Bounds of Waiting-Time in Nonlinear Diffusion

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Abstract—The solutions of the nonlinear diffusion equation \( h_t = (h^m h_x)_x \) may have a waiting-time, i.e., an initial finite time interval \( t_w \) in which the front is at rest before starting to move. The theory gives us the value of \( t_w \) only in a few special cases, when it is determined by the local behaviour near the front of the initial profile \( g(x) \). However, in many instances \( t_w \) depends on the global behaviour of \( g(x) \), and in these cases the theory provides only upper and lower bounds that frequently may not be very helpful to estimate the \( t_w \). Here we discuss some global attributes that influence \( t_w \). Then we employ the values of \( t_w \) obtained numerically for initial profiles of the power law type to obtain, for general initial profiles, bounds more stringent than those given by the current theory. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

There are many processes described by the nonlinear diffusion equation (also called the porous media equation), that in one dimension can be written as

\[ h_t = (h^m h_x)_x . \] (1)

Here \( h \equiv h(x, t) \), \( m > 0 \), \( x \) is a Cartesian coordinate, \( t \) is the time, and the meaning of \( h \) and the value of \( m \) depend on the specific problem under consideration. Some examples are: unconfined groundwater flow [1–3] \( (m = 1 \) and \( h \) is the height\); dispersion of biological populations [4] \( (m \geq 1 \) and \( h \) is the population density\); gas flow in a porous media [5] \( (m \geq 1 \) and \( h \) is the

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The solutions of equation (1) may have moving fronts (also called interfaces) that move with a finite velocity. More precisely, if an initial profile \( g(x) = h(x, 0) \) has a finite support, then for all \( t > 0 \), \( h(x, t) \) will also have a finite support. The position \( x_f(t) \) of a front is a continuous monotonic function of time. Under certain conditions the front may initially remain motionless for a time interval \( t_w > 0 \), after which it starts to move and never stops. This behavior is called the waiting-time phenomenon and is closely connected with the occurrence during the waiting stage of corner layers (small intervals \( \Delta x \) in which \( h_x \) varies rapidly) in the solutions of equation (1).

For any given \( m \), the theory provides an upper and a lower bound [15] of \( t_w \) that we call \( t_A \) and \( t_B \), but only in the very special cases when \( t_A = t_B \) the exact value of \( t_w \) is obtained. Another upper bound that we call \( t_\infty \) has been derived by Vázquez [16] by means of the Shifting Comparison Principle (SCP). In many instances \( t_\infty \) is a more stringent bound than \( t_A \). However, for general \( g(x) \) these theoretical bounds usually do not provide good estimates of \( t_w \).

Here we present values of \( t_w \) obtained solving numerically equation (1) for seven values of \( m \) in the interval \( 1/2 < m < 9 \) and for initial profiles \( g_\alpha(x) \) given by

\[
g_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ (1 + \alpha q)x^\alpha q, & \text{if } 0 \leq x \leq 1, \end{cases}
\]

with \( q = 2/m \) and \( \alpha \geq 1 \) (this condition ensures that \( t_w > 0 \) [17,18]). For \( t > 0 \), we assume the no-flow condition \( h_x(1, t) = 0 \) at \( x = 1 \). From these numerical solutions we determine the waiting-times \( t_w(m, \alpha) \). Notice that as (2) is only one type among all conceivable initial conditions leading to waiting-time solutions, our results may seem uninteresting. However, we shall show how to use \( t_w(m, \alpha) \) in conjunction with the SCP to find bounds of the waiting-time for general initial conditions, and that these new bounds may be better than the previously mentioned theoretical bounds.

In Section 2, we briefly review the theoretical bounds of \( t_w \), and we discuss some global features of the initial profile that influence the waiting-time. The values of \( t_w \) obtained numerically are presented in Section 3. In Section 4, we explain by means of an example how to use them to improve the theoretical bounds. Section 5 contains the discussion.

2. THE THEORETICAL BOUNDS OF WAITING-TIME

It has been shown by Aronson, Caffarelli and Kamin [15] that if \( g(x) \) satisfies the condition

\[
g(x) = A |x - x_f|^q, \quad \text{near } x = x_f,
\]

and

\[
g(x) \leq B |x - x_f|^q, \quad \text{everywhere},
\]

where \( A \) and \( B \) (the smallest such \( B \), i.e., \( B = \sup_{x>x_f} g(x)/|x - x_f|^q \)) are positive constants, then

\[
\frac{m}{2(m+2)B^m} = t_B(m) \leq t_w \leq t_A(m) = \frac{m}{2(m+2)A^m}.
\]

When \( A = B \), the exact value of waiting-time is given by \( t_A(m) \). There are two kinds of waiting-times:

(a) those that depend on the local behaviour of \( g(x) \) at the front, for which \( t_w = t_A \geq t_B \), and
(b) those that depend on the global behaviour of \( g(x) \), whose exact value is not known from theory.

