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Modelling of periodic electromagnetic structures bianisotropic materials with memory effects

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Abstract

We study the behavior of the electromagnetic field of a medium presenting periodic microstructures made of bianisotropic material. We reconsider the classical multi-scale homogenization technique by giving a new approach based upon the periodic unfolding method. The limiting homogeneous constitutive law is thus rigorously justified both in the time domain and in the frequency domain. In particular we show that the limit law differs from the initial one regarding the convolution term accounting for the memory effects.

Résumé

On propose une justification rigoureuse de la loi de comportement limite d'un matériau électromagnétique présentant une structure périodique bianisotrope. Cette étude est menée sur les formulations temporelle, puis fréquentielle, des équations de Maxwell en appliquant la méthode de l'éclatement périodique. Nous montrons en particulier que la loi limite comporte un terme de convolution supplémentaire qui rend compte de certains effets de mémoire.

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1. Introduction

The question of replacing spatially periodic material with respect to Maxwell equations presenting *heterogeneous microstructures* by a "new material" characterized by *homogeneous parameters* has been addressed long time ago as an challenging one (Bossavit [2], El Feddi [7]). The objective of the *micro–macro* approach is twofold: on the one hand, by replacing a problem with *periodically varying coefficients* by an *homogeneous* problem, one can save computation time, and on the other hand by selecting an "optimal design" of the microstructures one can improve the performance of the *effective parameters* of the homogeneous equivalent material.

From a mathematical point of view, Maxwell equations with *time-independent* coefficients have been recently studied by Wellander [8] with a "two-scale" approach and by Barbatis [1] with two-scale and "compactness compensated" approaches. In this paper we revisit this problem using the novel *periodic unfolding method* introduced by D. Cioranescu, A. Damlamian and G. Griso in the abstract framework of stationary elliptic equations [4]. The study is conducted in the general case of constitutive laws that take into account *bianisotropy* and *chiral symmetry*, *thermal* and *memory effects*. The treatment of *time-dependent* coefficients we consider here presents several technical difficulties and yields a limit constitutive law *different* from the original one, in particular the convolution operator that accounts for memory effects is replaced by a more complex Hilbert–Schmidt operator.

The paper is organized as follows: in the first section we give the expression of the generalized Maxwell equations for bianisotropic materials, in the second section we derive the limiting homogenized evolution problem, finally in the third section we adopt the frequency point of view by considering the stationary Maxwell equations and derive the associated limit homogeneous model. For the sake of clearness all proofs are gathered in Appendix A.

These results have been first presented in [3].

1.1. Equilibrium equations and constitutive law

We consider a time $T > 0$ and a domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. Under the action of exterior source $(j^{\mathbb{E}}, j^{\mathbb{H}})$, the electric field $\mathbb{E}$ and the magnetic field $\mathbb{H}$ are solutions to the following evolution problem posed on $(0, T) \times \Omega$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{D}(t,x) = \operatorname{curl}\mathbb{H}(t,x) - j^{\mathbb{E}}(t,x), \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}(t,x) = -\operatorname{curl}\mathbb{E}(t,x) - j^{\mathbb{H}}(t,x), \end{cases}$$

with the initial condition,

$$\mathbb{E}(0,x) = \mathbb{E}^0(x), \quad \mathbb{H}(0,x) = \mathbb{H}^0(x) \quad \text{in } \Omega,$$

and the ideal conductor boundary condition,

$$n(x) \times \mathbb{E}(t,x) = 0 \quad \text{on } [0,T] \times \partial\Omega,$$

where n is the unit normal vector to $\partial\Omega$. The electric displacement $\mathbb{D}$ and the magnetic induction $\mathbb{B}$ are related to $\mathbb{E}$ and $\mathbb{H}$ by a constitutive law whose properties are described below.

In the case of a *linear* dielectric media the general constitutive law takes the convolution form:

$$\begin{cases} \mathbb{D}(t,x) = (\varepsilon * \mathbb{E})(t,x), \\ \mathbb{B}(t,x) = (\mu * \mathbb{H})(t,x), \end{cases}$$

where the parameters are the permittivity ε and permeability μ of the material under study,

$$\varepsilon = \varepsilon_0 \varepsilon_r \delta + \sigma^{\mathbb{E}} \mathbb{Y} + \nu^{\mathbb{E}}, \qquad \mu = \mu_0 \mu_r \delta + \sigma^{\mathbb{H}} \mathbb{Y} + \nu^{\mathbb{H}},$$

ε_0 is the permittivity of free space, ε_r the relative permittivity of the media, μ_0 the permeability of free space, μ_r the relative permeability of the media, $\sigma^{\mathbb{E}} > 0$ the electric conductivity that characterizes the current density, $\nu^{\mathbb{H}}$ is the magnetic susceptibility, the other quantities are introduced for the sake of symmetry; as functions of time δ is the Dirac distribution and $\mathbb{Y}$ the Heaviside function. When we use the expressions of ε, μ given before we get the general form of the constitutive law:

$$\begin{cases} \mathbb{D}(t,x) = \varepsilon_0 \varepsilon_r(x) \mathbb{E}(t,x) + \int_0^t \sigma^{\mathbb{E}}(x) \mathbb{E}(\tau,x)\, d\tau + \int_0^t \nu^{\mathbb{E}}(t-\tau) \mathbb{E}(\tau,x)\, d\tau, \\ \mathbb{B}(t,x) = \mu_0 \mu_r(x) \mathbb{H}(t,x) + \int_0^t \sigma^{\mathbb{H}}(x) \mathbb{H}(\tau,x)\, d\tau + \int_0^t \nu^{\mathbb{H}}(t-\tau) \mathbb{H}(\tau,x)\, d\tau. \end{cases}$$

In the *homogeneous case* parameters ε and μ are independent of the space variable $x \in \Omega$. In the *isotropic* case parameters $\varepsilon_r, \mu_r, \sigma^{\mathbb{E}}, \sigma^{\mathbb{H}}, \nu^{\mathbb{E}}, \nu^{\mathbb{H}}$ are scalars or diagonal matrices. In this paper we study the general case of *chiral* materials whose constitutive law is symmetric in $\mathbb{E}$ and $\mathbb{H}$,

$$\begin{cases} \mathbb{D} = \varepsilon_1 * \mathbb{E} + \varepsilon_2 * \mathbb{H}, \\ \mathbb{B} = \mu_1 * \mathbb{E} + \mu_2 * \mathbb{H}, \end{cases}$$

where $\varepsilon_1, \varepsilon_2, \mu_1, \mu_2$ are of the same forms as ε and μ given previously.

To be able to take into account *thermal effects* we assume that all the parameters are time-dependent. In the next section we rewrite the Maxwell equations in a more compact form.

