f(\xi, \zeta) \subseteq \xi \quad$ and

(ii) $$\forall D \in [[\omega_2]^{<\omega}]^{\omega_1}\ \exists a, b \in D\, \forall \xi \in (a \setminus b)\, \forall \zeta \in (b \setminus a)\, \forall \nu \in (a \cap b)$$
$$(\xi, \zeta > \nu \to \nu \in f(\xi, \zeta))\ \&\ (\nu < \zeta \to f(\xi, \nu) \subseteq f(\xi, \zeta))\ \&\ (\nu < \xi \to f(\zeta, \nu) \subseteq f(\xi, \zeta)).$$

These functions are closely related to $T$-functions (as was asserted in [22]).

**Proposition 8.** *If there is a $T$-function satisfying* 5(a)–(c) *then there is a $\Delta$-function.*

**Proof.** Let $g$ be the $T$-function. Set $f(\xi, \zeta) =_{\text{def}} \{\delta < \xi \mid g(\delta, \xi) \leqslant g(\xi, \zeta)\}$. Note $g(\delta, \zeta) \leqslant g(\xi, \zeta)$ if and only if $g(\delta, \xi) \leqslant g(\xi, \zeta)$. Suppose that $D$ is an uncountable subset of $[\omega_2]^{<\omega}$. Without loss of generality suppose that $D$ is a $\Delta$-system, and list it as $\langle a_\tau \mid \tau < \omega_1 \rangle$ with $a_\tau \cap a_\sigma = a$ and $\max(a_\tau) < \min(a_\sigma \setminus a)$ for all $\tau < \sigma < \omega_1$. Now apply (5(c)) to $D$. Let $\tau < \sigma < \omega_1$ be such that given $\nu \in a$, $\xi \in a_\tau$ and $\zeta \in a_\sigma$ both $g(\nu, \xi)$ and $g(\nu, \zeta) \leqslant g(\xi, \zeta)$. Clearly, $\nu \in f(\xi, \zeta)$. So suppose $\delta \in f(\nu, \xi)$. The aim is to show that $\delta \in f(\xi, \zeta)$. Both $g(\delta, \nu)$ and $g(\delta, \xi) \leqslant g(\nu, \xi)$, and so $g(\delta, \xi)$, $g(\delta, \nu) \leqslant g(\nu, \xi) \leqslant g(\xi, \zeta)$. The proof that $f(\nu, \zeta) \subseteq f(\xi, \zeta)$ is symmetrical. Thus $f$ has property $\Delta$.

Baumgartner and Shelah obtained a function with property $\Delta$ by forcing. In their "Notes added in proof" they observe that their forcing "is really just adding a certain kind of simplified morass." In fact their forcing adds a simplified morass with continuous paths (cf. [17]), a stronger concept than that of a simplified morass on its own. (See Section 2 for further details on this point.) They use both the paths and the fact that the function is added generically to check its properties. However, Proposition 8 shows that this is unnecessary as both an $(\omega_1, 1)$-simplified morass and (its consequence) $\square_{\omega_1}$ guarantee the existence of a $\Delta$-function.

It was remarked upon earlier that the subject of the true extent of the overlap between morasses and $\square$ remains of interest. This is so even in the case $\kappa = \omega_1$, when much is known. For example, while $\square_{\omega_1}$ holds if there is a $(\omega_1, 1)$-simplified morass, it is consistent relative to an inaccessible that $\square_{\omega_1}$ holds and there is no $(\omega_1, 1)$-simplified morass. To see this is so one may collapse the inaccessible to $\omega_2$ while simultaneously adding a $\square_{\omega_1}$ sequence by using conditions consisting of closed subsets of countably many limit ordinals smaller than the inaccessible with a suitable coherence property. Silver's argument that there are no Kurepa trees in the Levy collapse of an inaccessible to $\omega_2$ can be adapted to this forcing. Alternatively, one may argue that if there is an inaccessible in $V$ there must be a least one in $L$. Taking $L$ as the ground model, Levy collapsing this inaccessible over $L$ to $\omega_2$ guarantees that there are no Kurepa trees and so no $(\omega_1, 1)$-simplified morasses, while $\square_{\omega_1}$ must hold since $\omega_2$ in the extension, being the least inaccessible in $L$, is not Mahlo in $L$.

Jensen showed that if $\square_{\omega_1}$ holds there is a ccc forcing to add a Kurepa tree, and Velickovic [22] gave a proof of this using $T$-functions. However Todorcevic (in personal communicaton) has observed that there can be no ccc forcing to add an $(\omega_1, 1)$-simplified morass unless the weak Chang conjecture, wCC($\omega_1$), fails. But wCC($\omega_1$) is similarly independent of the existence of $\square_{\omega_1}$ and hence of the existence of a $T$-function. It would be interesting to find conditions that did suffice ensure that there was a ccc forcing to add an $(\omega_1, 1)$-simplified morass, or indeed to add a $\square_{\omega_1}$ sequence.

## 2. Continuity and coherence principles for morasses

It is frequently useful in constructions involving a simplified morass to have a greater degree of control over the limit stages than that afforded by the simplified morass alone. In this section various ways of augmenting the notion of a simplified morass which putatively facilitate limit steps of inductions will be examined. The idea is to enable morasses to be used in building objects from collections of pieces which are not closed under full directed limits but do satisfy some weaker closure notion. Equivalently, the forcing axioms corresponding to these augmented simplified morasses cover a wider class of forcings than the axiom corresponding to simplified morasses alone.

Let $\mathscr{M} = \langle \langle \theta_\alpha \mid \alpha \leqslant \kappa \rangle, \langle \mathscr{F}_{\alpha\beta} \mid \alpha \leqslant \beta \leqslant \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass. Immediately below a rather general formalism is set up which facilitates discussion of various restrictions on the limits of $\mathscr{M}$.

**Definition 1.** If $\alpha < \kappa$ and $\lim(\alpha)$ a *simple limit at* $\alpha$ is a club set $D_\alpha \subseteq \alpha$ and a collection of maps $\langle g^\alpha_\beta \mid \beta \in D_\alpha \rangle$ such that $g^\alpha_\beta \in \mathscr{F}_{\beta\alpha}$ for each $\beta \in D_\alpha$,

$$\forall \beta, \gamma \in D_\alpha\, (\beta < \gamma \to \exists f \in \mathscr{F}_{\beta\gamma}\, g^\alpha_\beta = g^\alpha_\gamma \cdot f) \quad \text{and} \tag{1}$$

$$\forall \beta < \alpha\, \forall f \in \mathscr{F}_{\beta\alpha}\, \exists \gamma \in D_\alpha \setminus \beta\, \exists f' \in \mathscr{F}_{\beta\gamma}\, (f = g^\alpha_\gamma \cdot f'). \tag{2}$$

A *path* at $\alpha$ is a simple limit such that $D_\alpha = \alpha$. A simple path or limit at successor $\alpha$ satisfies Definition 1 with clause (2) omitted.

For any set $C$ which is club in some ordinal, let $C'$ be the set of limit points of $C$.

**Definition 2.** (a) $\mathscr{M}$ has *small limits* if $\overline{\overline{\theta_\alpha}} = \overline{\overline{\alpha}}$ for each $\lim(\alpha) < \kappa$.

(b) $\mathscr{M}$ has *simple limits* if it has a simple limit at each $\lim(\alpha) < \kappa$.

(c) $\mathscr{M}$ has *continuous limits* if it has simple limits such that for each $\lim(\alpha) < \kappa$,

$$\forall \gamma \in D'_\alpha\, \forall \beta < \gamma\, \forall f \in \mathscr{F}_{\beta\gamma}\, \exists \delta \in D_\alpha \cap [\beta, \gamma)\, \exists f' \in \mathscr{F}_{\beta\delta}\, (g^\alpha_\gamma \cdot f = g^\alpha_\delta \cdot f'). \tag{3}$$

(d) $\mathscr{M}$ has *linear limits* if it has a simple limit at each $\lim(\alpha) < \kappa$ such that

$$\forall \gamma \in D'_\alpha\, (D_\gamma = D_\alpha \cap \gamma \quad \text{and} \quad \forall \beta \in D_\gamma\, g^\alpha_\beta = g^\alpha_\gamma \cdot g^\gamma_\beta). \tag{4}$$

In Definition 2(b)–(d) each reference to simple limits can be replaced by a reference to paths, leading to the concepts of $\mathscr{M}$ having *paths*, *continuous paths* and *linear paths*. (The terminology would probably be improved if one could use coherent limits for linear limits and linear limits for simple limits, but this would conflict with the usage in [24] in which linear limits were introduced.)

There are several ways in which each pair of properties could be related. It could be that every simplified morass with property (x) has property (y), or that if there is a simplified morass with (x) then there is one with (y), or it is consistent that if there is a simplified morass with (x) there is one with (y). Note that the definition is arranged in order of (formally) increasing strength, and that each notion for paths is formally stronger than that for simple limits, in both cases in the first sense above. It is also clear that in this first sense none of the properties imply any of those below them.

If $\kappa = \omega_2$ and $\mathscr{M}$ is a $(\kappa, 1)$-simplified morass with small limits the first $\omega_1 \cdot 2$ levels of the morass can be dispensed with and replaced by any simplified morass segment of height $\omega_1 + \omega$ with at least $\omega_1$ distinct maps in $\mathscr{F}_{\omega_1\omega_1+\omega}$. In the revised simplified morass $\mathrm{cf}(\bigcup\{\mathrm{rge}(f) \mid f \in \mathscr{F}_{\omega_1\omega_1+\omega}\}) = \omega_1$ and thus there can be no simple path at $\omega_1 + \omega$, since if there were the union would have cofinality $\omega$. Nevertheless, the morass will continue to enjoy small limits.

If $\mathscr{M}$ has linear limits is simple to alter some of the limits maps so that the revised limits are continuous but not linear: for example take any $\alpha$ and $\gamma \in D'_\alpha$ such that $\mathrm{otp}(D_\gamma) = \omega$ and replace the map at the smallest element of $D_\gamma$ by any other map which factors through the map at the next element. Similarly it is easy to obtain a simplified morass with simple but not continuous paths.

But it is not clear a priori what the relationship is between, say, continuous paths and linear limits in the other two senses. In fact this particular question was the starting point of these investigations.

Passing to less trivial points, in [17] it was shown that if $\mathcal{M}$ has linear limits it also has continuous paths. (In fact the definition of continuous paths in [17] specifies that there should be a path for successor $\alpha < \kappa$ as well as for limits, but we can simply use induction on $\kappa$ to fill the gaps. If $\alpha = \alpha' + 1$ we can take $g^{\alpha}_{\alpha'}$ to be any map in $\mathscr{F}_{\alpha'\alpha}$ and set $g^{\alpha}_{\beta} = g^{\alpha}_{\alpha'} \cdot g^{\alpha'}_{\beta}$ for each $\beta < \alpha'$.) By analysing the proof one obtains that if $\mathcal{M}$ has simple limits it has simple paths, and if it has continuous limits it has continuous paths. However, as Velleman showed in [17], no $(\kappa, 1)$-simplified morass has linear paths. This latter fact was also implicitly proved earlier by Donder in the proof of Lemma 1 of [6]. That lemma shows that linear limits cannot be "trivialised on a cofinal set": i.e. there can be no cofinal set $S \subseteq \kappa$ such that for all $\alpha$ and $\beta$ in $S$ if $\alpha < \beta$ then $D_\alpha = D_\beta \cap \alpha$, or, equivalently, such that for all $\alpha$ in $S$, $D_\alpha = S \cap \alpha$ (whence $S$ would be club). Linear paths are trivialised by the well-known cofinal subset of $\kappa$, $\kappa$ itself, while the consequence Donder drew was that if $\kappa$ is weakly compact some club would trivialise any putative limits and so there is no $(\kappa, 1)$-simplified morass with linear limits.

