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A Note on the Boundary Stabilization of a Compactly Coupled System of Wave Equations

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Abstract—We obtain decay estimates of the energy of solutions to compactly coupled wave equations with a nonlinear boundary dissipation which is weak as $|u'|$ tends to infinity.

Keywords—Boundary stabilization, Multiplier method, Wave equation, Integral inequalities.

1. INTRODUCTION

In this paper, we are concerned with the decay property of the solutions to the evolutionary system

$$u_1'' - \Delta u_1 + \alpha(u_1 - u_2) = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$u_2'' - \Delta u_2 + \alpha(u_2 - u_1) = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u_i = 0, \qquad \text{on } \Gamma_0 \times \mathbb{R}_+, \quad i = 1, 2, \tag{1.3}$$

$$\frac{\partial u_i}{\partial \nu} + a_i u_i + g_i(u_i') = 0, \qquad \text{on } \Gamma_1 \times \mathbb{R}_+, \quad i = 1, 2, \tag{1.4}$$

$$u_i(0) = u_{i0} \quad \text{and} \quad u_i'(0) = u_{i1}, \qquad \text{in } \Omega, \quad i = 1, 2, \tag{1.5}$$

where Ω is bounded open domain in $\mathbb{R}^n$, $\{\Gamma_0, \Gamma_1\}$ is a partition of its boundary Γ, ν is the outward unit normal vector to Γ, $\mathbb{R}_+ = [0, +\infty)$, and $\alpha : \Omega \to \mathbb{R}$, $a_1, a_2 : \Gamma_1 \to \mathbb{R}$, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are some given functions.

The problem of proving the energy decay rates for solutions of systems of evolution equations with dissipation at the boundary has been treated by several authors. Indeed, in the case of wave or plate equations we can mention Conrad and Rao [1], Komornik [2], Komornik and Zuazua [3], Lagnese [4], Lasiecka [5], Lasiecka and Tataru [6], Lions [7], and Zuazua [8] among others.

Very little is known for the compactly coupled wave equations. To our knowledge, uniform decay estimates for the one-dimensional case and by applying a linear boundary feedback was

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studied by Najafi, Sahrangi and Wang [9], and quite recently Komornik and Rao [10] have obtained exponential (or polynomial) decay in the multidimensional case when the boundary dissipation satisfies:

$$c_1|x|^p \le |g_i(x)| \le c_2|x|^{1/p}, \qquad \text{if } |x| \le 1, \tag{1.6}$$

$$c_3|x| \le |g_i(x)| \le c_4|x|, \qquad \text{if } |x| \ge 1, \tag{1.7}$$

where c_1, c_2, c_3, and c_4 are four positive constants.

These works [9,10] have a serious drawback from the point of view of physical applications: they never apply for bounded functions g_i because $c_3 > 0$ in (1.7). The purpose of this paper is to obtain a variant of Najafi *et al.* and Komornik and Rao's results for functions such that

$$-\infty < \lim_{x\to-\infty} g_i(x) < \lim_{x\to+\infty} g_i(x) < +\infty. \tag{1.8}$$

If $g_i(x)$ satisfies at most (1.8), the dissipation effect of $g_i(u_i')$ is weak as $|u_i'|$ is large, and for convenience we call such a term weak dissipation. The most typical example is $g_i(x) = x/\sqrt{1+x^2}$. Let us note that the case of single wave equation with internal damping $g(x)$ satisfying (1.8) was studied by Nakao [11].

2. THE MAIN RESULT

Throughout the paper we shall make the following assumptions.

(H1) The domain Ω is of class C^2.

(H2) The partition of Γ satisfies the condition $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

(H3) There exists a point $x_0 \in \mathbb{R}^n$ such that, putting $m(x) = x - x_0$, we have

$$m\cdot\nu \le 0, \quad \text{on } \Gamma_0 \qquad \text{and} \qquad \inf_{\Gamma_1} m\cdot\nu > 0.$$

(H4) The coefficients a_i are nonnegative and they belong to $C^1(\Gamma_1)$. Moreover, either $\Gamma_0 \ne \emptyset$ or $\inf_{\Gamma_1} a_1 > 0$ and $\inf_{\Gamma_1} a_2 > 0$.

(H5) The function α is nonnegative and belongs to $L^\infty(\Omega)$.