1.2. Generalized Maxwell equations

We introduce the following notations:

- $j : (0,T) \times \Omega \to \mathbb{R}^6$, $j = (j^{\mathbb{E}}, j^{\mathbb{H}})$, is the exterior source,

- $u:(0,T)\times\Omega\to\mathbb{R}^6$, $u=(u_1,u_2)$, $u_1=\mathbb{E}$, $u_2=\mathbb{H}$, is the electromagnetic field,
- M is the Maxwell operator

$$M: v=(v_1,v_2)\in L^2(\Omega;\mathbb{R}^6)\to Mv=\begin{pmatrix}\operatorname{curl} v_2\\ -\operatorname{curl} v_1\end{pmatrix},$$

- L is the operator associated to the most general constitutive law,

$$Lu(t,x)=A(t,x)u(t,x)+\int_0^t G(t,s,x)u(s,x)\,\mathrm{d}s,$$

  where A, G are 6×6 matrices, not necessarily diagonal.

Hence the evolution problem reads:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\big(A(t,x)u(t,x)+\int_0^t G(t,s,x)u(s,x)\,\mathrm{d}s\big)=Mu(t)-j(t) & \text{in } (0,T)\times\Omega,\\ u(0,x)=u^0(x) & \text{in } \Omega,\\ n(x)\times u_1(t,x)=0 & \text{on } (0,T)\times\partial\Omega.\end{cases}\tag{1}$$

Let us remark that matrix A contains the permittivity and permeability data and matrix G, when it is under the form $G(t,s,x)=B(s,x)+C(t-s,x)$, takes into account conduction effects via matrix B and memory effects via matrix C.

The aim of this paper is to study the behavior of the electromagnetic field, solution to problem (1), when the structure (that occupies the domain Ω) presents periodic microstructures leading to matrices (A,G) with oscillatory coefficients (with respect to the space variable x). When the data A depends upon the time variable t we address this question in the framework of a *time-domain* formulation. In the more classical situation of time-independent data we consider the *frequency-domain* formulation.

2. Time-dependent formulation

2.1. Evolution problem

Let us state the appropriate functional framework to problem (1) by introducing some definitions and assumptions. We denote:

- $H(\operatorname{curl},\Omega)=\{v\in L^2(\Omega;\mathbb{R}^3);\ \operatorname{curl} v\in L^2(\Omega;\mathbb{R}^3)\}$ equipped with the norm $\|v\|^2=|v|^2+|\operatorname{curl} v|^2$,
- $H_0(\operatorname{curl},\Omega)=\{v\in H(\operatorname{curl},\Omega);\ n(x)\times v(x)=0 \text{ in } H^{-1/2}(\partial\Omega;\mathbb{R}^3)\}$,
- $H(\operatorname{curl},\operatorname{div},\Omega)=\{v\in H(\operatorname{curl},\Omega);\ \operatorname{div} v\in L^2(\Omega)\}$,
- $H(\operatorname{curl},\operatorname{div}0,\Omega)=\{v\in H(\operatorname{curl},\operatorname{div},\Omega);\ \operatorname{div} v=0\}$,
- $V(\Omega)=H_0(\operatorname{curl},\Omega)\times H(\operatorname{curl},\Omega)$,

- $\mathcal{S}$ and D the following triangle and line,

$$\mathcal{S} = \{(t,s) \in [0,T]^2;\ s \leqslant t\}, \qquad D = \{(t,t);\ t \in]0,T[\,\},$$

- $\mathcal{W}^{1,p}(\mathcal{S};X) = \{\phi \in L^p(\mathcal{S};X);\ \frac{\partial\phi}{\partial t} \in L^p(\mathcal{S};X),\ \phi|_D \in L^p(0,T;X)\}$,
- $\mathcal{W}^{2,p}(\mathcal{S};X) = \{\phi \in \mathcal{W}^{1,p}(\mathcal{S};X);\ \frac{\partial\phi}{\partial t} \in \mathcal{W}^{1,p}(\mathcal{S};X),\ \phi|_D \in W^{1,p}(0,T;X)\}$, where X is a Banach space, the spaces $\mathcal{W}^{1,p}(\mathcal{S};X)$ and $\mathcal{W}^{2,p}(\mathcal{S};X)$ are respectively equipped with the norms:

$$\|\phi\|_{1,p;X} = \left\|\frac{\partial\phi}{\partial t}\right\|_{L^p(\mathcal{S};X)} + \|\phi|_D\|_{L^p(0,T;X)},$$

$$\|\phi\|_{2,p;X} = \left\|\frac{\partial\phi}{\partial t}\right\|_{1,p;X} + \|\phi|_D\|_{W^{1,p}(0,T;X)}.$$

In the sequel we assume that matrix A is *symmetric and uniformly coercive*, i.e., there exists $\gamma > 0$ such that for all vector fields $\phi \in \mathbb{R}^6$,

$$\forall (t,x) \in [0,T] \times \Omega, \quad {}^t\phi A(t,x)\phi \geqslant \gamma|\phi|^2.$$

Hence we can establish the existence and uniqueness of the solution to problem (1).

Proposition 1. *Let $A \in W^{2,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ and $G \in \mathcal{W}^{2,1}(\mathcal{S};L^\infty(\Omega;\mathbb{R}^{36}))$ be two matrices. With the initial condition $u^0 \in V(\Omega)$ and the exterior source $j \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))$. The problem,*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(A(t,x)u(t,x) + \int_0^t G(t,s,x)u(s)\,\mathrm{d}s\right) = Mu(t,x) - j(t,x) & \textit{in } (0,T)\times\Omega, \\ u(0,x) = u^0(x) \quad \textit{in } \Omega, \\ n(x) \times u_1(t,x) = 0 \quad \textit{on } (0,T)\times\partial\Omega, \end{cases}$$

has a unique solution $u \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^6)) \cap L^\infty(0,T;V(\Omega))$ that satisfies the following bounds:

$$\|u\|_{L^\infty(0,T;V(\Omega))} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))} \leqslant c\left(\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \left\|u^0\right\|_{V(\Omega)}\right).$$

The regularity of the solution is therefore,

$$u \in C^1\left([0,T];L^2\left(\Omega;\mathbb{R}^6\right)\right) \cap C^0\left([0,T];V(\Omega)\right).$$

Remark. It is important to note that when the matrix G reads $G(t,s) = B(s) + C(t-s)$ with $B \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$, $C \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$, the solution u of problem (1) satisfies the more precise bound:

$$\forall t \in [0, T], \quad \|u\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^6))} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^6))}$$
$$\leqslant c_0 \mathrm{e}^{c_1 t}\big\{\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \|u^0\|_{V(\Omega)}\big\}.$$

This will allow, in Section 3, the study of the frequency formulation.