In Case (b), a corner layer develops during the waiting stage, that ends when the corner layer overtakes the front becoming a corner shock. In Case (a), no corner layer appears for \( t < t_w \).

This distinction was discussed in [17] for the case \( m \ll 1 \) and later in [19] for arbitrary \( m \).

Conditions (3),(4) can be stated in an equivalent form defining

\[
t^*(x) = \frac{m}{2(m+2)} \frac{(x-x_f)^2}{g(x)^m}.
\]

Let us call \( x^* \) the point where \( t^*(x) \) is minimum. Then \( t^*(x_f) = t_A(m) \) and \( t^*(x^*) = t_B(m) \).

If \( t^*(x_f) = 0 \), there is no waiting-time, if \( t^*(x_f) = \infty \), the waiting-time is determined by the global data; if \( t^*(x_f) \) is finite and nonvanishing, the waiting-time depends on the local data when \( t^*(x_f) = t^*(x^*) \), otherwise, it depends on the global data and is bounded by \( t^*(x_f) \) and \( t^*(x^*) \). In consequence, a plot of \( t^*(x) \) allows us at once to find which is the case for any \( g(x) \) of interest. However, it is not easy to evaluate from the shape of \( t^*(x) \) how \( t_w \) depends on the global data. Roughly speaking, \( t_w \) will be closer to \( t_B(m) \) the smaller \( |x_f - x^*| \) is and the closer \( t^*(x) \) is to \( t^*(x^*) \) in the domain of interest, as we found by numerical calculations in a number of cases (we omit details). Based on this evidence, it appears likely that \( |x_f - x^*| \) and some appropriate "average value" of \( t^*(x) \) are the main attributes of the global behaviour that control waiting-times. Both these attributes can be easily appreciated from the plot of \( t^*(x) \).

Vázquez [16] has derived a different upper bound of \( t_w \) by means of the SCP, that in the case we are considering can be stated as follows: if for every \( x \in (-\infty, 1] \), we have

\[
M_1(x,0) = \int_{-\infty}^{x} g_1(x) dx \leq \int_{-\infty}^{x} g_2(x) dx = M_2(x,0),
\]

then for every \( t > 0 \), one has

\[
M_1(x,t) = \int_{-\infty}^{x} h_1(x,t) dx \leq \int_{-\infty}^{x} h_2(x,t) dx = M_2(x,t),
\]

i.e., if initially the "mass" of \( h_1 \) is shifted to the right with respect to that of \( h_2 \), this situation is preserved for every later time. As an application of this principle, let us assume that \( g_1,2(x \leq 0) = 0 \) and \( h_2 \) is a waiting-time solution; then, taking \( x = 0 \) in (8), it can be seen that \( h_1 \) is also a waiting-time solution with \( t_{w1} \geq t_{w2} \), a result that shall be useful later. To obtain the upper bound of Vázquez, let us consider a waiting-time solution \( h_2 \) with \( M_2(1,0) = M \) whose waiting-time is \( t_{w2} \). Now we take \( g_1 = 2M\delta(x-1) \), that is, an initial condition for which equation (1) has a known analytical solution, according to which \( M_1(0,t) = 0 \) for \( t < t_\infty \) given by

\[
t_\infty (m,M) = \frac{m}{M^m(2m+4) \left[ \frac{\sqrt{\pi} \Gamma(1+1/m)}{2\Gamma(3/2+1/m)} \right]^m}.
\]

Then according to the SCP we obtain \( t_{w2} \leq t_\infty (m,M) \), which is the desired upper bound.

In conclusion, for generic \( m \) there are no analytical formulæ for \( t_w \), except for the special cases when \( A = B \), and only the bounds given by equations (5) and (9) are available.