The main result of [6] complements this conclusion, showing that if $V = L$ and $\kappa$ is not weakly compact there is a $(\kappa, 1)$-simplified morass with linear limits. At the other end of the scale, if $\kappa$ is ineffable there is no $(\kappa, 1)$-simplified morass with small limits, since the inherent tree of a $(\kappa, 1)$-simplified morass with small limits obviously witnesses the $\kappa$-Kurepa Hypothesis, which must fail if $\kappa$ is ineffable. Perhaps surprisingly, in $L$ at least, this dichotomy encompasses all of the concepts mentioned so far. For it will be shown in this section that if $\kappa$ is weakly compact there is no $(\kappa, 1)$-simplified morass with continuous limits, and if $\kappa$ is not ineffable there is a $(\kappa, 1)$-simplified morass with simple limits. Consequently $L$ fails to distinguish between seemingly different notions: small and simple limits and linear and continuous limits.

First of all it is established, as promised, that simple and continuous limits give simple and continuous paths, respectively, and hence that one may concentrate attention on whichever of the versions of these continuity concepts is more convenient for each particular purpose.

**Lemma 3.** *Suppose $\mathcal{M}$ (a simplified morass) has simple limits. Then $\mathcal{M}$ has simple paths. Moreover if the simple limits are continuous the paths shown to exist will also be continuous.*

**Proof.** The construction used will be that of [17, Lemma 2.1], in which linear limits were used to obtain continuous paths. The weaker hypotheses of this lemma will suffice to prove its conclusions. Suppose $\langle \langle g^{\alpha}_{\beta} \mid \beta \in D_\alpha \rangle \mid \lim(\alpha) < \kappa \rangle$ is a collection of simple limits for $\mathcal{M}$. It will be proven by induction on $\alpha$ (including successors) that given any $\beta < \alpha$ and $f \in \mathscr{F}_{\beta\alpha}$ there are simple paths at $\alpha$ such that $f^{\alpha}_{\beta} = f$. If $\alpha$ is a successor it is easy to "fill in" the gap between $\alpha$ and the largest limit smaller than it as indicated above, and then use composition with the path at this ordinal to complete the definition. So assume $\alpha$ is a limit ordinal. First choose $\zeta \in D_\alpha$ such that there is some $f' \in \mathscr{F}_{\beta\zeta}$ such that $f = g^\alpha_\zeta \cdot f'$. By the inductive hypothesis choose a simple path $\langle f^\zeta_\gamma \mid \gamma < \zeta \rangle$ at $\zeta$ such that $f^\zeta_\beta = f'$. Now, again by the inductive hypothesis, choose simple paths $\langle \langle f^\delta_\gamma \mid \gamma < \delta \rangle \mid \delta \in D_\alpha \setminus \zeta + 1 \rangle$ such that if $\delta$ is the successor of $\delta^*$ in $D_\alpha$, then $g^\alpha_\delta \cdot f^\delta_{\delta^*} = g^\alpha_{\delta^*}$. Next one combines these paths to get a simple path at $\alpha$. For $\gamma < \alpha$ set $f^\alpha_\gamma = g^\alpha_\delta \cdot f^\delta_\gamma$, where $\delta$ is the least element of $E = (D_\alpha \setminus \zeta)$ greater than $\gamma$.

Clearly, the definition ensures $f^\alpha_\beta = f$. Now one must check that $\langle f^\alpha_\gamma \mid \gamma < \alpha \rangle$ is a simple path at $\alpha$.

Firstly, check that (1) holds. Suppose $\beta < \gamma < \alpha$, so that $f^\alpha_\beta = g^\alpha_\delta \cdot f^\delta_\beta$ and $f^\alpha_\gamma = g^\alpha_\varepsilon \cdot f^\varepsilon_\gamma$, for some $\delta \leqslant \varepsilon$ in $E$. If $\delta = \varepsilon$, (1) holds since it does for the path at $\delta$. So suppose $\delta < \varepsilon$, in which case $\delta \leqslant \varepsilon^*$. By (1) for the original limits $f^\alpha_\beta = g^\alpha_{\varepsilon^*} \cdot h$ for some $h \in \mathscr{F}_{\beta\varepsilon^*}$. Since $g^\alpha_{\varepsilon^*} = g^\alpha_\varepsilon \cdot f^\varepsilon_{\varepsilon^*}$, it follows from (1) for the path at $\varepsilon$ that $f^\alpha_\beta = g^\alpha_\varepsilon \cdot f^\varepsilon_\gamma \cdot k$ for some $k \in \mathscr{F}_{\beta\gamma}$. As $g^\alpha_\varepsilon \cdot f^\varepsilon_\gamma = f^\alpha_\gamma$ by definition, $f^\alpha_\beta = f^\alpha_\gamma \cdot k$ as required.

Secondly, check (2) for $\lim(\alpha)$. If $\gamma < \alpha$ and $f \in \mathscr{F}_{\gamma\alpha}$, there is some $\delta \geqslant \zeta$ such that $f = g^\alpha_\delta \cdot f'$ for some $f' \in \mathscr{F}_{\gamma\delta}$. But by the definition $f^\alpha_\delta = g^\alpha_\delta$, for $\delta \in D_\alpha \setminus \zeta$, so $f = f^\alpha_\delta \cdot f'$.

Finally, suppose that the original limits were continuous limits, and, inductively, that each time one choses a simple path, a continuous path is chosen. To check that (3) holds at $\alpha$, suppose $\beta < \lim(\gamma) < \alpha$. If $\alpha$ is a successor and $\gamma$ is the largest limit ordinal smaller than it, (3) holds trivially by the definition of the paths at $\alpha$. So assume $\lim(\alpha)$ and let $f \in \mathscr{F}_{\beta\gamma}$. For $\delta < \alpha$ let $\xi_\delta$ be the least element of $E$ greater than $\delta$.

If there is some $\delta < \gamma$ such that $\xi_\gamma = \xi_\delta, = \xi$ say, then $f^\alpha_\gamma \cdot f = g^\alpha_\xi \cdot f^\xi_\gamma \cdot f = g^\alpha_\xi \cdot f^\xi_\delta \cdot f'$, for some $f' \in \mathscr{F}_{\beta\delta}$, by the inductive hypothesis that (3) holds for the path at $\xi$. But $g^\alpha_\xi \cdot f^\xi_\delta \cdot f' = f^\alpha_\delta \cdot f'$, as needed.

If there is no $\delta \in [\beta, \gamma)$ such that $\xi_\delta = \xi_\gamma$, then $\gamma \in D_\alpha \setminus \zeta$ and $f^\alpha_\gamma = g^\alpha_\gamma$. Suppose firstly that $\gamma \in D_\alpha \setminus (D'_\alpha \cup \zeta)$, and let $\gamma^*$ be the predecessor of $\gamma$ in $D_\alpha$. Then for $\delta \in [\gamma^*, \gamma)$, $f^\alpha_\delta = g^\alpha_\gamma \cdot f^\gamma_\delta$. By the inductive hypothesis (2) holds for $\gamma$, since $\gamma$ is a limit ordinal, so $f = f^\gamma_\delta \cdot f'$ for some $\delta < \gamma$ and $f' \in \mathscr{F}_{\beta\delta}$, where without loss of generality $\beta, \gamma^* \leqslant \delta$. Thus $f^\alpha_\gamma \cdot f = g^\alpha_\gamma \cdot f = g^\alpha_\gamma \cdot f^\gamma_\delta \cdot f' = f^\alpha_\delta \cdot f'$.

The remaining case is that $\gamma \in D'_\alpha \setminus \zeta$. By (3) for the original limits, there is some $\delta \in D_\alpha \cap [\beta, \gamma)$ such that $g^\alpha_\gamma \cdot f = g^\alpha_\delta \cdot f'$, and (by (1)) one may assume that $\zeta < \delta$. Observing that $f^\alpha_\delta = g^\alpha_\delta$ finishes the proof.

Now turn to the resemblances between continuous and linear limits. It is more or less immediate that when (2) holds, (3) is equivalent to:

$$\forall \gamma \in D'_\alpha\, \forall \beta \in D_\gamma\, \exists \delta \in D_\alpha \cap [\beta, \gamma)\, \exists f \in \mathscr{F}_{\beta\delta}\, (g^\alpha_\gamma \cdot g^\gamma_\delta \cdot f = g^\alpha_\delta \cdot f \text{ and } g^\gamma_\delta \cdot f = g^\gamma_\beta). \tag{3$'$}$$

**Proof.** ($\Rightarrow$) Suppose $\alpha$, $\beta$ and $\gamma$ are as in the hypothesis of (3′). Applying (3) to the map $g^\gamma_\beta$ gives some $\delta \in [\beta, \gamma)$ and $f \in \mathscr{F}_{\beta\delta}$ such that $g^\alpha_\gamma \cdot g^\gamma_\beta = g^\alpha_\delta \cdot f$. By (1), $g^\gamma_\beta = g^\gamma_\delta \cdot f'$ for some $f' \in \mathscr{F}_{\beta\delta}$, and thus $g^\alpha_\gamma \cdot g^\gamma_\delta \cdot f' = g^\alpha_\delta \cdot f$. But then $f = f'$. (Since given any two pairs of maps $h_i$, $k_i$ ($i = 0, 1$) in a simplified morass, if $h_0 \cdot k_0 = h_1 \cdot k_1$ and the

$h_i$ have the same domain, which contains the ranges of the $k_i$, one has $h_0 = h_1$ on $\mathrm{rge}(k_0) \cup \mathrm{rge}(k_1)$, and so $k_0 = k_1$.

($\Leftarrow$) Let $\alpha$, $\beta$, $\gamma$ and $f$ be given as in the hypothesis of (3). By (2) there is some $\delta \in D_\gamma \setminus \beta$ and some $f' \in \mathscr{F}_{\beta\delta}$ such that $f = g^\gamma_\delta \cdot f'$. By (3′) there is some $\varepsilon \in [\delta, \gamma)$ and some $f'' \in \mathscr{F}_{\delta\varepsilon}$ such that $g^\gamma_\delta = g^\gamma_\varepsilon \cdot f''$ and $g^\alpha_\varepsilon \cdot f'' = g^\alpha_\gamma \cdot g^\gamma_\varepsilon \cdot f'' = g^\alpha_\gamma \cdot g^\gamma_\delta$. Consequently $g^\alpha_\gamma \cdot f = g^\alpha_\varepsilon \cdot f'' \cdot f'$, as required.)

This observation will be particularly useful when considering forcing axiom characterisations of augmentations of morasses in Section 3.

The following lemma relates two variants of continuous limits.

**Lemma 4.** *If $\mathscr{M}$ has simple limits then for each* $\lim(\alpha) < \kappa$ *and* $\gamma \in D'_\alpha$,

$$\forall \beta < \gamma \, \forall f \in \mathscr{F}_{\beta\gamma} \, \exists \delta \in D_\alpha \cap D_\gamma \setminus \beta \, \exists f' \in \mathscr{F}_{\beta\delta} \, (g^\alpha_\gamma \cdot f = g^\alpha_\delta \cdot f') \iff \tag{5}$$

$$\forall \beta \in D_\gamma \, \exists \delta \in D_\alpha \cap D_\gamma \setminus \beta \, \exists f \in \mathscr{F}_{\beta\delta} \, (g^\alpha_\gamma \cdot g^\gamma_\delta \cdot f = g^\alpha_\delta \cdot f \text{ and } g^\gamma_\delta \cdot f = g^\gamma_\beta). \tag{5′}$$

*And if the limits are paths then* (3), (3′) *and their analogues derived from* (5) *and* (5′) *are all equivalent.*

Suppose that $\mathscr{M}$ has continuous limits $\langle g^\alpha_\beta \mid \beta \in D_\alpha \rangle$. For convenience set $g^\alpha_\alpha = \mathrm{id} \mid \theta_\alpha$. For each limit $\alpha < \kappa$, let $h^\alpha_{\beta\gamma} = (g^\alpha_\gamma)^{-1} \cdot g^\alpha_\beta$ for $\beta \leqslant \gamma$ and $\beta, \gamma \in D_\alpha \cup \{\alpha\}$. Note that (1) in the definition of simple limit guarantees that each $h^\alpha_{\beta\gamma}$ is an element of $\mathscr{F}_{\beta\gamma}$.