(H6) The functions g_i are continuous, nondecreasing, and

$$g_i(x) = 0 \Longleftrightarrow x = 0.$$

Furthermore, there exists a constant c such that

$$|g_i(x)| \le 1 + c|x|, \qquad \forall x \in \mathbb{R}.$$

REMARKS.

(1) If $n = 1$, then Ω is bounded open interval, say $\Omega = (x_1, x_2) \subset \mathbb{R}$, and hence hypothesis (H1) is always satisfied. Furthermore, hypotheses (H2),(H3) are satisfied in each of the following three cases:
   (i) $\Gamma_0 = \emptyset$, $\Gamma_1 = \{x_1, x_2\}$,
   (ii) $\Gamma_0 = \{x_1\}$, $\Gamma_1 = \{x_2\}$,
   (iii) $\Gamma_0 = \{x_2\}$, $\Gamma_1 = \{x_1\}$.

(2) If $n \ge 2$ and Ω is star-shaped with respect to some point $x_0 \in \Omega$ (i.e., $m\cdot\nu \ge 0$ on Γ), then hypotheses (H2),(H3) are satisfied with $\Gamma_0 = \emptyset$, $\Gamma_1 = \Gamma$.

(3) If $n \ge 2$ and $\Omega = \Omega_1 - \bar{\Omega}_0$ where Ω_0, Ω_1 are star-shaped domains with respect to some point $x_0 \in \Omega_0$ and $\bar{\Omega}_0 \subset \Omega_1$, then hypotheses (H2),(H3) are satisfied with $\Gamma_0 = \partial\Omega_0$, $\Gamma_1 = \partial\Omega_1$.

(4) If $n \geq 2$ and Ω is not of the form mentioned in the preceding two examples, then in general there is no point x_0 satisfying simultaneously (H2) and (H3). By applying an approximational method of Grisvard [12], one could considerably weaken assumptions (H2),(H3), at least in dimensions $n = 2, 3$, by adapting an analogous argument given in [3] for the wave equation. We note that Moussaoui [13] has recently generalized Grisvard's results for arbitrary space dimension n. We do not pursue these matters here.

(5) The condition $\inf_{\Gamma_1} m \cdot \nu > 0$ of (H3) is unnecessary if we use the weight $m \cdot \nu$ in the boundary damping as in [3].

If $u = (u_1, u_2)$ is a solution of problems (1.1)–(1.5), then we define its energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ by the following formula:

$$E(t) := \frac{1}{2} \int_\Omega (u_1')^2 + |\nabla u_1|^2 + (u_2')^2 + |\nabla u_2|^2 + \alpha(u_1 - u_2)^2 + \frac{1}{2} \int_{\Gamma_1} a_1 u_1^2 + a_2 u_2^2.$$

It is well known (see [10, Theorem 1]) that under hypotheses (H1)–(H6), given $(u_{i0}, u_{i1}) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ $(i = 1, 2)$ arbitrarily, problems (1.1)–(1.5) has a unique weak solution satisfying

$$u_i \in C\left(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right), \qquad i = 1, 2,$$

where we set $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}$.

Our main result is the following.

MAIN THEOREM. *Let $(u_{i0}, u_{i1}) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)$ $(i = 1, 2)$ such that*

$$\frac{\partial u_{i0}}{\partial \nu} + a_i u_{i0} + g_i(u_{i1}) = 0, \qquad \text{on } \Gamma_1, \quad i = 1, 2.$$

Assume that there exists a number p such that

$$p = 1, \qquad \text{if } n = 1, \tag{2.1}$$

$$p > 1, \qquad \text{if } n = 2, \tag{2.2}$$

$$p \geq n - 1, \qquad \text{if } n \geq 3, \tag{2.3}$$

and four positive constants c_1, c_2, c_3, c_4 such that

$$c_1|x|^p \leq |g_i(x)| \leq c_2|x|^{1/p}, \qquad \text{if } |x| \leq 1, \tag{2.4}$$

$$c_3 \leq |g_i(x)| \leq c_4|x|, \qquad \text{if } |x| \geq 1, \tag{2.5}$$

and assume that α is a nonnegative constant, then the strong solution of (1.1)–(1.5) satisfies the estimates:

$$E(t) \leq cE(0)e^{-\omega t}, \qquad \forall t \geq 0, \quad \text{if } p = 1, \tag{2.6}$$

where c, ω are positive constants and

$$E(t) \leq \frac{c}{(1+t)^{2/(p-1)}}, \qquad \forall t \geq 0, \quad \text{if } p > 1, \tag{2.7}$$

where c is a positive constant.