2.2. Conservation law

In the case of "perfect" media with constitutive law $\mathbb{D} = \varepsilon\mathbb{E}$, $\mathbb{B} = \mu\mathbb{H}$, the electromagnetic energy density defined by $\mathcal{E}(t,x) = (\mathbb{E}(t,x)\cdot\mathbb{D}(t,x) + \mathbb{H}(t,x)\cdot\mathbb{B}(t,x))/2$ satisfies the evolution law:

$$\frac{\partial\mathcal{E}}{\partial t} + (\operatorname{curl}\mathbb{E}\cdot\mathbb{H} - \mathbb{E}\cdot\operatorname{curl}\mathbb{H}) = -\mathbb{E}\cdot j^{\mathbb{E}}.$$

This yields, when the current density $j^{\mathbb{E}}$ vanishes, the conservation law:

$$\int_\Omega \frac{\partial\mathcal{E}}{\partial t}(t,x)\,\mathrm{d}x = 0, \qquad \int_\Omega \mathcal{E}(t,x)\,\mathrm{d}x = \int_\Omega \mathcal{E}(0,x)\,\mathrm{d}x.$$

In the framework of this study, which takes into account more complexity in the constitutive law, we can still write a *conservation law* as follows (see Remark (i) of the proof of Proposition 1):

$$\frac{1}{2}\int_\Omega A(t)u(t)\cdot u(t) + \frac{1}{2}\int_0^t\int_\Omega \frac{\mathrm{d}A}{\mathrm{d}t}(s)u(s)\cdot u(s)\,\mathrm{d}s + \int_0^t\int_0^{t_1}\int_\Omega \frac{\partial G}{\partial t}(t_1,s)u(s)\cdot u(t_1)\,\mathrm{d}s\,\mathrm{d}t_1$$
$$+ \int_0^t\int_\Omega G(s,s)u(s)\cdot u(s)\,\mathrm{d}s + \int_0^t\int_\Omega j(s)\cdot u(s)\,\mathrm{d}s = \frac{1}{2}\int_\Omega A(0)u^0\cdot u^0. \tag{2}$$

2.3. Homogenization

Now we consider a material whose periodic structure is characterized by an *elementary microstructure* with size $\alpha > 0$ supposed to go to zero (the "small parameter" generally denoted by ε in the literature, is denoted here by α to avoid any confusion with permittivity ε). The constitutive parameters (A^α, G^α) and the data $(u^{\alpha,0}, j^\alpha)$ that depend on α, are supposed to have the regularity needed by Proposition 1. Hence, for $\alpha > 0$, we obtain a family of electromagnetic fields u^α (indexed by α) solutions to evolution problems:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(A^\alpha(t,x)u^\alpha(t,x) + \int_0^t G^\alpha(t,s,x)u^\alpha(s,x)\,\mathrm{d}s\big) \\ \quad = Mu^\alpha(t,x) - j^\alpha(t,x) \quad \text{in } (0,T)\times\Omega, \\ u^\alpha(0,x) = u^{\alpha,0}(x) \quad \text{in } \Omega, \\ n(x)\times u_1^\alpha(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{3}$$

This paper aims at finding the asymptotic behavior of the solution u^α when α goes to zero; for doing that, we need to precise the asymptotic behavior of the initial data and of the source since they depend upon α. This is done below under the following assumptions (with strong convergences in both cases):

$$\begin{cases} u^{\alpha,0} \to u^0 & \text{in } V(\Omega), \\ j^\alpha \to j & \text{in } W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6)). \end{cases} \tag{4}$$

2.3.1. Periodic geometry

In this section, for the sake of simplicity, we restrict the microstructure to be of *cubic form* and we denote by $Y = [0,1[^3$ the reference cell. For all $z \in \mathbb{R}^3$ let $[z]$ be the unique integer such that $z - [z] \in Y$, so that we may write $z = [z] + \{z\}$ for all $z \in \mathbb{R}^3$. Consequently, for all $\alpha > 0$, we get the unique decomposition:

$$x = \alpha\left(\left[\frac{x}{\alpha}\right] + \left\{\frac{x}{\alpha}\right\}\right) \quad \forall x \in \mathbb{R}^3.$$

To be consistent with this geometry, the constitutive parameters (A^α, G^α) are assumed to be *periodic with period* α; more precisely we assume that according to the previous decomposition there exists two matrices (A, G) such that

$$A^\alpha(t,x) = A\left(t, \left\{\frac{x}{\alpha}\right\}\right), \qquad G^\alpha(t,s,x) = G\left(t,s,\left\{\frac{x}{\alpha}\right\}\right). \tag{5}$$

2.3.2. Periodic unfolding operator

We study the limit, when α goes to 0, of the family u^α, by using *the periodic unfolding method* [4]. The periodic unfolding operator $\mathcal{T}_\alpha : v \in L^2(\Omega;\mathbb{R}) \to L^2(\Omega \times Y;\mathbb{R})$ is defined by:

$$\mathcal{T}_\alpha(v)(x,y) = v\left(\alpha\left[\frac{x}{\alpha}\right] + \alpha y\right), \quad x \in \Omega,\ y \in Y.$$

Hence the periodicity of the constitutive parameters yields,

$$\mathcal{T}_\alpha(A^\alpha)(t,x,y) = A(t,y), \quad \mathcal{T}_\alpha(G^\alpha)(t,s,x,y) = G(t,s,y) \quad \text{a.e. in }]0,T[\times\Omega\times Y.$$

For our purpose all functions defined in $L^2(\Omega)$ are extended by 0 outside Ω and we denote by $H^1_{\text{per}}(Y)$ the space of periodic functions with vanishing mean value.

Some properties of the operator $\mathcal{T}_\alpha$, essential in the sequel, are stated in the next theorem.

Theorem 2. (i) *For all $v \in L^2(\Omega)$, we have the strong convergence*:

$$\mathcal{T}_\alpha(v) \to v \quad \textit{in } L^2(\Omega \times Y).$$

(ii) *Let v^α be a family of functions uniformly bounded in $L^2(\Omega)$. There exists $v \in L^2(\Omega \times Y)$ such that, up to a subsequence, we have the weak convergence*:

$$\mathcal{T}_\alpha(v^\alpha) \rightharpoonup v \quad \text{in } \in L^2(\Omega \times Y).$$

(iii) *Let v^α be a family of functions uniformly bounded in $H(\mathrm{curl}, \Omega)$. There exists three fields,*

$$v \in H(\mathrm{curl}, \Omega), \qquad \bar{v} \in L^2\big(\Omega, H^1_{\mathrm{per}}(Y; \mathbb{R})\big),$$
$$\bar{\bar{v}} \in L^2\big(\Omega, H^1_{\mathrm{per}}\big(Y; \mathbb{R}^3\big)\big), \quad \mathrm{div}_y\big(\bar{\bar{v}}\big) = 0,$$

such that, up to a subsequence, we have the weak convergences:

$$\begin{cases} v^\alpha \rightharpoonup v & \text{in } H(\mathrm{curl}, \Omega), \\ \mathcal{T}_\alpha(v^\alpha) \rightharpoonup v + \nabla_y \bar{v} & \text{in } L^2(\Omega \times Y; \mathbb{R}^3), \\ \mathcal{T}_\alpha(\mathrm{curl}\, v^\alpha) \rightharpoonup \mathrm{curl}_x\, v + \mathrm{curl}_y\, \bar{\bar{v}} & \text{in } L^2(\Omega \times Y; \mathbb{R}^3). \end{cases}$$

Notation. We extend to $\mathbb{R}^6$ some notation defined in $\mathbb{R}^3$.