**Lemma 5.** *For each limit $\alpha < \kappa$, $\langle \langle h^\alpha_{\beta\gamma} \mid \beta \in D_\alpha \cap \gamma \rangle \mid \gamma \in D'_\alpha \cup \{\alpha\} \rangle$ is a collection of fake linear limits, that is, obeys* (1), (2) *and* (4) *with $D_\alpha \cup \{\alpha\}$ in place of $\kappa$ (changing the other bound variables suitably of course).*

**Proof.** Fix $\beta$, $\gamma$ and $\delta \in D_\alpha \cup \{\alpha\}$ with $\beta \leqslant \gamma \leqslant \delta$. Firstly, by (1),

$$h^\alpha_{\beta\delta} = (g^\alpha_\delta)^{-1} \cdot g^\alpha_\beta = (g^\alpha_\delta)^{-1} \cdot g^\alpha_\gamma \cdot f = h^\alpha_{\gamma\delta} \cdot f$$

for some $f \in \mathscr{F}_{\beta\gamma}$. Secondly, if $\delta \in D'_\alpha \cup \{\alpha\}$, then, by (3) (resp. (2) for $\delta = \alpha$), for each $f \in \mathscr{F}_{\beta\delta}$ there is some $\varepsilon \in [\beta, \delta) \cap D_\alpha$ such that

$$g^\alpha_\delta \cdot f = g^\alpha_\varepsilon \cdot f' \text{ and so } f = (g^\alpha_\delta)^{-1} \cdot g^\alpha_\varepsilon \cdot f' = h^\alpha_{\varepsilon\delta} \cdot f'$$

for some $f' \in \mathscr{F}_{\beta\varepsilon}$. Lastly,

$$h^\alpha_{\beta\delta} = (g^\alpha_\delta)^{-1} \cdot g^\alpha_\beta = (g^\alpha_\delta)^{-1} \cdot g^\alpha_\gamma \cdot (g^\alpha_\gamma)^{-1} \cdot g^\alpha_\beta = h^\alpha_{\gamma\delta} \cdot h^\alpha_{\beta\gamma},$$

so this strengthening of (4) holds. (It is stronger because there is no demand that $\delta \in D'_\alpha$.)

Note that if the continuous limits are paths the fake linear limits given by Lemma (5) have limits at each limit $\gamma \leqslant \alpha$.

**Lemma 6.** *If $\kappa$ is weakly compact there is no simplified morass with continuous paths.*

**Proof.** Consider the following tree: the elements are fake linear limits through $\mathscr{M}$, ordered by extension, that is for limit $\alpha$ and $\bar{\alpha}$,

$$\langle\langle\psi^\gamma_\beta \mid \beta \in D_\gamma\rangle \mid \gamma \leqslant \alpha\rangle \prec \langle\langle\bar{\psi}^\gamma_\beta \mid \beta \in D_\gamma\rangle \mid \gamma \leqslant \bar{\alpha}\rangle$$

if $\alpha < \bar{\alpha}$ and for $\beta \in D_\gamma$ and $\gamma \leqslant \alpha$ one has $\psi^\beta_\gamma = \bar{\psi}^\beta_\gamma$. If $\mathscr{M}$ has continuous paths and $\kappa$ is inaccessible then, as shown in Lemma 5, this tree is a $\kappa$-tree. Consequently if $\kappa$ is weakly compact there is a branch of length $\kappa$ through the tree. But this branch (or more strictly the union of its elements) then constitutes a collection of (true) linear limits, a contradiction to Donder's Lemma 1 from [6]. (There is similarly a branch through the restriction of the tree to those fake linear limits generated by the continuous paths as in Lemma 5, which constitutes a collection of linear paths, contradicting Velleman's result in Section 2.3 of [17].)

**Corollary 7.** ($V = L$) *If there is a* $(\kappa, 1)$*-simplified morass with continuous paths there is one with linear limits.*

**Proof.** By Lemma (6), if $\mathscr{M}$ has continuous paths, $\kappa$ is not weakly compact. Theorem (2) of [6] asserts that if $V = L$ and $\kappa$ is not weakly compact there is a $(\kappa, 1)$-simplified morass with linear limits.

I had hoped to improve on Corollary 7 by showing the result held outright. However Dieter Donder found a serious error in my purported proof, and I should like to thank him for this and for convincing me that my various efforts to patch the proof were to no avail. My plan was to use condensation properties of the model $L[M]$ of [10] and [11] extending those given there to obtain an unsimplified morass with linear limits and from that a simplified one. However my argument broke down if $L[M, A]$ is not "reshaped" – in fine structural argot – for some $A \subseteq \kappa$ where $L[M, A] \subseteq V$. Thus the question as to whether there is necessarily a $(\kappa, 1)$-simplified morass with linear limits if there is one with continuous paths is still open. (Indeed it may be that one needs to assume $\square(\kappa)$ as well, or possibly something even stronger.) It seems that $L[M]$ will be sufficiently re-shaped if $\kappa^+$ is accessible in $L$, and so it seems one will need some large cardinal hypothesis to separate the existence of linear limits and continuous paths.

**Theorem 8.** ($V = L$) *There is a* $(\kappa, 1)$*-simplified morass with small limits if and only if there is a* $(\kappa, 1)$*-simplified morass with simple limits* (*if and only if* $\kappa$ *is not ineffable*).

**Proof.** Assume $V = L$. The proof is a very mild modification of [6, Theorem 2], and is presumably what Donder had in mind in his note at the end of [6] (p. 236) If $\kappa$ is not ineffable one can thin the initial $\mathscr{S}$ of [6] as in [5], p. 43], rather than by using weak compactness as in [6, pp. 231–232]. Continuing in the style of [6], the resultant morass will have simple limits, but the analogue (12)(c) will in general fail for $\alpha$ as in Case 1, so the limits will not cohere.

I elaborate slightly on the comment before Proposition 1.3.

**Definition 9.** $\mathscr{M}$ is a $(\kappa, 1)$-simplified morass with *authentic small limits* if $\lhd$ has no $\nu$-Cantor subtrees for any infinite $\nu < \kappa$. (Recall that a tree $T$ is $\nu$-Cantor if there is some $\mu \leqslant \nu$ such that the height of $T$ is $\mu + 1$, each level $T_\alpha$ has size at most $\nu$ for $\alpha < \nu$, and $\overline{\overline{T_\mu}} > \nu$, while no two elements of $T_\mu$ have the same set of predecessors in $T$.)

**Observation 10.** *It is implicit in the proof of* [24, 4.3] *that a* $(\kappa, 1)$*-simplified morass with simple limits has authentic small limits. And it is clear that a* $(\kappa, 1)$*-simplified morass with authentic small limits has small limits.*

**Lemma 11.** *If* $\mathscr{M}$ *is a neat* $(\kappa, 1)$*-simplified morass with authentic small limits,* $b(= b_\kappa)$ *demonstrates* $\kappa^+ \not\rightarrow [\lambda]^2_{\kappa, <\lambda}$ *for each* $\lambda \leqslant \kappa$.

**Proof.** Clearly it suffices to prove the lemma for successor $\lambda$. (To see this, suppose that the lemma fails for some limit cardinal $\lambda$. Then there is some $A \in [\kappa]^\lambda$ such that $\overline{\overline{b''[A]^2}} = \mu$ for some $\mu < \lambda$. Hence $\overline{\overline{b''[B]^2}} = \mu$ for any $B \in [A]^{\mu^+}$, and any such $B$ contradicts the lemma for $\mu^+$.) So assume $\lambda = \tau^+$.

The hypothesis is far stronger than is necessary. The assumption of neatness is simply to prevent $\lambda$ many points having the same "date-of-birth" and distinct ancestors at that level, and could be weakened to specify that. (I leave the trivial thinning argument to the reader.) The conclusion of the lemma is also true of the branching function of any $\kappa$-Kurepa tree with no $\nu$-Cantor subtrees for each infinite $\nu < \kappa$ (and this shows the lemma itself.) I prove the contrapositive of this assertion.

Suppose $b$ is the braching function of a $\kappa$-Kurepa tree $T$. Suppose $A$ is a set of points at level $\kappa$ in $T$, with $\overline{\overline{A}} = \lambda < \kappa$ and $\overline{\overline{b''[A]^2}} = \mu < \lambda$, so that the partition property holds. (To be pedantic, I should thin $T$ to a subtree whose set of points at level $\kappa$ has size $\kappa^+$ and containing $A$, let me re-cycle notation and also call this subtree $T$.)

Let $\beta < \kappa$ be least such that there is some set $B$, of size $\lambda$, of points at level $\beta$ which are predecessors in $T$ of points in $A$. Note that $\beta$ is a limit ordinal of cofinality $\rho \leqslant \mu$. Let $S$ be the subtree of $T \mid \beta + 1$ whose underlying set is $\{ y \mid \exists x \in B\; yTx \;\&\; \exists w, z \in B\; \exists \bar{w}, \bar{z} \in A (wT\bar{w},\; zT\bar{z} \;\&\; b(\bar{w}, \bar{z}) = \mathrm{ht}(y))\}$. Then $S$ is a $\tau$-Cantor subtree of $T$.

**Note.** In [25, 2.6] Velleman proves the combinatorial principle $H_3(\omega_3)$, due to Hajnal and Komjath, holds in $L$. (It is not important to know what this principle and its variants mentioned below actually are for the purposes of the present discussion.) Explicitly he uses a (neat) $(\omega_2, 1)$-simplified morass with linear limits to show a principle called $wH_2(\omega_3)$ holds and then uses some auxiliary lemmas that hold in $L$ to complete the proof. But inspection of the proof shows that all he uses of the linear limits is that at each $\gamma$ of cofinality $\omega$ there is a sequence $\langle \beta_i \mid i < \omega \rangle$ cofinal in $\gamma$ and a sequence of maps $\langle h_i \mid i < \omega \rangle$ with each $h_i \in \mathscr{F}_{\beta_i \gamma}$ and $\bigcup h_i'' \theta_{\beta_i} = \theta_\gamma$. Thus an

$(\omega_2, 1)$-simplified morass with simple limits (at points of cofinality $\omega$) clearly suffices for his proof. Inspection also reveals that his proof shows $wH_2(\omega_{n+1})$ holds using an $(\omega_n, 1)$-simplified morass with simple limits at points of cofinality $\omega$.

Now the auxiliary lemmas Velleman uses are

**Lemma 12.** *CH, $wH(\kappa)$ and $\kappa^+ \not\to [\omega_1]^{<\omega}_{\omega_1,\omega}$ imply $H(\kappa)$; the former and the latter hypotheses hold in $L$; and $H(\omega_n)$ implies $H_n(\omega_n)$ for $n \geqslant 1$.*

One also knows that $\omega_1 \not\to [\omega_1]^2_{\omega_1}$, and that $\kappa \not\to [\omega_1]^r_{\omega_1,}$ implies $2^\kappa \not\to [\omega_1]^{r+1}_{\omega_1,}$ if there is a $\kappa$-Kurepa tree with no $\omega_1$-Aronszajn or $\omega$-Cantor subtrees (by the main result of [20] and by a special case of [18,9.3], respectively). Inspection of the proof of [24,4.3] shows that such a tree exists (for $\kappa = \omega_{n+1}$) if there is an $(\omega_n, 1)$-simplified morass with linear limits at levels of cofinality $\leqslant \omega_1$. So $H_{n+1}(\omega_{n+1})$ holds if there is an $(\omega_m, 1)$-simplified morass with linear limits at levels of cofinality $\leqslant \omega_1$ for all $m \leqslant n$ and $CH$ holds. It is clear from [6] that such simplified morasses are available in $L$.