REMARK. If (2.4) holds, then it holds also for any $q \geq p$. Thus as soon as (2.4) holds for some p, the main theorem applies and provides the decay rate corresponding to the least $q \geq p$ satisfying the assumptions of the theorem.

To end this section, let us recall the following useful lemma.

LEMMA 2.1. *(See [14, Theorem 9.1].) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and assume that there are two constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^\infty E^{\alpha+1}(s)\, ds \leq TE(0)^\alpha E(t), \qquad \forall t \in \mathbb{R}_+. \tag{2.8}$$

Then we have

$$E(t) \leq E(0) \left(\frac{T + \alpha t}{T + \alpha T}\right)^{-1/\alpha}, \qquad \forall t \geq T. \tag{2.9}$$

3. PROOF OF THE MAIN THEOREM

Multiplying (1.1) by u_1', (1.2) by u_2', integrating their sum by parts in $\Omega \times (0,T)$, and finally eliminating the normal derivatives by using the boundary conditions (1.3) and (1.4), we obtain easily that

$$E(0) - E(T) = \int_0^T \int_{\Gamma_1} u_1' g_1(u_1') + u_2' g_2(u_2')\, d\gamma\, dt \tag{3.1}$$

for every positive number T. Being the primitive of an integrable function, hence E is locally absolutely continuous and

$$E' = -\int_{\Gamma_1} u_1' g_1(u_1') + u_2' g_2(u_2')\, d\gamma \tag{3.2}$$

is satisfied almost everywhere in $\mathbb{R}_+$.

We are going to prove that the energy of the strong solution

$$u_i \in L^\infty\left(\mathbb{R}_+, H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \cap W^{1,\infty}\left(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)\right) \cap W^{2,\infty}\left(\mathbb{R}_+, L^2(\Omega)\right), \qquad i = 1,2,$$

satisfies the estimate

$$\int_S^T E^{(p+1)/2}(t)\, dt \le cE(S) \tag{3.3}$$

for all $0 \le S < T < +\infty$. Here and in the sequel we shall denote by c various positive constants, depending only on the initial energy $E(0)$ and on the four constants c_i appearing in assumptions (2.4),(2.5).

According to Komornik and Rao [10, Lemma 4.1], we have by multiplying equation (1.1) by $E^{(p-1)/2}(2(x-x_0)\cdot\nabla u_1 + (n-\varepsilon)u_1)$ that

$$\int_S^T E^{(p+1)/2}\, dt \le cE(S) + c\sum_{i=1}^2 \int_S^T E^{(p-1)/2} \int_{\Gamma_1} (u_i')^2 + g_i(u_i')^2 + (u_i)^2. \tag{3.4}$$

In order to get rid of the term u_i^2 in estimate (3.4), we apply a method introduced by Conrad and Rao [1]. Thanks to [10, pp. 353–355], we arrive at

$$\begin{aligned}\int_S^T E^{(p-1)/2} \int_{\Gamma_1} |u|^2\, d\gamma\, dt \\ \le cE(S) + c\int_S^T E^{(p-1)/2} \int_{\Gamma_1} |u_1 g_1(u_1')| + |u_2 g_2(u_2')|\, d\gamma\, dt + c\int_S^T E^{p/2} \|u'\|_{L^2(\Gamma_1)}\, dt\end{aligned}$$

where $|u|^2 = u_1^2 + u_2^2$ and $|u'|^2 = (u')_1^2 + (u')_2^2$.

Now, we want to estimate the last term of the right-hand side of the above estimate.

LEMMA 3.1. *It holds that*

$$\|u'\|_{L^2(\Gamma_1)} \le c|E'|^{1/2}, \qquad \text{if } n = 1,$$

and

$$\|u'\|_{L^2(\Gamma_1)} \le c|E'|^{1/(p+1)}, \qquad \text{if } n \ge 2. \tag{3.5}$$

PROOF OF LEMMA 3.1. Using the growth assumption (2.4), we have

$$\begin{aligned}\int_{|u_1'|\le 1} (u_1')^2\, d\gamma &\le c\int_{|u_1'|\le 1} \left(u_1' g_1(u_1')\right)^{2/(p+1)} d\gamma \le c\left(\int_{|u_1'|\le 1} u_1' g_1(u_1')\, d\gamma\right)^{2/(p+1)} \\ &\le c\left(\int_{\Gamma_1} u_1' g_1(u_1')\, d\gamma\right)^{2/(p+1)} \le c|E'|^{2/(p+1)}.\end{aligned}$$