Let $v_1, v_2 \in \mathbb{R}^3$, $v = (v_1, v_2)$, $w_1, w_2 \in \mathbb{R}$, $w = (w_1, w_2)$, $n \in \mathbb{R}^3$, then

$$\mathrm{curl}\, v := (\mathrm{curl}\, v_1, \mathrm{curl}\, v_2), \qquad \mathrm{div}\, v := (\mathrm{div}\, v_1, \mathrm{div}\, v_2),$$
$$n \times v := (n \times v_1, n \times v_2), \qquad \mathrm{grad}\, w := (\mathrm{grad}\, w_1, \mathrm{grad}\, w_2).$$

For $\alpha > 0$ let u^α be the solution to (3); as established in Proposition 1, it satisfies the uniform bound:

$$\|u^\alpha\|_{L^\infty(0,T;V(\Omega))} + \left\| \frac{\mathrm{d}u^\alpha}{\mathrm{d}t} \right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))} \leqslant c\big(\|j^\alpha\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \|u^{\alpha,0}\|_{V(\Omega)}\big),$$

therefore we are in position to prove the convergence of the family u^α and to identify its limit.

Theorem 3. *Let $A^\alpha \in W^{2,1}(0, T; L^\infty(\Omega; \mathbb{R}^{36}))$ and $G^\alpha \in \mathcal{W}^{2,1}(\mathcal{S}; L^\infty(\Omega; \mathbb{R}^{36}))$, be two matrices given by* (5). *With the initial condition $u^{\alpha,0}$ and the source j^α satisfying assumptions* (4)*, let u^α be the solution to* (3). *There exists three fields,*

$$\begin{cases} u \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^6)) \cap L^\infty(0, T; V(\Omega)), \\ \bar{u} \in W^{1,\infty}(0, T; L^2(\Omega; H^1_{\mathrm{per}}(Y; \mathbb{R}^2))), \\ \bar{\bar{u}} \in L^\infty(0, T; L^2(\Omega; H^1_{\mathrm{per}}(Y; \mathbb{R}^6))),\ \mathrm{div}_y(\bar{\bar{u}}) = 0, \end{cases}$$

(i) *which are limits defined as follows*:

$$\begin{cases} u^\alpha \rightharpoonup u & \textit{weakly * in } L^\infty(0,T;V(\Omega)),\\ \mathcal{T}_\alpha(u^\alpha)\to u+\nabla_y\bar{u} & \textit{strongly in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^6)),\\ \mathcal{T}_\alpha(\operatorname{curl}_x u^\alpha)\to \operatorname{curl}_x u+\operatorname{curl}_y\bar{\bar{u}} & \textit{strongly in } L^2(\,]0,T[\,\times\Omega\times Y;\mathbb{R}^6); \end{cases}$$

(ii) *which solve the evolution problem*:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(A(t,y)(u(t,x)+\nabla_y\bar{u}(t,x,y)) + \int_0^t G(t,s,y)(u(s,x)+\nabla_y\bar{u}(s,x,y))\,\mathrm{d}s\big)\\ \quad = M_x u(t,x)+M_y\bar{\bar{u}}(t,x,y)-j(t,x) \quad \textit{in } (0,T)\times\Omega\times Y,\\ u(0)+\nabla_y\bar{u}(0)=u^0 \quad \textit{in } \Omega\times Y,\\ n\times u_1=0 \quad \textit{on } (0,T)\times\partial\Omega. \end{cases} \tag{6}$$

Problem (6) has a unique solution $u,\bar{u},\bar{\bar{u}}$, the field $u+\nabla_y\bar{u}$ and its derivative with respect to time satisfy a conservation law analogous as (2):

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\times Y} A(t,y)\big(u(t,x)+\nabla_y\bar{u}(t,x,y)\big)\cdot\big(u(t,x)+\nabla_y\bar{u}(t,x,y)\big)\\ &\quad+\frac{1}{2}\int_0^t\int_{\Omega\times Y}\frac{\mathrm{d}A}{\mathrm{d}t}(s,y)\big(u(s,x)+\nabla_y\bar{u}(s,x,y)\big)\cdot\big(u(s,x)+\nabla_y\bar{u}(s,x,y)\big)\,\mathrm{d}s\\ &\quad+\int_0^t\int_0^{t_1}\int_{\Omega\times Y}\frac{\partial G}{\partial t}(t_1,s)\big(u(s,x)+\nabla_y\bar{u}(s,x,y)\big)\cdot\big(u(t_1,x)+\nabla_y\bar{u}(t_1,x,y)\big)\,\mathrm{d}s\,\mathrm{d}t_1\\ &\quad+\int_0^t\int_{\Omega\times Y} G(s,s)\big(u(s,x)+\nabla_y\bar{u}(s,x,y)\big)\cdot\big(u(s,x)+\nabla_y\bar{u}(s,x,y)\big)\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega j(s)\cdot u(s)\,\mathrm{d}s=\frac{1}{2}\int_\Omega A(0,y)u^0(x)\cdot u^0(x). \end{aligned}$$

2.4. Limit model

In this section we show that the *limit solution* u given by (6) solves a *global Maxwell problem* posed in $(0,T)\times\Omega$, while the correctors $\bar{u}$ and $\bar{\bar{u}}$ solve *local diffusion problems* posed in $(0,T)\times Y$.

Theorem 4. *Let* $A^\alpha\in W^{2,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ *and* $G^\alpha\in\mathcal{W}^{2,1}(\mathcal{S};L^\infty(\Omega;\mathbb{R}^{36}))$ *be two matrices given by* (5). *With the initial condition* $u^{\alpha,0}$ *and the source* j^α *satisfying assumptions* (4). *There exists a unique limit electromagnetic field,*

$$u = (\mathbb{E}, \mathbb{H}) \in W^{1,\infty}\big(0, T; L^2(\Omega)\big) \cap L^\infty\big(0, T; V(\Omega)\big),$$

solution to the homogenized problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{A}(t)u(t,x) + \int_0^t \mathcal{G}(t,s)u(s,x)\,\mathrm{d}s\big) \\ \quad = Mu(t,x) - j(t,x) - \mathcal{J}^0(t,x) \quad \text{in } (0,T)\times\Omega, \\ u(0) = u^0 \quad \text{in } \Omega, \\ n \times u_1 = 0 \quad \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{7}$$

where $\mathcal{A}, \mathcal{G}$ *are two matrices independent of the space variable* x, $\mathcal{J}^0$ *is an extra source which depends only upon the initial condition* u^0 (*their expressions and regularity are given in* (A.5)–(A.6) *below*).