However as Velleman remarks in his "notes added in proof", Todorcevic has proven that each $H_{n+1}(\omega_{n+1})$, for $n \geqslant 1$ holds in $L$, using $\square_{\omega_n}$. In fact Todorcevic's proofs use only $T$-functions and so it follows immediately from the results of Section 1 that one could use $(\omega_n, 1)$-simplified morasses alone (i.e., without worrying about what limits these morasses have) instead of $\square_{\omega_n}$. This raises the (open) question as to whether if there is a $(\kappa, 1)$-simplified morass there is one with linear limits at points of cofinality $\leqslant \omega_1$. Presumably this question is related to that of removing the assumption of $V = L$ from Corollary 7 and Theorem 8.

This seems an appropriate place to note that any simplified morass with the variant of small limits that $\mathrm{cf}(\alpha) = \mathrm{cf}(\theta_\alpha)$ and $\bar{\bar{\alpha}} = \overline{\overline{\theta_\alpha}}$ at regular cardinals $\alpha$ almost has simple limits at regular cardinals by a crude cardinality argument: properties (1) and (2) of the definition of simple limit are satisfied but the $D_\alpha$ need not necessarily be club. To see this observe that $\mathscr{F}_\alpha = \bigcup_{\beta<\alpha} \mathscr{F}_{\beta\alpha}$ has size $\bar{\bar{\alpha}}$ as $\overline{\overline{\theta_\alpha}} = \bar{\bar{\alpha}}$. Hence a simple limit can be defined inductively, ensuring that each of the $\bar{\bar{\alpha}}$ many elements of $\mathscr{F}$ can be factored through it in turn. At limit ordinals below $\alpha$ one uses the following Lemma which is a mild generalisation of [23,3.5] and which is proved in exactly the same way as that corollary.

**Lemma 13.** *Suppose $\mathscr{M}$ is a $(\kappa, 1)$-simplified morass, $\alpha \leqslant \kappa$, $\mu < \mathrm{cf}(\alpha)$. $\mathrm{cf}(\theta_\alpha)$ and for each $\delta < \mu$ there is some $\beta_\delta < \alpha$ and $f_\delta \in \mathscr{F}_{\beta_\delta\alpha}$. Then there is some $\gamma < \alpha$ and maps $g \in \mathscr{F}_{\gamma\alpha}$ and $f'_\delta \in \mathscr{F}_{\beta_\delta\gamma}$ such that for each $\delta < \mu$, $f_\delta = g \cdot f'_\delta$.*

## 3. Forcing axioms corresponding to the augmentations of morasses

The fundamental work on forcing axiom characterisations of morasses was carried out by Velleman in [23] and [24]. His forcing axioms are named after the particular closure notions they satisfy, the other properties being common to them all. In this section forcing axiom characterisations of the notions introduced in the previous section are given in a similar style. In summary the situation is as follows: small limits correspond to a variant of the $\kappa$-directed closed forcing axiom, simple limits correspond to the $\kappa$-closed forcing axiom and continuous limits correspond to a new kind of $\kappa$-strategically closed forcing axiom, in which a strategy gives a reaction to the position in which a player finds themself, rather than the sequence of prior moves in the game. Where convenient a unified account is given.

**Definition 1.** Let $\boldsymbol{P}$ be a partial order and $\mathscr{D} = \langle D_\alpha \mid \alpha < \kappa^+ \rangle$ be a family of $\kappa^+$ dense open subsets of $\boldsymbol{P}$.

(a) $\forall p \in \boldsymbol{P}\ \mathrm{rlm}(p) =_{\mathrm{def}} \{\alpha < \kappa^+ \mid p \in D_\alpha\}$.

(b) $\forall \alpha < \kappa^+\ \boldsymbol{P}_\alpha =_{\mathrm{def}} \{p \in \boldsymbol{P} \mid \mathrm{rlm}(p) \subseteq \alpha\}$.

(c) $\boldsymbol{P}^* =_{\mathrm{def}} \bigcup_{\alpha<\kappa} \boldsymbol{P}_\alpha$.

Suppose for each $\alpha < \kappa$, $\gamma < \kappa^+$ and order-preserving $f : \alpha \to \gamma$ there is $\pi_f : \boldsymbol{P}_\alpha \to \boldsymbol{P}_\gamma$. Consider the following conditions on $\mathscr{D}$ and the $\pi$, for each $\alpha < \kappa$, $\gamma < \kappa^+$ and order-preserving $f : \alpha \to \gamma$.

(i) $\forall p \in \boldsymbol{P}_\alpha (\mathrm{rlm}(\pi_f(p)) = f''\,\mathrm{rlm}(p))$.

(ii) $\forall p, q \in \boldsymbol{P}_\alpha (p \leqslant q \Rightarrow \pi_f(p) \leqslant \pi_f(q))$.

(iii) $\forall \alpha_0 < \alpha_1 < \kappa\ \forall \alpha_2 < \kappa^+\ \forall f_0 : \alpha_0 \to_{\mathrm{o.p.}} \alpha_1\ \forall f_1 : \alpha_1 \to_{\mathrm{o.p.}} \alpha_2$
$(\pi_{f_1 \cdot f_0} = \pi_{f_1} \cdot \pi_{f_0})$.

(iv.W) $\boldsymbol{P}^* \neq \emptyset \,\&\, \forall \alpha < \kappa (D_\alpha \cap \boldsymbol{P}^*$ is dense in $\boldsymbol{P}^*)$.

(iv.S) $\forall \alpha < \beta < \kappa (\boldsymbol{P}_\beta \neq \emptyset \,\&\, (D_\alpha \cap \boldsymbol{P}_\beta$ is dense in $\boldsymbol{P}_\beta)$.

(X.v.W) $\boldsymbol{P}^*$ is a forcing notion satisfying $X$.

(X.v.S) $\forall \alpha < \kappa\ \boldsymbol{P}_\alpha$ is a forcing notion satisfying $X$.

(vi.W) $\forall \beta < \alpha < \gamma < \kappa\ \forall f : \alpha \to_{\mathrm{o.p.}} \gamma\ (f \restriction \beta = \mathrm{id} \,\&\, f(\beta) \geqslant \alpha \,\&\, p \in \boldsymbol{P}_\alpha \Rightarrow$
$p$ and $\pi_f(p)$ are compatible in $\boldsymbol{P}^*$).

(vi.S) $\forall \beta < \alpha < \gamma < \kappa\ \forall f : \alpha \to_{\mathrm{o.p.}} \gamma\ (f \restriction \beta = \mathrm{id} \,\&\, f(\beta) \geqslant \alpha \,\&\, p \in \boldsymbol{P}_\alpha \Rightarrow$
$p$ and $\pi_f(p)$ are compatible in $\boldsymbol{P}_\gamma$).

(d) $\mathscr{D}$ is *$X$-weakly $\kappa$-indiscernible* if it satisfies (i)–(iii), (iv.W), (X.v.W) and (vi.W), and $\mathscr{D}$ is *$X$-strongly $\kappa$-indiscernible* if it satisfies (i)–(iii), (iv.S), (X.v.S) and (vi.S).

(e) $\boldsymbol{P}$ satisfies the *$X$-strong forcing axiom* (the $X\mathrm{SFA}_\kappa$) if for each $X$-weakly $\kappa$-indiscernible set $\mathscr{D}$ there is some $G \subseteq \boldsymbol{P}$ such that $\alpha < \kappa (G \cap D_\alpha \neq \emptyset)$ and $G$ is $\kappa$-complete.
$\boldsymbol{P}$ satisfies the *$X$-weak forcing axiom* (the $X\mathrm{WFA}_\kappa$) if for each $X$-strongly $\kappa$-indiscernible set $\mathscr{D}$ there is some $G \subseteq \boldsymbol{P}$ such that $\alpha < \kappa (G \cap D_\alpha \neq \emptyset)$ and $G$ is $\kappa$-complete.

The properties $X$ which are most interesting are closure properties of various kinds, in particular $\kappa$-directed closure, $\kappa$-closure and various strategic closure properties. Write $\mathrm{DWFA}_\kappa$ and $\mathrm{DSFA}_\kappa$ for the $\kappa$-directed closed forcing axioms.

Velleman showed that there is a $(\kappa, 1)$-simplified morass if and only if $\mathrm{DWFA}_\kappa$ holds if and only if $\mathrm{DSFA}_\kappa$. The next theorem has a very similar proof, which is given in full for the reader's convenience.

**Theorem 2.** *The following are equivalent*
(a) *There is a $(\kappa, 1)$-simplified morass with simple paths.*
(b) *$\kappa$-closed-$SFA_\kappa$ holds.*
(c) *$\kappa$-closed-$WFA_\kappa$ holds.*

**Proof.** (a)⇒(b) Let $\boldsymbol{P}$ be a partial order and $\mathscr{D} = \langle D_\alpha \mid \alpha < \kappa^+\rangle$ a $\kappa$-closed weakly $\kappa$-indiscernible set of dense open subsets of $\boldsymbol{P}$ (under the functions $\{\pi_f \mid f \in {}^\alpha\gamma$ is o.p. & $(\alpha,\gamma) \in \kappa \times \kappa^+\}$.) Suppose $\mathscr{M} = \langle\langle\theta_\alpha \mid \alpha \leqslant \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} \mid \alpha \leqslant \beta \leqslant \kappa\rangle\rangle$ is a simplified morass and $\langle\langle g^\alpha_\beta \mid \beta < \alpha\rangle \mid \alpha < \kappa\rangle$ is a collection of simple paths for it. Suppose also that each $\theta_\alpha$ is a successor. (Any $(\kappa, 1)$-simplified morass can be modified so this is true without destroying the property of having simple paths by the methods of [23,Theorem 3.10]). Write $C$ for $\kappa$-closed, so, for example, the forcing axiom is $\mathrm{CSFA}_\kappa$

Inductively $p_\alpha \in \boldsymbol{P}^*$, $\gamma_\alpha < \kappa$ and $f^* \in \mathscr{F}^*_{\beta\alpha}$ for each $\alpha < \kappa$ and $f \in \mathscr{F}_{\beta\alpha}$ will be defined satisfying the following conditions:

(a) $\quad \forall\alpha < \kappa\,(\gamma_\alpha \cdot \theta_\alpha \subseteq \mathrm{rlm}(p_\alpha) \subseteq \gamma_{\alpha+1})$.

(b) $\quad \forall\beta < \alpha < \kappa\,(\gamma_\beta < \gamma_\alpha < \kappa)$.

(c) $\quad \forall\beta < \alpha \leqslant \kappa\,\forall f \in \mathscr{F}_{\beta\alpha}\,\forall\tau < \gamma_{\beta+1}$ (if $\tau = \gamma_\beta \cdot \zeta + \xi$, and $\xi < \gamma_\beta$ if $\zeta + 1 < \theta_\beta$ then $f^*(\tau) = \gamma_\alpha \cdot f(\zeta) + \xi$).

So $f^*$ is order-preserving and $(f \cdot g)^* = f^* \cdot g^*$. And if $\mathscr{F}_{\alpha\alpha+1}$ is a pair $\{h, j\}$, then $h^* \mid (\gamma_\alpha \cdot \sigma_\alpha) = j^* \mid (\gamma_\alpha \cdot \sigma_\alpha)$ and $h^{*\prime\prime}\gamma_{\alpha+1} \subseteq j^*(\gamma_\alpha \cdot \sigma_\alpha)$.

(d) $\quad \forall\beta < \alpha < \kappa\,\forall f \in \mathscr{F}_{\beta\alpha}\,(p_\alpha \leqslant \pi_{f^*}(p_\beta))$.