For \( m = \epsilon \ll 1 \), Kath and Cohen [17] have derived a leading order approximation of \( t_w \), given by (\( f(x) = g^*(x) \))

\[
t^{(0)}_w = \epsilon \frac{f(\xi)}{f^2(\xi)},
\]

where \( \xi \) satisfies \( f'(\xi)(\xi - x_f) = 2f(\xi) \). Equation (10) reduces to

\[
t^{(0)}_w = \epsilon \frac{1}{2f''(\xi)},
\]

if \( f(\xi) \) and \( f'(\xi) \) are both zero.
3. NUMERICAL VALUES OF WAITING-TIME

The numerical values of $t_w(m, \alpha)$ are plotted in Figure 1 (for details about the numerical procedure, see [20]; tables can be provided if requested). It can be noted that $t_w(m, \alpha)$ increases with $\alpha$ and $m$. The theory predicts lower and upper bounds of $t_w(m, \alpha)$. The lower bound corresponds to $\alpha = 1$ and is given by $t_B(m)$ with $B = 1 + q$ (see (5)). The upper bound, corresponding to $\alpha \to \infty$, is $t_\infty(m, 1)$. Because our initial condition (2) tends to a Dirac delta $2\delta(x - 1)$ as $\alpha \to \infty$, it is obvious that $t_w(m, \alpha) \to t_\infty(m, 1)$ in this limit. For fixed $m$, as $\alpha$ increases, $t_w(m, \alpha)$ rapidly approaches $t_\infty(m, 1)$.

![Figure 1. Values of waiting-times for all m and \alpha considered in this work.](image)

4. BOUNDS OF THE WAITING-TIME FOR OTHER INITIAL CONDITIONS

Our numerical investigation is limited to a particular class of initial profiles, and the reader may wonder what happens in problems involving initial conditions more general than (2). In this connection it must be pointed out that the results of Section 3 may be used in combination with the previously mentioned SCP, to find useful lower and upper bounds of $t_w$. Indeed, given any initial profile $g(x)$, if we can find $\alpha_1$, $k_1 > 0$, and $\alpha_2$, $k_2 > 0$ such that for every $x \in (-\infty, 1]$

$$\int_{-\infty}^{x} k_1 g_{\alpha_1}(x) dx \leq \int_{-\infty}^{x} g(x) dx \leq \int_{-\infty}^{x} k_2 g_{\alpha_2}(x) dx,$$  \tag{12}

where $g_{\alpha_1}(x)$ and $g_{\alpha_2}(x)$ are of the form (2), then

$$\frac{t_w(\alpha_2)}{k_2^m} \leq t_w \leq \frac{t_w(\alpha_1)}{k_1^m}.$$ \tag{13}

Notice that the behaviour of $g(x)$ near the front restricts $\alpha_1$ and $\alpha_2$. In effect, it can easily be seen that

if $g(x) \propto |x - x_f|^{\beta}$, then $1 \leq \alpha_2 \leq \beta \leq \alpha_1$. \tag{14}

In the special case $\beta = 1$, it follows from (14) that $\alpha_2 = 1$, which means that the lower bound in (13) coincides with $t_B(m)$; in this case, the present method does not lead to an improvement of this bound.
As an example of how to employ the numerical values \( t_w(m, \alpha) \) plotted in Figure 1 to derive bounds for \( t_w \) for a given initial profile, let us consider the initial condition
\[
g(x) = \begin{cases} (1 - \theta) \sin^2 \left( \frac{\pi x}{2} \right) + \theta \sin^4 \left( \frac{\pi x}{2} \right) \right]^{1/m}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0, \end{cases}
\] (15)
where \( 0 \leq \theta \leq 1 \). This initial profile has been studied (see [15,21,22]) to test the theoretical bounds. From equation (5), it can be found that if \( 0 \leq \theta \leq 1/4 \), then \( t_w = 2m[\pi^2(m+2)(1-\theta)]^{-1} \), but if \( 1/4 < \theta \leq 1 \), \( t_B(m) \leq t_w \leq 2m[\pi^2(m+2)(1-\theta)]^{-1} \) (notice that \( t_B(m) \) must be found numerically and that the upper bound diverges as \( \theta \to 1 \)).