The homogenized constitutive law which expresses the limit electric displacement $\mathbb{D}$ and the limit magnetic induction $\mathbb{B}$ in terms of the electromagnetic fields $\mathbb{E}$ and $\mathbb{H}$, $(\mathbb{D}, \mathbb{B}) = \mathcal{L}(\mathbb{E}, \mathbb{H})$, takes the form:

$$\mathcal{L}u(t,x) = \mathcal{A}(t)u(t,x) + \int_0^t \mathcal{G}(t,s)u(s,x)\,\mathrm{d}s.$$

The computation of u relies on the first corrector $\bar{u}$ whose expression is given in (A.5). The expression of the second corrector $\bar{\bar{u}}$ is given in (A.8) below.

Remarks. (i) Let us first consider the most general physical situation which corresponds to a matrix G^α that reads $G^\alpha(t,s) = B^\alpha(s) + C^\alpha(t-s)$, with an obvious definition of associated matrices B, C. In this case the initial constitutive law has a *convolution* term

$$L^\alpha u^\alpha(t,x) = A^\alpha(t,x)u^\alpha(t,x) + \int_0^t B^\alpha(s,x)u^\alpha(s,x)\,\mathrm{d}s + \int_0^t C^\alpha(t-s,x)u^\alpha(s,x)\,\mathrm{d}s$$

and the limit constitutive law takes the form:

$$\mathcal{L}u(t,x) = \mathcal{A}(t)u(t,x) + \int_0^t \mathcal{B}(s)u(s,x)\,\mathrm{d}s + \int_0^t \mathcal{C}(t,s)u(s,x)\,\mathrm{d}s,$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are given in (A.9). The homogenized kernel $\mathcal{C}$ depends upon two variables, hence it *does not take the convolution form*! Moreover it is not obvious to give a physical meaning (in terms of electric or magnetic susceptibility) to the homogenized matrix $\mathcal{C}$.

(ii) However, when A^α and B^α *do not depend* on t we show that the new homogenized constitutive law is of the convolution form, i.e., $\mathcal{C}$ depends upon only *one* variable,

$$\mathcal{L}u(t,x) = \mathcal{A}u(t,x) + \mathcal{B}\int_0^t u(s,x)\,\mathrm{d}s + \int_0^t \mathcal{C}(t-s)u(s,x)\,\mathrm{d}s,$$

the expression of $\mathcal{C}$ is given in (18).

(iii) In the case where neither A^α nor B^α *depend* upon t and *no memory effects* are taken into account, i.e., $C^\alpha = 0$, we recover the classical results (even though slightly different from [8]).

(iv) *Strong convergence.* Theorem 3 suggests the formal asymptotic expansion:

$$u^\alpha(x) = u(x) + \nabla_y \bar{u}\left(x, \frac{x}{\alpha}\right) + \alpha \bar{\bar{u}}\left(x, \frac{x}{\alpha}\right)\cdots.$$

Hence the computation of the term of order 0 (with respect to α) has to take into account the first corrector $\bar{u}$. Under assumptions of Theorem 3 and by following the same approach as in [4], we obtain the strong convergences of the electromagnetic field:

$$\begin{cases} u^\alpha - (u + \mathcal{U}_\alpha(\nabla_y \bar{u})) \to 0 & \text{in } H^1(0,T; L^2(\Omega; \mathbb{R}^6)), \\ \mathrm{curl}_x\, u^\alpha - (\mathrm{curl}_x\, u + \mathcal{U}_\alpha(\mathrm{curl}_y\, \bar{\bar{u}})) \to 0 & \text{in } L^2(\,]0,T[\,\times \Omega; \mathbb{R}^6), \end{cases}$$

where $\mathcal{U}_\alpha$ is the averaging operator:

$$\mathcal{U}_\alpha(v)(x) = \frac{1}{|Y|}\int_Y v\left(\alpha\left[\frac{x}{\alpha}\right] + \alpha z, \left\{\frac{x}{\alpha}\right\}\right)\mathrm{d}z, \quad \forall v \in L^2(\Omega \times Y).$$

3. Frequency formulation

3.1. Stationary Maxwell equations

In order to study the *stationary problem* we assume, in this section, that the constitutive law reads:

$$L^\alpha u^\alpha(t,x) = A^\alpha(x)u^\alpha(t,x) + \int_0^t B^\alpha(x)u^\alpha(s,x)\,\mathrm{d}s + \int_0^t C^\alpha(t-s,x)u^\alpha(s,x)\,\mathrm{d}s,$$

where matrices A^α and B^α are independent of t; in other words the permittivity, the permeability and the electric conductivity of the material only depends upon the space variable. We denote by $\hat{v}(p) = \int_0^\infty \mathrm{e}^{-pt} v(t)\,\mathrm{d}t$ the Laplace transform of any function $v \in L^1(0,\infty)$. According to the bounds given in Proposition 1, there exists a constant $p_0 > 0$ that depends only on the data such that the solution u^α to problem (3) has a Laplace transform for all $p \geqslant p_0$. Therefore, we may consider the following equilibrium equation: For all $p \geqslant p_0$:

$$\begin{cases} (pA^\alpha(x) + B^\alpha(x) + p\widehat{C}^\alpha(p,x))\hat{u}^\alpha(p,x) \\ \quad = M\hat{u}^\alpha(p,x) - \hat{j}^\alpha(p,x) + A^\alpha(x)u^{\alpha,0}(x) \quad \text{in } \Omega, \\ n \times \hat{u}_1^\alpha = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{8}$$

and the associated constitutive law is:

$$\left(\widehat{\mathbb{D}}^\alpha(p,x),\widehat{\mathbb{B}}^\alpha(p,x)\right)=\left(A^\alpha(x)+\frac{1}{p}B^\alpha(x)+\widehat{C}^\alpha(p,x)\right)\hat{u}^\alpha(p,x).$$

With the same kind of assumptions on the data as those needed for the evolution problem, we can state an existence and uniqueness result analogous to that of Proposition 1.