Initially let $\gamma_0 = 1$ and by (iv.W) and (C.v.W) choose $p_0 \in \boldsymbol{P}^*$ so that $\theta_0 \subseteq \mathrm{rlm}(p_0)$. Next suppose that $\gamma_\beta$, $\mathscr{F}^*_{\delta\beta}$ and $p_\beta$ have been defined for all $\delta \leqslant \beta \leqslant \alpha$. Choose $\gamma_{\alpha+1}$ so that $\mathrm{rlm}(p_\alpha) \subseteq \gamma_{\alpha+1}$. Inductive condition (c) gives $f^*$ for each $f \in \mathscr{F}_{\alpha\alpha+1}$. If $\mathscr{F}_{\alpha\alpha+1}$ is a singleton it is easy to choose $p_{\alpha+1}$ such that $\gamma_{\alpha+1} \,.\, \theta_{\alpha+1} \subseteq \mathrm{rlm}(p_{\alpha+1})$ and $p_{\alpha+1} \leqslant \pi_{f^*}(p_\alpha)$ by (iv.W) and (C.v.W). If on the other hand $\mathscr{F}_{\alpha\alpha+1}$ is the pair $\{h_\alpha, j_\alpha\}$, let $A = h^{*\prime\prime}_\alpha\gamma_{\alpha+1} \cup j^{*\prime\prime}_\alpha\gamma_{\alpha+1} \subseteq \varepsilon$, $\delta = \mathrm{otp}(A)$ and $k : \delta \to A$ be the order preserving bijection.

Then $k^{-1} \cdot h^*_\alpha = \mathrm{id}$, $k^{-1} \cdot j^*_\alpha \mid \gamma_\alpha \cdot \sigma_\alpha = \mathrm{id}$ and $k^{-1} \cdot j^*_\alpha(\gamma_\alpha \cdot \sigma_\alpha) = \gamma_{\alpha+1}$, and (vi.W) is applicable to $k^{-1} \cdot j^*_\alpha$. Let $q = \pi_{\mathrm{id}}(p_\alpha)$. By (iii) $\pi_{h^*_\alpha}(p) = \pi_k(q)$ and $\pi_{j^*_\alpha}(p) = \pi_k(\pi_{k^{-1} \cdot j^*_\alpha}(q))$. By (vi.W) let $r \in \boldsymbol{P}^*$ be such that $r \leqslant q, \pi_{k^{-1} \cdot j^*_\alpha}(q)$ and suppose $r \in \boldsymbol{P}_{\varepsilon'}$, where $\varepsilon < \varepsilon'$ and extend $k$ to $k' : \delta' \to \varepsilon' < \kappa$. Then $h^*_\alpha = k' \cdot \mathrm{id}$ and $j^*_\alpha = k' \cdot (k^{-1} \cdot j^*_\alpha) \cdot \mathrm{id}$, so $\pi_{k'} \leqslant \pi_{h^*_\alpha}(p)$, $\pi_{j^*_\alpha}(q)$. It is then easy to pick $p_{\alpha+1}$ and such that $\gamma_{\alpha+1} \,.\, \theta_{\alpha+1} \subseteq \mathrm{rlm}(p_{\alpha+1})$ and $p_{\alpha+1} \leqslant \pi_{k'}(r)$. Clearly the inductive hypotheses are maintained.

The limit stage is dealt with by showing that it is only important to take into account the images of the $p_\beta$ under the simple path maps at $\alpha$ in order to preserve the inductive hypotheses, and that this can be done by $\kappa$-closure. Specifically, if $\lim(\alpha)$ and $\gamma_\beta$, $\mathscr{F}^*_{\delta\beta}$

and $p_\beta$ have been picked for all $\delta \leqslant \beta < \alpha$, let $\gamma_\alpha = \bigcup_{\beta<\alpha}\gamma_\beta$. By (c), $f^*$ is defined for each $f \in \mathscr{F}_{\beta\alpha}$. By the definition of simple paths, for each $f \in \mathscr{F}_{\beta\alpha}$ there is some $\varepsilon \in (\beta,\alpha)$ and some $f' \in \mathscr{F}_{\beta\varepsilon}$ such that $f = g^\alpha_\varepsilon \cdot f'$. Thus, by inductive hypothesis (d) applied to $\delta$, $\forall f \in \mathscr{F}_{\beta\alpha}\, \exists\delta \in (\beta,\alpha)\,(\pi_{(g^\alpha_\delta)^*}(p_\delta) \leqslant \pi_{f^*}(p_\beta))$.

Also, for any $\beta < \varepsilon < \alpha$ there is, by the definition of simple paths, some $f \in \mathscr{F}_{\beta\varepsilon}$ such that $g^\alpha_\beta = g^\alpha_\varepsilon \cdot f$. So by inductive condition (d) at $\varepsilon$, $\pi_{(g^\alpha_\beta)^*}(p_\beta) = \pi_{(g^\alpha_\varepsilon)^*}(\pi_{f^*}(p_\beta)) \geqslant \pi_{(g^\alpha_\varepsilon)^*}(p_\varepsilon)$.

As $\boldsymbol{P}^*$ is $\kappa$-closed there is some $r \in \boldsymbol{P}^*$ such that $(r \leqslant \pi_{(g^\alpha_\beta)^*}(p_\beta))$, and so for all $\beta < \alpha$ and $f \in \mathscr{F}_{\beta\alpha}$, $r \leqslant \pi_{f^*}(p_\beta)$. Once more $r$ can be extended to some $p_\alpha$ such that $\gamma_\alpha \,.\, \theta_\alpha \subseteq \mathrm{rlm}(p_\alpha)$.

Now let $\gamma_\kappa = \bigcup_{\alpha<\kappa}\gamma_\alpha = \kappa$ and let $G$ be the upward closure of $\{\pi_{f^*}(p_\alpha) \mid \alpha < \kappa \;\&\; f \in \mathscr{F}_{\alpha\kappa}\}$. $G$ is directed, for given $\pi_{f_0^*}(p_{\alpha_0})$ and $\pi_{f_1^*}(p_{\alpha_1})$ there are $\gamma \in (\alpha_0 \cup \alpha_1, \kappa)$, $g \in \mathscr{F}_{\gamma\kappa}$, $f_0' \in \mathscr{F}_{\alpha_0\kappa}$ and $f_1' \in \mathscr{F}_{\alpha_1\kappa}$ such that $f_0 = g \cdot f_0'$ and $f_1 = g \cdot f_1'$. But $\pi_{f_i^*}(p_{\alpha_i}) = \pi_{g^*}(\pi_{f_i'}(p_{\alpha_i}))$ and $p_\gamma \leqslant \pi_{f_i}(p_{\alpha_i})$, for $i = 0, 1$, hence $\pi_{f_0^*}(p_{\alpha_0}), \pi_{f_1^*}(p_{\alpha_1}) \geqslant \pi_{g^*}(p_\gamma) \in G$.

$G$ really is sufficiently generic, for if $\tau < \kappa^+$ there are $\zeta < \kappa^+$ and $\xi < \kappa$ such that $\tau = \kappa \,.\, \zeta + \xi$. One can then pick $\alpha < \kappa$, $f \in \mathscr{F}_{\alpha\kappa}$ and $\zeta' < \theta_\alpha$ such that $\gamma_\alpha > \xi$ and $f(\zeta') = \zeta$. Thus $\tau' = \gamma_\alpha \,.\, \zeta' + \xi \in \mathrm{rlm}(p_\alpha)$, and so $\tau \in \mathrm{rlm}(\pi_{f^*}(p_\alpha))$ and $\pi_{f^*}(p_\alpha) \in G \cap D_\tau$.

Finally $G$ is $\kappa$-complete. If $X \subseteq G$ and $\overline{\overline{X}} = \lambda < \kappa$, one may without loss of generality take $X = \{\pi_{f_\delta^*}(p_{\beta_\delta}) \mid \delta < \lambda\}$, where for each $\delta < \lambda$, $\beta_\delta < \kappa$ and $f_\delta \in \mathscr{F}_{\beta_\delta\kappa}$. By the directedness of simplified morasses there is some $\gamma < \kappa$, $g \in \mathscr{F}_{\gamma\kappa}$ and $f_\delta' \in \mathscr{F}_{\beta_\delta\gamma}$ such that $f_\delta = g \cdot f_\delta'$ for each $\delta < \lambda$. Hence $p_\gamma \leqslant \pi_{(f_\delta')^*}(p_{\beta_\delta})$ and so $\pi_{g^*}(p_\gamma) \in G$ extends each $\pi_{f_\delta^*}(p_{\beta_\delta})$.

(b)$\Rightarrow$(c) is trivial.

(c)$\Rightarrow$(a). The natural conditions for adding a $(\kappa,1)$-simplified morass with simple paths are of the form $p = \langle\langle\theta_\alpha \mid \alpha \leqslant \mu\rangle, \langle\mathscr{F}_{\alpha\beta} \mid \alpha \leqslant \beta \leqslant \mu\rangle, \langle g^\alpha_\beta \mid \beta < \alpha \leqslant \mu\rangle, B\rangle$, where

(i) $\mu < \kappa$,

(ii) $\langle\theta_\alpha \mid \alpha \leqslant \mu\rangle \in [\kappa]^{\mu+1}$ and
$\forall\alpha \leqslant \beta \leqslant \mu\,(\mathscr{F}_{\alpha\beta}$ is a set of order preserving maps from $\theta_\alpha$ to $\theta_\beta)$,

(iii) $\forall\alpha \leqslant \mu\,(\mathscr{F}_{\alpha\alpha} = \{\mathrm{id}\}$ and $\mathscr{F}_{\alpha\alpha+1}$ is either a singleton or a pair $\{h_\alpha, \psi_\alpha\}$
such that for some $\sigma_\alpha$ less than $\theta_\alpha$, $h_\alpha \restriction \sigma_\alpha = j_\alpha \restriction \sigma_\alpha \;\&\; h_\alpha{}''\theta_\alpha \subseteq j_\alpha(\sigma_\alpha))$,

(iv) $\forall\alpha \leqslant \beta \leqslant \gamma \leqslant \mu\,(\mathscr{F}_{\alpha\gamma} = \{g \cdot f \mid g \in \mathscr{F}_{\beta\gamma} \;\&\; f \in \mathscr{F}_{\alpha\beta}\})$,

(v) $\forall\,\mathrm{lim}(\alpha) \leqslant \mu\, \forall\beta_0, \beta_1 < \alpha\, \forall f_0 \in \mathscr{F}_{\beta_0\alpha} \forall f_1 \in \mathscr{F}_{\beta_1\alpha} \exists\gamma \in (\beta_0 \cup \beta_1, \alpha)$
$\exists g \in \mathscr{F}_{\gamma\alpha}\, \exists f_0' \in \mathscr{F}_{\beta_0\gamma}\, \exists f_1' \in \mathscr{F}_{\beta_1\gamma}\,(f_0 = g \cdot f_0' \;\&\; f_1 = g \cdot f_1')$,

(so $\langle\langle\theta_\alpha \mid \alpha \leqslant \mu\rangle, \langle\mathscr{F}_{\alpha\beta} \mid \alpha \leqslant \beta \leqslant \mu\rangle\rangle$ is a $(\kappa,1)$-simplified morass segment) and for each limit $\alpha \leqslant \mu$

(vi) $\forall\beta < \alpha\,(g^\alpha_\beta \in \mathscr{F}_{\beta\alpha})$,

(vii) $\forall\beta < \gamma < \alpha\, \exists f \in \mathscr{F}_{\beta\gamma}\,(g^\alpha_\beta = g^\alpha_\gamma \cdot f)$,

(viii) $\forall\beta<\alpha\,\forall f\in\mathscr{F}_{\beta\alpha}\,\exists\delta\in(\beta,\alpha)\,\exists f'\in\mathscr{F}_{\beta\delta}\,(f=g^\alpha_\delta\cdot f')$, and

(ix) $B$ is an order preserving function from $\theta_\mu$ into $\kappa^+$.