To test our bounds, we considered the cases \( m = 1/2, 1, 3 \). The results are shown in Figure 2 in which we plot the bounds derived from equations (5) and (9), and our bounds. For comparison we also plot the exact values of \( t_w \) obtained with our numerical code, and for \( m = 1/2 \) the
leading order approximation of [17]. When \( x \to 0^+ \), for the profile (15) \( g(x) \propto x^\theta \) for \( \theta < 1 \), and \( g(x) \propto x^{2\theta} \) for \( \theta = 1 \). Then, according to (14), we cannot improve the lower bound except for \( \theta = 1 \), when we find a lower bound considerably larger than that derived from (5). It can be seen that the upper bound we derive is more stringent than (5) for \( \theta \gtrsim 0.55 \) and is always better than (9). The fact that our bound is stronger than that given by (5) implies the presence of corner behaviour of the solutions of (1) and (15) for \( \theta \gtrsim 0.55 \), which means that \( t_w \) is determined by the global properties of (15). Actually, the corner behaviour is easily observed in the numerical solutions for \( \theta > 0.3 \). This can be appreciated in Figure 3a in which we plot \( h^{1/2} \) in the neighbourhood of the front, at different equally spaced times close to \( t_w \), for an initial profile of the form (15) with \( \theta = 0.9 \). It can be seen that the thickness of the corner layer decreases towards zero faster than its distance from the front, and that its speed is nearly equal to the velocity of the front for \( t > t_w \) as anticipated in [19]. We strongly suspect that this corner behaviour occurs in the whole interval \( 1/4 < \theta \leq 1 \). Details of the formation and the evolution of the corner layer for initial conditions of the form (2) can be found in [23].

\[
\begin{align*}
\text{(a)} & \quad \text{Figure 3. Plots of } h^{1/2} \text{ in the neighbourhood of the front, at different times close to } t_w \text{ and spaced by } 0.02t_w, \text{ for an initial profile of the form (15) with (a) } \theta = 0.9, \\
\text{(b)} & \quad \text{and (b) } \theta = 0.1. \text{ The thick lines correspond to the initial profile and the profile at } t = t_w.
\end{align*}
\]

In contrast, Figure 3b shows a similar sequence of profiles as in Figure 3a, but for \( \theta = 0.1 \), when the waiting-time is determined by local properties of \( g(x) \) near the front. No corner layer is observed during the waiting stage, and the front velocity is continuous at \( t = t_w \).

For \( m = 1/2 \), we can compare our results with the leading order approximation for \( m \equiv \epsilon \ll 1 \) to the waiting-time given by the appropriate formula (10) or (11) and tabulated in [17]. Surprisingly, this approximation is quite good in the interval \( 0.5 < \theta < 1 \). However, it fails for \( 0 < \theta < 0.25 \), when \( t_w = t_A = t_B \) is determined by the local behaviour of (15) near the front. Notice also that for the initial conditions (15), the first-order approximation mentioned in [17] coincides with \( t_B \) for all \( \theta \).

### 5. DISCUSSION

We have shown how our \( t_w(\alpha, m) \) (obtained numerically for initial profiles of the power law type (2)) can be used to derive bounds that can be more stringent than those given by the theory. Of course, other families of initial data can be used in place of (2) in conjunction with the SCP to find bounds that might perhaps turn out to be better than the present ones.

It must be noticed that condition (3) singles out a unique class of initial profiles. For any other \( g(x) \) leading to a waiting-time solution, \( t_A(m) \) diverges, and the only meaningful theoretical
upper bound is $t_\infty(m, M)$. On the other hand, as long as $t_{\infty}(m, M)$ is a more stringent bound than $t_A(m)$, with adequate choices of $\alpha_1$ and $k_1$ in (12), we can achieve a better upper bound, since our set of comparison functions is broader than that used by Vázquez (that corresponds to the special choice $\alpha_2 = \infty$ in (12)). When condition (3) is satisfied, we must compare the upper bound derived by our method with $t_A(m)$ to verify which is better. In this sense, the example shown in the previous section provides a quite severe test of the quality of our upper bound. With respect to the lower bound, clearly this example is the worst possible case, since, for any $g(x)$ that does not satisfy condition (3), we should be able to obtain a lower bound better than $t_B(m)$ as is suggested by the comparison of $t_{\alpha}(m, \alpha)$ with the corresponding $t_B(m)$ (see, for example, [20, Figure 6b]).

As previously said, frequently the theoretical bounds (5) and (9) are not good estimators of the waiting-time. Thus, if one wants to know $t_w$, one must compute the numerical solution of equation (1) with the initial conditions under consideration, and then analyze its evolution. This procedure requires not only the development of an adequate numerical code, but in addition accurate techniques to extract from the numerical solutions the exact moment when the front starts moving, which is not a trivial matter. All this implies a cumbersome task that many people interested in applications would gladly avoid. In this situation, our results provide a simpler alternative that consists in estimating the waiting-time by means of the bounds derived from our numerical $t_w(m, \alpha)$.

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