Proposition 5. *Let $A^\alpha\in L^\infty(\Omega;\mathbb{R}^{36})$, $B^\alpha\in L^\infty(\Omega;\mathbb{R}^{36})$, $C^\alpha\in W^{1,1}(\mathbb{R}^+;L^\infty(\Omega;\mathbb{R}^{36}))$ be three matrices, let $u^{\alpha,0}\in V(\Omega)$ be the initial condition and $j^\alpha\in W^{1,1}(\mathbb{R}^+;L^\infty(\Omega;\mathbb{R}^6))$ be the source. There exists $p_0>0$ such that problem* (8) *has a unique solution $\hat{u}^\alpha=(\widehat{\mathbb{E}}^\alpha,\widehat{\mathbb{H}}^\alpha)\in L^\infty(p_0,\infty;V(\Omega))$. This solution satisfies the uniform bounds*:

$$\left\|\hat{u}^\alpha(p)\right\|_{L^2(\Omega)}\leqslant\frac{c}{p},\qquad\left\|\operatorname{curl}\hat{u}^\alpha(p)\right\|_{L^2(\Omega)}\leqslant c\quad\forall p>p_0.$$

The estimate of the solution and the strong convergence (4) yield the existence of a limit electromagnetic field $\hat{u}$ and of two fields of correctors $\underline{u},\underline{\underline{u}}$ which are given in Lemma 6 and Theorem 7.

3.2. Homogenization and limit model

We use the same notations as in Section 2.2.1 and assume that there exists three matrices A,B,C such that

$$A^\alpha(x)=A\left(\left\{\frac{x}{\alpha}\right\}\right),\qquad B^\alpha(x)=B\left(\left\{\frac{x}{\alpha}\right\}\right),\qquad C^\alpha(p,x)=C\left(p,\left\{\frac{x}{\alpha}\right\}\right).\tag{9}$$

Hence we can establish, with an approach analogous to the time formulation, the convergence of the sequence $\hat{u}^\alpha$.

Lemma 6. *Let $A^\alpha\in L^\infty(\Omega;\mathbb{R}^{36})$, $B^\alpha\in L^\infty(\Omega;\mathbb{R}^{36})$, $C^\alpha\in W^{1,1}(\mathbb{R}^+;L^\infty(\Omega;\mathbb{R}^{36}))$ be three matrices given by* (9)*. With the initial condition $u^{\alpha,0}$ and the source j^α satisfying assumptions* (4)*, let $\hat{u}^\alpha$ be the solution to* (8)*. There exists three fields*:

$$\begin{cases}\hat{u}\in L^\infty(p_0,\infty;V(\Omega)),\\ \underline{u}\in L^\infty(p_0,\infty;L^2(\Omega;H^1_{\text{per}}(Y;\mathbb{R}^2))),\\ \underline{\underline{u}}\in L^\infty(p_0,\infty;L^2(\Omega;H^1_{\text{per}}(Y;\mathbb{R}^6))),\ \operatorname{div}_y(\underline{\underline{u}})=0,\end{cases}$$

(i) *which are limits defined as follows*:

$$\begin{cases}\hat{u}^\alpha\rightharpoonup\hat{u} & \textit{weakly} * \textit{in } L^\infty(p_0,\infty;V(\Omega)),\\ \mathcal{T}_\alpha(\hat{u}^\alpha)\to\hat{u}+\nabla_y\underline{u} & \textit{strongly in } L^2(p_0,\infty\times\Omega\times Y;\mathbb{R}^6),\\ \mathcal{T}_\alpha(\operatorname{curl}_x\hat{u}^\alpha)(p)\to\operatorname{curl}_x\hat{u}(p)+\operatorname{curl}_y\underline{\underline{u}}(p) & \textit{strongly in } L^2(\Omega\times Y;\mathbb{R}^6),\end{cases}$$

(ii) *which solve the evolution problem*:

$$\begin{cases} (pA(y) + B(y) + p\widehat{C}(p)(y))(\hat{u}(p,x) + \nabla_y \underline{u}(p,x,y)) \\ \quad = M_x \hat{u}(p,x) + M_y \underline{\underline{u}}(p,x,y) - \hat{j}(p,x) + A(y)u^0(x) \quad \text{in } \Omega \times Y,\ \forall p > p_0, \\ n \times \hat{u}_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In the next theorem we give the expression of the homogenized equilibrium equations.

Theorem 7. *Let* $A^\alpha \in L^\infty(\Omega;\mathbb{R}^{36})$, $B^\alpha \in L^\infty(\Omega;\mathbb{R}^{36})$, $C^\alpha \in W^{1,1}(\mathbb{R}^+;L^\infty(\Omega;\mathbb{R}^{36}))$ *be three matrices given by* (9), *with the initial condition* $u^{\alpha,0}$ *and the source* j^α *satisfying assumptions* (4). *There exists a unique limit electromagnetic field* $\hat{u} \in L^\infty(p_0,\infty;V(\Omega))$ *solution to the homogenized problem*:

$$\begin{cases} (p\mathcal{A} + \mathcal{B} + p\widehat{\mathcal{C}}(p))\hat{u}(p,x) \\ \quad = M\,\hat{u}(p,x) - \hat{j}(p,x) - \widehat{\mathcal{J}}^0(p,x) + \int_Y A(y)u^0(x) \quad \text{in } \Omega,\ \forall p > p_0, \\ n \times \hat{u}_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The homogenized effective *matrices* $\mathcal{A},\mathcal{B},\widehat{\mathcal{C}}$ *and the extra source* $\widehat{\mathcal{J}}^0$ *are described in* (A.12) *below.*

We note that in accordance with Remark (ii) in Section 2.4, the limiting constitutive law takes in this case, the same form as the initial one

$$\big(\widehat{\mathbb{D}}(p,x), \widehat{\mathbb{B}}(p,x)\big) = \left(\mathcal{A} + \frac{1}{p}\mathcal{B} + \widehat{\mathcal{C}}(p)\right)\hat{u}(p,x).$$

The corrector $\underline{u}$ is given in (A.11), the second corrector $\underline{\underline{u}}$ can be computed as previously in terms of $\hat{u}$. It is easy to check that $\underline{u}$, $\underline{\underline{u}}$, $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{J}}^0$ are the Laplace transforms of respectively, $\bar{u}$, $\bar{\bar{u}}$, $\mathcal{C}$ and $\mathcal{J}^0$.

4. Comments

Let us conclude this paper with some remarks.

(i) *Stability of the constitutive law.* When the initial constitutive law is of form,

$$Lu(t,x) = A(t,x)u(t,x) + \int_0^t B(s,x)u(s,x)\,\mathrm{d}s + \int_0^t C(t-s,x)u(s,x)\,\mathrm{d}s,$$

we have established that the corresponding homogenized one is:

$$\mathcal{L}u(t,x) = \mathcal{A}(t)u(t,x) + \int_0^t \mathcal{B}(s)u(s,x)\,\mathrm{d}s + \int_0^t \mathcal{C}(t,s)u(s,x)\,\mathrm{d}s,$$

where the initial convolution function C has been replaced by the Hilbert–Schmidt kernel $\mathcal{C}$. However Theorem 4, states that under some regularity on the data, the previous form is *stable by the homogenization procedure*. This allows multi-step scalings without need of new mathematical tools.