When considering more than one condition adorn each associated notion of each condition by adding a superscript of the name of the condition. The conditions are ordered in the obvious way: $p\leqslant q$ if

(i) $\mu^q\leqslant\mu^p$,

(ii) $\forall\alpha\leqslant\mu^q\,(\theta^p_\alpha=\theta^q_\alpha)$,

(iii) $\forall\alpha\leqslant\beta\leqslant\mu^q\,(\mathscr{F}^p_{\alpha\beta}=\mathscr{F}^q_{\alpha\beta})$,

(iv) $\forall\alpha\leqslant\mu^q\,(\langle(g^\alpha_\beta)^p\mid\beta<\alpha\rangle=\langle(g^\alpha_\beta)^q\mid\beta<\alpha\rangle)$,

(v) $\exists f\in\mathscr{F}^p_{\mu^q\mu^p}\,(B^q=B^p\cdot f)$.

Let $D_\alpha=\{\,p\in\boldsymbol{P}\mid\alpha\in\mathrm{rge}(B^p)\}$ for each $\alpha<\kappa^+$. Clearly each $D_\alpha$ is a dense open subset of $\boldsymbol{P}$ and so the notation of Definition 1(a)–(c) is applicable to $\mathscr{D}=\langle D_\alpha\mid\alpha<\kappa^+\rangle$. If $\alpha<\kappa$, $\gamma<\kappa^+$, $f:\alpha\to\gamma$ is an order preserving map, and $p\in\boldsymbol{P}_\alpha$, define the condition $\pi_f(p)\in\boldsymbol{P}_\gamma$ by setting $\pi_f(p)=\langle\langle\theta^p_\varepsilon\mid\varepsilon\leqslant\mu^p\rangle,\langle\mathscr{F}^p_{\varepsilon\beta}\mid\varepsilon\leqslant\beta\leqslant\mu^p\rangle,\langle(g^\varepsilon_\beta)^p\mid\beta<\varepsilon\leqslant\mu^p\rangle,f\cdot B^p\rangle$. Then $\pi_f:\boldsymbol{P}_\alpha\to\boldsymbol{P}_\gamma$ for each $\alpha$, $\gamma$ and $f$. Clearly it suffices to check the six conditions for the $\kappa$-closed strong $\kappa$-indiscernibility of $\mathscr{D}$, for then, by the forcing axiom, a sufficiently generic set exists, and consequently a $(\kappa,1)$-simplified morass with simple paths.

(i) $\mathrm{rlm}(\pi_f(p))=\{f(\beta)\mid\beta\in\mathrm{rlm}(p)\}=f''\mathrm{rlm}(p)$.

(ii) If $p\leqslant q$ then $B^q=B^p\cdot g$ for some $g\in\mathscr{F}^p_{\mu^q\mu^p}$, so $f\cdot B^q=f\cdot B^p\cdot g$. Thus $\pi_f(p)\leqslant\pi_f(q)$.

(iii) Clearly $\pi_{f_1\cdot f_0}=\pi_{f_1}\cdot\pi_{f_0}$.

(iv) If $\alpha<\beta<\kappa$ and $p\in\boldsymbol{P}_\beta\setminus D_\alpha$ let $\gamma\leqslant\mu^p$ be such that $\alpha\in(B^p(\gamma),B^p(\gamma+1))$, if possible, $\gamma=\mu^p$ if $\mathrm{rge}(B^p)\subseteq\alpha$, and $\gamma=-1$ if $\gamma<\bigcap\mathrm{rge}(B^p)$. Define $q\in\boldsymbol{P}$ by setting $\mu^q=\mu^p+1$, $\theta^q_\xi=\theta^p_\xi$ for $\xi\leqslant\mu^p$, $\mathscr{F}^q_{\xi\zeta}=\mathscr{F}^p_{\xi\zeta}$ for $\xi\leqslant\zeta\leqslant\mu^p$,
$\langle(g^\zeta_\xi)^q\mid\xi<\lim(\zeta)\leqslant\mu^q\rangle=\langle(g^\zeta_\xi)^p\mid\xi<\lim(\zeta)\leqslant\mu^p\rangle$,
$\theta^q_{\mu^q}=\theta^p_{\mu^p}+1$, $\mathscr{F}^q_{\mu^p\mu^q}=\{h\}$, where $h\mid\gamma=\mathrm{id}$ and $h(\gamma+\xi)=\gamma+1+\xi$ for $\gamma+\xi<\theta^p_{\mu^p}$, and $B^q(\xi)=B^p(\xi)$ for $\xi<\gamma$, $B^q(\gamma)=\alpha$ and $B^q(\gamma+1+\zeta)=B^p(\gamma+\zeta)$ for $\gamma+1+\theta^q_{\mu^q}$.
Then $q\leqslant p$ and $q\in D_\alpha\cap\boldsymbol{P}_\beta$.

(v) Suppose $\langle p_\alpha\mid\alpha<\lambda\rangle$ is a descending sequence of conditions from $\boldsymbol{P}_\beta$, where $\lambda$ is a limit ordinal less than $\kappa$. Define $p_\lambda$ below each $p_\alpha$ piece by piece as follows. Let $\mu^\lambda=\mu$ be the $\sup\{\mu^\alpha\mid\alpha<\lambda\}$. For $\xi\leqslant\zeta<\mu$ let $\theta^\lambda$, $\mathscr{F}^\lambda_{\xi\zeta}$ be the (common) values of $\theta^\delta$, $\mathscr{F}^\delta_{\xi\zeta}$ for those $\delta<\lambda$ for which they are defined. Let $U=\bigcup_{\alpha<\lambda}\mathrm{rge}(B^\alpha)$, $\theta^p_\mu=\mathrm{otp}(U)$ and let $B^\lambda:\theta^\lambda_\mu\to U$ be the order preserving bijection. Let $g^\mu_{\mu^\alpha}=(B^\lambda)^{-1}\cdot B^\alpha$ for each $\alpha<\lambda$. Now define $\langle g^\mu_\xi\mid\xi<\mu\rangle$ as in Lemma 1 of Section 2, using $\{\mu^\alpha\mid\alpha<\lambda\}$ as the club set in $\mu$ and the paths already defined as the limits at smaller ordinals. Lastly, for each $\xi<\mu$, let $\mathscr{F}^\lambda_{\xi\mu}=\{g^\mu_\zeta\cdot f\mid\exists\zeta\in[\xi,\mu)\,f\in\mathscr{F}_{\xi\zeta}\}$. It is clear that (i)–(ix) all hold for $p^\lambda$, so $p^\lambda\in\boldsymbol{P}_\beta$. Equally it is clear that $\forall\alpha<\lambda\,(p^\lambda\leqslant p^\alpha)$.

(vi) If $f$, $p$ are as in the hypotheses define $q \leqslant \pi_f(p)$, $p$, as follows. Take $\mu^q = \mu^p + 1$, $\theta^q_\alpha = \theta^p_\alpha = \theta^{\pi_f(p)}_\alpha$ for $\alpha \leqslant \mu^p$, $\mathscr{F}^q_{\alpha\beta} = \mathscr{F}^p_{\alpha\beta} = \mathscr{F}^{\pi_f(p)}_{\alpha\beta}$ for $\alpha \leqslant \beta \leqslant \mu^p$, and $\langle (g^\beta_\alpha)^q \mid \alpha < \lim(\beta) \leqslant \mu^p \rangle = \langle (g^\beta_\alpha)^p \mid \alpha < \lim(\beta) \leqslant \mu^p \rangle = \langle (g^\beta_\alpha)^{\pi_f(p)} \mid \alpha < \lim(\beta) \leqslant \mu^p \rangle$. Let $U = \mathrm{rge}(B^p) \cup \mathrm{rge}(B^{\pi_f(p)}) = \mathrm{rge}(B^p) \cup f'' \mathrm{rge}(B^p)$, and take $\theta^q_{\mu^q} = \mathrm{otp}(U)$ and $B^q = c^{-1}$ where $c$ is the transitive collapse of $U$. Take $\mathscr{F}_{\mu^p\mu^q} = \{\mathrm{id}, h\}$, where $h \mid (c''\beta) = \mathrm{id}$ and $h(c''\beta + \xi) = (c''\alpha) + \xi$ for $c''\beta + \xi < \theta^p_{\mu^p}$, and $\mathscr{F}^q_{\alpha\mu^q} = \langle f \cdot g \mid f \in \mathscr{F}^q_{\mu^p\mu^q} \;\&\; g \in \mathscr{F}^q_{\alpha\mu^p} \rangle$ for $\alpha < \mu^p$.

Clearly $q$ is as desired.

Similar results obtain for the other augmentations of simplified morasses introduced in the previous section. The proofs each have the same format as the proof of Theorem 2, and so only the details of variations are given.

**Definition 3.** Suppose $\boldsymbol{P}$ is a partially ordered set. A *thin directed subset* of $\boldsymbol{P}$ is a directed subset of $\boldsymbol{P}$ such that $\overline{\overline{D}} \leqslant \mathrm{ht}(D)$, where $\mathrm{ht}(D)$ – the height of $D$ – is the supremum of the ordertypes of strictly descending chains in $D$. $\boldsymbol{P}$ is *$\kappa$-thin-directed closed* if every thin directed subset of $\boldsymbol{P}$ of size less than $\kappa$ has a lower bound.

**Theorem 4.** *The following are equivalent*:

(a) *There is a* $(\kappa, 1)$*-simplified morass with small limits.*

(b) $\kappa$*-thin-directed closed-*$SFA_\kappa$ *holds.*

(c) $\kappa$*-thin-directed closed-*$WFA_\kappa$ *holds.*

**Proof.** (a)⇒(b) The only variation from the proof of the analagous case of Theorem 2 is in the treatment of the limit case. Suppose $\alpha$ is a limit ordinal less than $\kappa$ and $\langle \gamma_\beta \mid \beta < \alpha \rangle$, $\langle \mathscr{F}^*_{\varepsilon\beta} \mid \varepsilon \leqslant \beta < \alpha \rangle$ and a descending sequence of conditions $\langle p_\beta \mid \beta < \alpha \rangle$ have been defined, obeying the inductive hypotheses. Then since the limits are small, $D = \{\pi_{f^*}(p_\beta) \mid \beta < \alpha \;\&\; f \in \mathscr{F}_{\beta\alpha}\}$ is a directed set of size $\overline{\overline{\alpha}}$, and of height $\alpha$, and hence is a thin directed set. As $\boldsymbol{P}^*$ is $\kappa$-thin-directed closed there is some $r \in \boldsymbol{P}^*$ such that $\forall f \in \mathscr{F}_{\beta\alpha}\, r \leqslant \pi_{f^*}(p_\beta)$. As usual $r$ can be extended to some $p_\alpha$ with $\gamma_\alpha \cdot \theta_\alpha \subseteq \mathrm{rlm}(p_\alpha)$.

(The reader might worry whether $D$ really has height $\alpha$. Clearly, this is the maximum height $D$ could have and $\langle \pi_{\mathrm{id}}(p_\beta) \mid \beta < \alpha \rangle$ is a strictly descending chain of height $\alpha$, by property (iii) of Definition 1, if $\mathrm{id} \in \mathscr{F}_{\beta\alpha}$ for each $\beta < \alpha$. It is standard that one can trade-in a morass for one with this property, but even this fact is not strictly necessary. Recall the standard notation that if $f \in \mathscr{F}_{\beta\alpha}$ and $\xi \leqslant \theta_\beta$ then $\bar{f}(\xi)$ is the least ordinal containing $f''\xi$.

**Lemma.** *Let* $\beta < \alpha \leqslant \kappa$, $f \in \mathscr{F}_{\beta\alpha}$ *and* $\xi \in \theta_\beta$. *Then there is some* $g \in \mathscr{F}_{\beta\alpha}$ *such that* $g \mid \xi = f \mid \xi$ *and* $g(\xi + \delta) =$ *the* $\delta$*th element of* $H_\beta \setminus \bar{f}(\xi)$, *for* $\xi + \delta < \theta_\beta$, *where* $H_\beta = \bigcup\{h''\theta_\beta \mid h \in \mathscr{F}_{\beta\alpha}\}$. *Moreover,* $\bar{g}(\theta_\beta)$ *is minimal for those* $g$ *with* $g \mid \xi = f \mid \xi$.