On the other hand, if $n = 1$, we have $u_1' \in H^1_{\Gamma_0}(\Omega) \subset L^\infty(\Omega)$ and then

$$\int_{|u_1'|\ge 1} (u_1')^2\, d\gamma \le c \int_{|u_1'|\ge 1} u_1' u_1' g_1(u_1')\, d\gamma$$
$$\le c\|u_1'\|_{L^\infty} \int_{\Gamma_1} u_1' g_1(u_1')\, d\gamma \le c|E'|.$$

If $n \ge 2$, set $\|\cdot\|_\beta := \|\cdot\|_{L^\beta(\Gamma_1)}$, $s := 2/(p+1)$, and $\alpha := (2-s)/(1-s)$, we have $0 < s < 1$ and $\alpha = 2p/(p-1) > 2$. Then

$$\begin{aligned}\int_{|u_1'|\ge 1} (u_1')^2\, d\gamma &\le c \int_{\Gamma_1} |u_1'|^{2-s} (u_1' g_1(u_1'))^s\, d\gamma\\
&\le \left\| |u_1'|^{2-s} \right\|_{1/(1-s)} \left\|(u_1' g_1(u_1'))^s\right\|_{1/s}\\
&= \|u_1'\|_\alpha^{(1-s)\alpha} \|u_1' g_1(u_1')\|_1^s\\
&\le \|u_1'\|_{2p/(p-1)}^{2p/(p+1)} |E'|^{2/(p+1)}\\
&\le c|E'|^{2/(p+1)};\end{aligned}$$

in the last step, we used the fact $H^1(\Omega) \subset L^{2p/(p-1)}(\Gamma)$ following from (2.3).

We obtain similar inequalities for u_2.

Proof of the Main Theorem Completed. For any fixed $\varepsilon > 0$ we have

$$c\,|u_i g_i(u_i')| \le \varepsilon u_i^2 + c\varepsilon^{-1} g_i(u_i')^2, \tag{3.6}$$

and

$$cE^{p/2}\|u_i'\|_{L^2(\Gamma_1)} \le \varepsilon E^{(p+1)/2} + c\varepsilon^{-p}|E'|. \tag{3.7}$$

From (3.5)–(3.7), and by choosing ε small enough, we have:

$$\int_S^T E^{(p-1)/2} \int_{\Gamma_1} |u|^2 \le \delta \int_S^T E^{(p+1)/2} + cE(S) + c\int_S^T E^{(p-1)/2} \int_{\Gamma_1} g_1(u_1')^2 + g_2(u_2')^2 \tag{3.8}$$

when $\delta > 0$ is an arbitrarily real number. By adding (3.4) and (3.8), we obtain easily

$$\int_S^T E^{(p+1)/2} \le cE(S) + c\int_S^T E^{(p-1)/2} \int_{\Gamma_1} |u'|^2 + g_1(u_1')^2 + g_2(u_2')^2\, d\gamma\, dt \tag{3.9}$$

for all $0 \le S < T < +\infty$.

From Lemma 3.1, (2.4)–(2.5), we have

$$\|g_i(u_i')\|_{L^2(\Gamma_1)} \le c|E'|^{1/(p+1)}, \qquad i = 1, 2.$$

Substituting into the right-hand side of (3.9), we obtain that

$$\int_S^T E^{(p+1)/2}\, dt \le cE(S) + c\int_S^T E^{(p-1)/2}|E'|^{2/(p+1)}\, dt.$$

Using the Young inequality, for any fixed $\varepsilon > 0$ we have

$$cE^{(p-1)/2}|E'|^{2/(p+1)} \le \varepsilon E^{(p+1)/2} + c\varepsilon^{(1-p)/2}|E'|.$$

Therefore,

$$(1-\varepsilon)\int_S^T E^{(p+1)/2}(t)\, dt \le c\left(1 + \varepsilon^{(1-p)/2}\right) E(S),$$

and choosing $0 < \varepsilon < 1$, it follows that

$$\int_S^T E^{(p+1)/2}(t)\, dt \le cE(S),$$

with $p = 1$ if $n = 1$.

Thanks to Lemma 2.1, we deduce (2.6),(2.7).

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