(ii) *Other geometries.* A straightforward extension is given by replacing cubic elementary cells by any cells having the $\mathbb{R}^3$ *paving property*. Another extension can be obtained with more general assumptions on the data. For example, let us assume that there exists two matrices $A \in W^{2,1}(0,T; L^\infty(\Omega \times Y; \mathbb{R}^{36}))$, $G \in \mathcal{W}^{2,1}(\mathcal{S}; L^\infty(\Omega \times Y; \mathbb{R}^{36}))$ which satisfy the strong convergences:

$$\mathcal{T}_\alpha(A^\alpha)(t,x,y) \to A(t,x,y),$$
$$\mathcal{T}_\alpha(G^\alpha)(t,x,y) \to G(t,x,y) \quad \text{a.e. in }]0,T[\times\Omega\times Y.$$

This yields a limiting constitutive law of the same form but in which *the global variable x still appears*. It is of interest to note that in addition to the classical situations [4], an easy way to get these convergences is obtained by letting:

$$A^\alpha(t,x) = \mathcal{U}_\alpha(A)(t,x), \qquad G^\alpha(t,s,x) = \mathcal{U}_\alpha(G)(t,s,x).$$

(iii) *Weak convergences.* Weak convergences instead of strong ones (4), would have led to small changes by adding extra sources in the right-hand side of the equilibrium equations.

Appendix A

In this section we give the complete proofs of our results.

A.1. Proof of Proposition 1

We begin with a preliminary lemma whose proof is based on Fredholm theory for Volterra equation.

Lemma 1.1. *Let $m \in \mathbb{N}^*$, $\tilde{A} \in W^{r,1}(0,T;\mathbb{R}^{m^2})$, $r \in \{1,2\}$, be a symmetric uniformly coercive matrix, $\widetilde{G} \in \mathcal{W}^{r,1}(\mathcal{S};\mathbb{R}^{m^2})$ be a square matrix and $\tilde{j} \in W^{r,1}(0,T;\mathbb{R}^m)$. There exists a unique function $g \in W^{r,1}(0,T;\mathbb{R}^m)$ such that*

$$\tilde{A}(t)g(t) + \int_0^t \widetilde{G}(t,s)g(s)\,\mathrm{d}s = \tilde{j}(t) \quad \textit{for all } t \textit{ in } [0,T].$$

The proof of Proposition 1 derives from Faedo–Galerkin method; for clearness it is broken into 3 steps.

First step. Existence of an approximate solution u_m of u. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m, \ldots$ be a sequence of linearly independent elements in $V(\Omega)$ whose linear finite combinations are dense in $V(\Omega)$, $V_m = \mathrm{Vect}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m)$. The scalar coefficients h_l^m of an approximation $u_m(t) = \sum_{l=1}^m h_l^m(t)\boldsymbol{e}_l$ of the solution to problem (1) are solution to the $m \times m$ system:

$$\int_\Omega \left(A(t)u_m(t) + \int_0^t \big(G(t,s)u_m(s) - Mu_m(s)\big)\,\mathrm{d}s \right) \cdot \boldsymbol{e}_k = \int_\Omega \left(A(0)u_m(0) - \int_0^t j(s)\,\mathrm{d}s \right) \cdot \boldsymbol{e}_k, \tag{A.1}$$

where $u_m(0) = \sum_{l=1}^m \alpha_l^m \boldsymbol{e}_l$ is the orthogonal projection of $u(0)$ on V_m for the scalar product of $V(\Omega)$. The existence of $u_m \in W^{2,1}(0,T;V_m)$ is a direct consequence of Lemma 1.1 with an obvious definition of $\tilde{A}$, $\widetilde{G}$, $\tilde{j}$ in terms of A, G, j.

Second step. Estimates. We differentiate (A.1) with respect to t, then we multiply it by $h_l^m(t)$ and add over l. This yields

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)u_m(t) + \int_0^t G(t,s)u_m(s)\,\mathrm{d}s \right) \cdot u_m(t) = \int_\Omega j(t) \cdot u_m(t).$$

We integrate this equality over $[0,t]$ and get the estimate:

$$\int_\Omega A(t)u_m(t).u_m(t) \leqslant \int_\Omega A(0)u_m(0).u_m(0) + \int_0^t \Omega(s)U_m^2(s)\,\mathrm{d}s + \|j\|_{L^1(0,t;L^2(\Omega;\mathbb{R}^6))} U_m(t),$$

where $U_m(t) = \sup_{s\in[0,t]} \|u_m(s)\|_{L^2(\Omega;\mathbb{R}^6)}$ and $\Theta \in L^1(0,T)$ is given by:

$$\Theta(t) = \frac{1}{2}\left\| \frac{\mathrm{d}A}{\mathrm{d}t}(t) \right\|_{L^\infty(\Omega;\mathbb{R}^{36})} + \big\|G(t,t)\big\|_{L^\infty(\Omega;\mathbb{R}^{36})} + \int_0^t \left\| \frac{\partial G}{\partial t}(t,s) \right\|_{L^\infty(\Omega;\mathbb{R}^{36})} \mathrm{d}s.$$

With Gronwall lemma we get:

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))} \leqslant c\big\{ \|j\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^6))} + \big\|u(0)\big\|_{V(\Omega)} \big\}.$$

The constant c only depends upon $\|A\|_{W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}$, $\|A^{-1}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}$ and $\|G\|_{1,1,L^\infty(\Omega;\mathbb{R}^{36})}$. To estimate $\frac{\mathrm{d}u_m}{\mathrm{d}t}$, we differentiate (10), and let $t=0$, hence

$$\left\|\frac{\mathrm{d}u_m}{\mathrm{d}t}(0)\right\|_{L^2(\Omega;\mathbb{R}^6)} \leqslant c\left\{\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \left\|u(0)\right\|_{V(\Omega)}\right\}.$$

Finally, we differentiate Eq. (A.1) twice, multiply by $\frac{\mathrm{d}h_l^m}{\mathrm{d}t}$, sum over l and integrate over $[0,t]$ to get:

$$\left\|\frac{\mathrm{d}u_m}{\mathrm{d}t}(t)\right\|_{L^2(\Omega;\mathbb{R}^6)} \leqslant c\left\{\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \left\|u(0)\right\|_{V(\Omega)}\right\}.$$

The constant c only depends upon $\|A\|_{W^{2,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}$, $\|A^{-1}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}$ and $\|G\|_{2,1;L^\infty(\Omega;\mathbb{R}^{36})}$.