**Proof.** Exactly as in [25, 2.4].

Now let $I_\beta$ be the map given when the lemma is applied to $\xi = 0$ and any $f \in \mathscr{F}_{\beta\alpha}$. Then it is, immediate that $I_\beta = I_\gamma \cdot f$ for each $\beta < \gamma < \alpha$ where $f \in \mathscr{F}_{\beta\gamma}$ is the map obtained when the lemma is applied to $\xi = 0$ and any map in $\mathscr{F}_{\beta\gamma}$. Finally, $\langle \pi_{I_\beta}(p_\beta) \mid \beta < \alpha \rangle$ is a strictly descending chain, by property (iii) of Definition 1.)

(b)$\Rightarrow$(c) Trivial.

(c)$\Rightarrow$(a) The proof is similar to the analogue in the proof of Theorem 2, save that references to the paths should be ignored and the conditions are merely $(\kappa, 1)$-simplified morass segements with the additional requirement that they should have small limits. The only place where the proof differs is in checking that the conditions satisfy ($\kappa$-thin-directed closed.v.S). Suppose $\{ p_\alpha \mid \alpha \in D \} \subseteq \boldsymbol{P}_\beta$ is a thin directed set of conditions. Define $p_\lambda$ below each $p_\alpha$ for $\alpha \in D$. Let $\mu^\lambda = \mu = \sup\{ \mu^\alpha \mid \alpha \in D \}$. For $\xi \leqslant \zeta < \mu$ let $\theta^\lambda$, $\mathscr{F}^\lambda_{\xi\zeta}$ be the (common) values of $\theta^\delta$, $\mathscr{F}^\delta_{\xi\zeta}$ for those $\delta \in D$ for which they are defined. Let $U = \bigcup_{\alpha \in D} \mathrm{rge}(B^\alpha)$, $\theta^p_\mu = \mathrm{otp}(U)$ and let $B^\lambda : \theta^\lambda_\mu \to U$ be the order preserving bijection. Let $f^\mu_\alpha = (B^\lambda)^{-1} \cdot B^\alpha$ for each $\alpha \in D$ and for each $\alpha < \mu$, let $\mathscr{F}^\lambda_{\alpha\mu} = \{ f^\mu_\varepsilon \cdot f \mid \exists \varepsilon \in [\alpha, \mu)\, f \in \mathscr{F}_{\alpha\varepsilon} \}$. Then $p^\lambda$ is a condition since $\overline{\overline{\theta^\lambda_\mu}} = \bigcup_{\alpha \in D} \theta^\alpha_{\mu_\alpha} = \bigcup_{\alpha \in D} \mu_\alpha = \overline{\overline{\mu}}$ (and $\langle \theta^\alpha_{\mu_\alpha} \mid \alpha \in D \rangle$ is a strictly increasing sequence.) Clearly $p_\lambda \leqslant p_\alpha$ for each $\alpha \in \bar{D}$, and $p \in \boldsymbol{P}_\beta$.

The direct forcing equivalent of continuous paths requires a strengthening of the notion of strategic closure. The details of this and of the original notion are elaborated in the following extended definition.

**Definition 5.** For $\boldsymbol{P}$ a partial order and $\alpha \in \mathrm{On}$ let $G^P_\alpha$ be the two player game in which the players construct a descending sequence of conditions from $\boldsymbol{P}$ of length $\alpha$. Player I chooses the conditions at limit stages and player II at successors. Player II wins a play of the game if player I is unable to move at some stage $\beta < \alpha$ in the play, otherwise player I wins.

For $\beta < \alpha$ let $[\boldsymbol{P}]^\beta$ be the set of decreasing sequences of elements of $\boldsymbol{P}$ of length $\beta$. A *quasistrategy* for player I is a function $S : \bigcup\{ [\boldsymbol{P}]^\beta \mid \lim(\beta) < \alpha \} \to \mathscr{P}(\boldsymbol{P})$ such that whenever $p \in S(\langle p_\gamma \mid \gamma < \beta \rangle)$, $\forall \gamma < \beta\ p \leqslant p_\gamma$.

Player I *plays by S up to stage* $\beta (< \alpha)$ in a run of $G^P_\alpha$ if for each limit $\delta$ less than $\beta$, $p_\delta \in S(\langle p_\gamma \mid \gamma < \delta \rangle)$. $S$ is a *winning quasi-strategy for player I* if for each limit $\beta$ and each partially completed run, $\langle p_\gamma \mid \gamma < \beta \rangle$ of $G^P_\alpha$ in which player I has played by $S$, $S(\langle p_\gamma \mid \gamma < \beta \rangle) \neq \emptyset$.

$S$ is a *winning strong quasi-strategy for player I* if for each limit $\beta$ and each partially completed run, $\langle p_\gamma \mid \gamma < \beta \rangle$, of $G^P_\alpha$ in which at each limit $\delta < \beta$ there is some sequence $\langle q^\delta_i \mid i < k \rangle$ cofinal with $\langle p_\gamma \mid \gamma < \delta \rangle$ such that $p_\delta \in S(\langle q^\delta_i \mid i < k \rangle)$, $S(\langle p_\gamma \mid \gamma < \beta \rangle) \neq \emptyset$.

Now suppose that $\boldsymbol{P}$, $\mathscr{D}$ and the $\pi_f$ are as defined in Definition 1. A quasi-strategy, $S$, for player I in $G^{P^*}_\kappa$ is *closed under the* $\pi_f s$ if whenever $\alpha, \gamma < \kappa$, $f : \alpha \to \gamma$ is order preserving, $\lim(\beta) < \kappa$, $\langle p_\delta \mid \delta < \beta \rangle \in [\boldsymbol{P}]^\beta$, and $p \in S(\langle p_\delta \mid \delta < \beta \rangle) \cap \boldsymbol{P}_\alpha$, then $\pi_f(p) \in S(\langle \pi_f(p_\delta) \mid \delta < \beta \rangle)$. $S$ is a *bounded realm quasi-strategy* if whenever $\lim(\beta) < \kappa$, $\alpha < \kappa$ and $\langle p_\delta \mid \delta < \beta \rangle \in [\boldsymbol{P}_\alpha]^\beta$, $S(\langle p_\delta \mid \delta < \beta \rangle) \subseteq \boldsymbol{P}_\alpha$.

Consider the conditions

(S.v.W.) Player I has a winning quasi-strategy for $G^{P^*}_\kappa$ which is closed under the $\pi_f$s, (S.v.S.) Player I has a bounded realm winning quasi-strategy for $G_\kappa^{P^*}$ which is closed under the $\pi_f$s,

(SS.v.W.) Player I has a winning strong quasi-strategy for $G_\kappa^{P^*}$ which is closed under the $\pi_f$s, and

(SS.v.S.) Player I has a bounded realm winning strong quasi-strategy for $G_\kappa^{P^*}$ which is closed under the $\pi_f$s.

These give rise to the axioms $SSFA_\kappa$, $SWFA_\kappa$, $SSSFA_\kappa$ and $SSWFA_\kappa$ as in Definition 1. (The initials SS standing for strongly strategic.) Velleman showed that the former two axioms hold iff and only if there is a $(\kappa, 1)$-simplified morass with linear limits.

**Theorem 6.** *The following are equivalent*:

(a) *There is a* $(\kappa, 1)$*-simplified morass with continuous paths,*

(b) $SSSFA_\kappa$ *holds,*

(c) $SSWFA_\kappa$ *holds.*

**Proof.** Once more the proof is as that of Theorem 2, save that some adjustment is necessary in the limit cases.

(a)$\Rightarrow$(b) Augment the inductive hypotheses with the following:

(e) $\lim(\alpha) \Rightarrow p_\alpha \in S(\langle \pi_{(g_\delta^\alpha)^*}(p_\delta) \mid \delta < \alpha \rangle)$.

If $\alpha = 0$ or is a successor the construction of $p_\alpha$ and $\gamma_\alpha$ are exactly as in Theorem 2. So suppose $\alpha$ is a limit ordinal less than $\kappa$, and $\langle \gamma_\beta \mid \beta < \alpha \rangle$, $\langle \mathscr{F}_{\varepsilon\beta}^* \mid \varepsilon \leqslant \beta < \alpha \rangle$ and a descending sequence of conditions $\langle p_\beta \mid \beta < \alpha \rangle$ have been defined, obeying the inductive hypotheses. Let $\gamma_\alpha = \bigcup_{\beta<\alpha} \gamma_\beta$ and define $f^*$ for each $f \in \mathscr{F}_{\beta\alpha}$ by inductive hypothesis (c).

Since continuous paths are simple, $\boldsymbol{p} = \langle \pi_{(g_\beta^\alpha)^*}(p_\beta) \mid \beta < \alpha \rangle \in [\boldsymbol{P}]^\alpha$ is a descending sequence, as shown in the proof of Theorem 2, and so can be thought of as a play in $G_\kappa^{P^*}$. If player I has played by $S$ so far, $p_\alpha$ can be obtained by applying $S$ to $\boldsymbol{p}$. So suppose $\lim(\delta) < \alpha$. By induction hypothesis (e), $p_\delta \in S(\langle \pi_{(g_\beta^\delta)^*}(p_\beta) \mid \beta < \delta \rangle)$, so $\pi_{(g_\delta^\alpha)^*}(p_\delta) \in S(\langle \pi_{(g_\delta^\alpha \cdot g_\beta^\delta)^*}(p_\beta) \mid \beta < \delta \rangle)$.

Consequently, the proof would be completed by showing that the sequences

$$\langle \pi_{(g_\beta^\alpha)^*}(p_\beta) \mid \beta < \delta \rangle \quad \text{and} \quad \langle \pi_{(g_\delta^\alpha)^* \cdot (g_\beta^\delta)^*}(p_\beta) \mid \beta < \delta \rangle \qquad \text{are cofinal in each other.}$$

Now, by Section 2(1) and Section 2(2), for each $\beta < \delta$, there is some $\varepsilon \in (\beta, \delta)$ and some $f' \in \mathscr{F}_{\beta\varepsilon}$ such that $g_\beta^\alpha = g_\delta^\alpha \cdot g_\varepsilon^\delta \cdot f'$. So $\pi_{(g_\beta^\alpha)^*}(p_\beta) = \pi_{(g_\delta^\alpha)^* \cdot (g_\varepsilon^\delta)^* \cdot (f')^*}(p_\beta)$. But inductive hypothesis (d) gives $p_\varepsilon \leqslant \pi_{(f')}(p_\beta)$ (since it gives this for any $f' \in \mathscr{F}_{\beta\varepsilon}$), thus for some $\varepsilon < \delta$ it is the case that $\pi_{(g_\beta^\alpha)^*}(p_\beta) \geqslant \pi_{(g_\delta^\alpha)^* \cdot (g_\varepsilon^\delta)^*}(p_\varepsilon)$.

Conversely, the continuity property from the definition of continuous paths (Section 2(3)) is that $g_\delta^\alpha \cdot g_\beta^\delta = g_\varepsilon^\alpha \cdot f$ for some $\varepsilon \in [\beta, \delta)$ and some $f \in \mathscr{F}_{\beta\varepsilon}$. Thus, by inductive hypothesis (d) again,

$$\pi_{(g_\delta^\alpha)^* \cdot (g_\beta^\delta)^*}(p_\beta) \geqslant \pi_{(g_\varepsilon^\alpha)^*}(p_\varepsilon), \text{ as required.}$$

(c)$\Rightarrow$(a) The proof is as in Theorem 2, except that the conditions must satisfy the additional requirement that

(x) For each limit $\alpha \leqslant \mu$ and $\beta < \lim(\gamma) < \alpha$, and for any $f \in \mathscr{F}_{\beta\gamma}$ there is some $\varepsilon \in [\beta, \gamma)$ and $f' \in \mathscr{F}_{\beta\varepsilon}$ such that $g^{\alpha}_{\gamma} \cdot f = g^{\alpha}_{\varepsilon} \cdot f'$.

The verification that the conditions are appropriate material for an application of the forcing axiom is as before, except that in checking (v.) it is necessary to show that $p_\lambda$ satisfies (x.). So take as a strategy, $S$, the construction as the checking of ((c)$\Rightarrow$(a))(v) in Theorem 2. By the proof of Lemma 1 of Section 2, it suffices to show that the structure given by the proof of Theorem 2((c)$\Rightarrow$(a))(v) prior to the conversion of the limit at $\mu$ into a path, has a continuous limit at $\mu$.

Fix a limit $\alpha < \lambda$. Suppose the sequence $\langle p_\beta \mid \beta < \alpha \rangle = \boldsymbol{p} \restriction \alpha$ is such that $p_\alpha \in S(\boldsymbol{q})$, where $\boldsymbol{q} = \langle q_i \mid i < k \rangle$ is a descending sequence of conditions cofinal with $\boldsymbol{p} \restriction \alpha$. Reserve roman letters for the superscripts notions associated with elements of $\boldsymbol{q}$ and greek letters for notions associated with elements of $\boldsymbol{p}$. (Thus $\mu^i$ is $\mu^{q_i}$, while $\mu^\delta$ is $\mu^{p_\delta}$.)

Then for any $\beta < \mu^\alpha$ there is some $i < k$ such that $\beta \leqslant \mu^i < \mu^\alpha$, and so $g^{\mu^\alpha}_{\beta} = g^{\mu^\alpha}_{\mu^i} \cdot f$ for some $f \in \mathscr{F}_{\beta\mu^i}$. By the strategy for obtaining $p_\alpha$ it is immediate that $g^{\mu^\alpha}_{\mu^i} = (B^{p_\alpha})^{-1} \cdot B^{q_i}$. Now, as the two sequences of conditions are cofinal there is some $\delta < \alpha$ such that $p_\delta \leqslant q_i$, so there is some $h \in \mathscr{F}_{\mu^i\mu^\delta}$ such that $B^{q_i} = B^{p_\delta} \cdot h$. Moreover the strategy for obtaining $p_\lambda$ gives that $g^{\mu^\lambda}_{\mu^\alpha} = (B^{p_\lambda})^{-1} \cdot B^{p_\alpha}$. Consequently,

$$\begin{aligned} g^{\mu^\lambda}_{\mu^\alpha} \cdot g^{\mu^\alpha}_{\beta} &= (B^{p_\lambda})^{-1} \cdot B^{p_\alpha} \cdot g^{\mu^\alpha}_{\mu^i} \cdot f = (B^{p_\lambda})^{-1} \cdot B^{p_\alpha} \cdot (B^{p_\alpha})^{-1} \cdot B^{q_i} \cdot f \\ &= (B^{p_\lambda})^{-1} \cdot B^{p_\delta} \cdot h \cdot f = g^{\mu^\lambda}_{\mu^\delta} \cdot (h \cdot f), \end{aligned}$$

where $\mu^\delta \in [\beta, \mu^\alpha)$. Now, $g^{\mu^\alpha}_{\beta} = g^{\mu^\alpha}_{\mu^\delta} \cdot k$ for some $k \in \mathscr{F}_{\beta\mu^\delta}$, by Section 2(1), since $p_\alpha$ is a condition. As all of the maps are morass maps, $k = h \cdot f$ (cf. (3) $\Longleftrightarrow$ (3′) in Section 2). Thus Section 2(3′), and so continuity, is verified as required.

In view of Theorem 8 of Section 2, the above theorem asserts that the axioms SSFA$_\kappa$, SWFA$_\kappa$, SSSFA$_\kappa$ and SSWFA$_\kappa$ are all equivalent.

*Mechanical strategic closure* is another modification of strategic closure of interest. This time a quasi-strategy is a function taking positions as arguments. That is instead of the usual (psychological) notion in which a player is given some possible responses to the prior sequence of moves in the game, a mechanical quasi-strategy gives a subset of the set of all continuations for the player at the given stage of the game, regardless of how the position was reached.

Expressing this formally, let $\sim$ be the equivalence relation on $\bigcup\{[\boldsymbol{P}]^\beta \mid \lim(\beta) < \alpha\}$ defined by

$$\langle p_\gamma \mid \gamma < \beta \rangle \sim \langle p'_\gamma \mid \gamma < \beta \rangle \text{ if } \{ p \in \boldsymbol{P} \mid \forall \gamma < \beta \; p \leqslant p_\gamma \} = \{ p \in \boldsymbol{P} \mid \forall \gamma < \beta \; p \leqslant p'_\gamma \}.$$

Let $[\langle p_\gamma \mid \gamma < \beta \rangle]$ be the $\sim$-equivalence class of $\langle p_\gamma \mid \gamma < \beta \rangle$.

A *mechanical quasi-strategy* for player I is a function $S : \bigcup\{[\boldsymbol{P}]^\beta \mid \lim(\beta) < \alpha\}/\!\sim \;\to \mathscr{P}(\boldsymbol{P})$ such that $p \in S([\langle p_\gamma \mid \gamma < \beta \rangle]) \Rightarrow \forall \gamma < \beta \; p \leqslant p_\gamma$. As descending sequence of conditions determine equivalence classes, a reasonable abuse of notation is to write a sequence itself as the argument of a mechanical quasi-strategy rather than its equivalence class. With this convention one can define "playing by $S$", "winning mechanical quasi-strategy for player I", "closed under the $\pi_f$s", and "bounded realm mechanical quasi-strategy" verbatim as in Definition 5. These allow the definition of a partial order satisfying

(MS.v.W.) Player I has a winning mechanical quasi-strategy for $G_\kappa^{P^*}$ which is closed under the $\pi_f$s, and

(MS.v.S) Player I has a bounded realm winning mechanical quasi-strategy for $G_\kappa^{P^*}$ which is closed under the $\pi_f$s,

and so the axioms $\mathrm{MSSFA}_\kappa$ and $\mathrm{MSWFA}_\kappa$ (the initials MS standing for mechanical strategic).

It is immediate that a partial order satisfying ($\kappa$-closed.v.W) satisfies (MS.v.W) and that a winning mechanical quasi-strategy is a winning strong quasi-strategy, and consequently that a $(\kappa, 1)$-simplified morass with continuous paths directly ensures that $\mathrm{MSSFA}_\kappa$ and $\mathrm{MSWFA}_\kappa$ hold, and they in turn ensure there is a $(\kappa, 1)$-simplified morass with simple paths. However it is open whether either converse holds.

In the case of each forcing axiom considered it is a simple matter of enumeration to show that the particular augmentation of a simplified morass under consideration guarantees the existence of a generic object meeting $\kappa$ many $X$-$\kappa$-indiscernible sets of dense open sets. Provided $2^{<\kappa} = \kappa$, it is possible, again by a simple enumeration argument, to guarantee the existence of a generic object meeting $X$-$\kappa$-indiscernible sets of dense open sets which are indexed by elements (homogeneously) $\mathscr{P}_{<\kappa}(\kappa^+)$ (cf. [15]).

Recently Cherlin, Hart and Laflamme have put a substantial amount of effort into preparing Shelah's work on his class of *uniform* partial orders for publication. As this seems likely to reach the light of day relatively soon as [14], and as Shelah's work has similarities with the morass inspired forcing theory discussed above, this may be an opportune place to comment on it.

Each $\kappa$-uniform partial order and single density-system satisfy the requirements for the $\kappa$-closedWFA$_\kappa$, save that the dense sets given by the density system are labelled by elements of $\mathscr{P}_{<\kappa}(\kappa^+)$. Consequently if there is a $(\kappa, 1)$-simplified morass with simple paths and $2^{<\kappa} = \kappa$ there will be a generic object meeting any $\kappa$-many density systems. This object will itself be indiscernible in the sense that it will be of the form $\langle \pi_{f^*}(p_\alpha) \mid \alpha < \kappa \,\&\, f \in \mathscr{F}_{\alpha\kappa} \rangle$. However the conditions for $\kappa$-uniformity are much more severe than those imposed on $\boldsymbol{P}_{\kappa^+}$ if it is to fall under the rubric of $\kappa$-closedWFA$_\kappa$, so much so that a generic object meeting any $\kappa$-many density systems can be found under the assumption $2^{<\kappa} = \kappa$ alone. (This object, which will not be indiscernible, is obtained, in the notation of [14], simply by enumerating for each successive $\xi < \kappa^+$ the $D_i(u, v)/G_{<\xi}$ such that $u \subseteq \beta < \kappa \cdot \xi$ in order type $\kappa$ and meeting them.)

Nevertheless, in various application $\diamondsuit_{\{\delta \mid \mathrm{cf}(\delta)=\kappa\}}$ is necessary to dictate appropriate density systems if $2^{<\kappa} = \kappa$ (or $\mathrm{D}\ell_\kappa$) alone is assumed, whereas in many cases a $(\kappa, 1)$-simplified morass with built-in-diamond (which does not make any demands on $2^\kappa$)

suffices. Some of these cases will be covered in the preliminaries to forthcoming papers on higher gap applications.

It is readily apparent that $\kappa$-directed-uniform can be defined in the spirit of $\kappa$-uniform, replacing each closure under unions with closure under directed sets. The examples of $\kappa$-uniform partial orders in [14] and [13] are all in fact $\kappa$-directed uniform. Of course, any use of simplified morasses with simple paths can then be replaced with a use of a plain simplified morass.

Finally Shelah has introduced another idea which should be included in the same circle: "historicisation of forcing conditions". He used this technique to obtain a function with property $\Delta$ (cf. Section 1.8), and it is used again, with greater relevance, in the paper [16]. Any historicised forcing, with $\mu^{\mathscr{P}} = \kappa$ and $\lambda^{\mathscr{P}} = \kappa$, in the terminology of Section 4 of [16], adds a $(\kappa^+, 1)$-simplified morass with continuous paths. Conversely, if there is a $(\kappa^+, 1)$-simplified morass with continuous paths and a complete amalgamation system for it (e.g. if $2^\kappa = \kappa^+$) then a generic object exists for any such historical set of conditions. This will be discussed further in the paper [12]. In [17] the major example of historicised forcing in [16] is replaced by a simplified morass construction. The proof in [17] uses, and seems to require, a simplified morass with continuous paths and a complete amalgamation system (to simulate genericity). For example it is unknown whether $\square$ (or even a simplified morass with simple paths) could be used instead of the morass.

**Questions.** I close by recapitulating, at the referees' suggestion, some of the unsolved problems mentioned in the course of the paper.

(1) Find conditions under which one can add an $(\omega_1, 1)$-simplified morass by ccc forcing.

(2) The same problem for $\square_{\omega_1}$.

(3) Is it consistent that there is a $(\kappa, 1)$-simplified morass with continuous paths but no simplified morass with linear limits? Are large cardinal hypotheses necessary to resolve this?

(4) Under what conditions are there $(\kappa, 1)$-simplified morasses but no $(\kappa, 1)$-simplified morass with linear limits at ordinals of cofinality $\leqslant \omega_1$?

(5) Separate small from simple limits.

(6) Where does mechanical strategic closure lie in the closure hierarchy? In particular, are $\mathrm{MSSFA}_\kappa$ and $\mathrm{SSSFA}_\kappa$ equivalent? Or are $\mathrm{MSSFA}_\kappa$ and $\kappa$-closed-$\mathrm{SFA}_\kappa$ equivalent?

(7) Can one show $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$ from less that a $(\omega_2, 1)$-simplified morass with continuous paths and $2^{\omega_1} = \omega_2$, for example using instead a $(\kappa, 1)$-simplified morass with simple limits or $\square_{\omega_2}$?

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