Third step. Existence of a solution. Sequences u_m and $\frac{\mathrm{d}u_m}{\mathrm{d}t}$ are bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))$. We extract subsequences, still denoted in the same way, and get the weak$*$ convergences:

$$\begin{cases} u_m \rightharpoonup u & \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^6)),\\ \frac{\mathrm{d}u_m}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} & \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^6)),\end{cases}$$

hence u is in $W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^6))$. For all $t\in[0,T]$ and all $l\leqslant m$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(A(t)u_m(t) + \int_0^t G(t,s)u_m(s)\,\mathrm{d}s\right)\cdot \boldsymbol{e}_l = -\int_\Omega u_m(t)\cdot M\boldsymbol{e}_j - \int_\Omega j(t)\cdot\boldsymbol{e}_j.$$

Since both sides of the equality weak$*$ converge in $L^\infty(0,T)$, we can pass to the limit and by density of V_m in $V(\Omega)$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(A(t)u(t) + \int_0^t G(t,s)u(s)\,\mathrm{d}s\right)\cdot v = -\int_\Omega u(t)\cdot Mv - \int_\Omega j(t)\cdot v, \quad \forall v\in V(\Omega).$$

On the one hand, we deduce that $u\in L^\infty(0,T;V(\Omega))$ and on the other hand that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(A(t)u(t) + \int_0^t G(t,s)u(s)\,\mathrm{d}s\right)\cdot v = \int_\Omega Mu(t)\cdot v - \int_\Omega j(t)\cdot v, \quad \forall v\in V(\Omega).$$

By density of $V(\Omega)$ in $L^2(\Omega;\mathbb{R}^6)$ the previous equality is still valid for all $v\in L^2(\Omega;\mathbb{R}^6)$, and finally (1) is satisfied for almost all $t\in[0,T]$.

From the equality $Mu(t) = \frac{\mathrm{d}}{\mathrm{d}t}(A(t)u(t) + \int_0^t G(t,s)u(s)\,\mathrm{d}s) + j(t)$ we deduce the following bound on the curl-field:

$$\|\operatorname{curl} u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c\big(\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \|u^0\|_{V(\Omega)}\big).$$

Remarks. (i) By multiplying the equilibrium equation (1), and noting that $\int_\Omega Mv\cdot v\,\mathrm{d}x = 0$ for all $v \in V(\Omega)$, we get the conservation law (2).

(ii) Under weaker assumptions: $A \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$, $u(0) \in L^2(\Omega;\mathbb{R}^6)$, $j \in L^1(0,T;L^2(\Omega;\mathbb{R}^6))$ and $G \in \mathcal{W}^{1,1}(\mathcal{S};L^\infty(\Omega;\mathbb{R}^{36}))$, problem (1) has a weak solution $u \in \mathcal{C}^0([0,T];L^2(\Omega;\mathbb{R}^6))$ that solves the variational equation:

$$\int_\Omega \left(A(t)u(t) + \int_0^t G(t,s)u(s)\,\mathrm{d}s\right)\cdot v = \int_\Omega \left(\int_0^t \big(M\big(u(s)\big) - j(s)\big)\,\mathrm{d}s + A(0)u(0)\right)\cdot v, \quad \forall v \in L^2\big(\Omega;\mathbb{R}^6\big),$$

and satisfies the same conservation law (2).

(iii) Let G be of the particular form $G(t,s) = B(s) + C(t-s)$.

With the assumptions $B \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$, $C \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ the solution u to (1) still belongs to the space $W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^6)) \cap L^\infty(0,T;V(\Omega))$ and satisfies the same bounds. In addition, a straightforward computation leads to the more precise estimate:

$$\forall t \in [0,T], \quad \|u\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^6))} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^6))} \leqslant c_0 \mathrm{e}^{c_1 t}\big\{\|j\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \|u(0)\|_{V(\Omega)}\big\}.$$

The constants c_0, c_1 are strictly positive and depend only on the data A, B and C. Let us also notice that when A and B are independent of t, and $T = +\infty$, we may define the Laplace transform $\hat{u}(p) = \int_0^\infty \exp^{-pt} u(t)\,\mathrm{d}t$ for all $p > c_1$.

A.2. Proof of Theorem 2

The proofs of (i) and (ii) are given in [4]. The proof of part (iii) relies on the following decomposition (see [6]):

$$H(\mathrm{curl},\Omega) = \nabla H_0^1(\Omega) \oplus H(\mathrm{curl},\mathrm{div}\,0,\Omega).$$

Since the space $H(\mathrm{curl},\mathrm{div},\Omega)$ does not coincide with the space $H^1(\Omega)$, we introduce in Lemma 2.1 a new space $H_\rho^1 \subset H_{\mathrm{loc}}^1$ to study the convergences of unfolded bounded sequences and we establish in Lemma 2.2 the inclusion $H(\mathrm{curl},\mathrm{div},\Omega) \subset H_\rho^1(\Omega;\mathbb{R}^3)$. Let us first introduce some notation:

- $\mathcal{O}$ is a domain (open, connected subset) in $\mathbb{R}^n$ with Lipschitz boundary,
- $\rho(x) = \mathrm{dist}(x, \partial\mathcal{O})$ is the distance of x to the boundary of $\mathcal{O}$,
- $H^1_\rho(\mathcal{O}) = \{\phi \in H^1_{\mathrm{loc}}(\mathcal{O});\ \phi \in L^2(\mathcal{O})$ and $\rho\nabla\phi \in L^2(\mathcal{O};\mathbb{R}^n)\}$ equipped with the norm

$$\|\phi\|_\rho = \sqrt{\|\phi\|^2_{L^2(\mathcal{O})} + \|\rho\nabla\phi\|^2_{L^2(\mathcal{O};\mathbb{R}^n)}},$$

- $L^2_\rho(\mathcal{O}) = \{\phi \in L^2_{\mathrm{loc}}(\mathcal{O});\ \rho\phi \in L^2(\mathcal{O})\}$,
- $L^2_\rho(\mathcal{O}; L^2(Y)) = \{\phi \in L^2_{\mathrm{loc}}(\mathcal{O}; L^2(Y));\ \int_{\mathcal{O}\times Y} |\phi(x,y)|^2\rho^2(x) < +\infty\}$.

We denote by $\mathcal{O}_\alpha$ the bounded set:

$$\mathcal{O}_\alpha = \{x \in \mathcal{O};\ \rho(x) > \alpha\}, \quad \alpha > 0.$$

Lemma 2.1. *Let $(v^\alpha)_\alpha$ be a bounded sequence in $H^1_\rho(\mathcal{O})$. After extraction of a subsequence, we get the weak convergences*:

$$\begin{cases} v^\alpha \rightharpoonup v & \textit{in } H^1_\rho(\mathcal{O}),\\ \mathcal{T}_\alpha(v^\alpha) \rightharpoonup v & \textit{in } L^2(\mathcal{O}\times Y),\\ \mathcal{T}_\alpha(\nabla_x v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}) \rightharpoonup \nabla_x v + \nabla_y \bar{v} & \textit{in } L^2_\rho(\mathcal{O}; L^2(Y;\mathbb{R}^n)), \end{cases}$$

where $\bar{v}$ belongs to $L^2_\rho(\mathcal{O}; H^1_{\mathrm{per}}(Y))$.

Proof. First we show that the sequence $\mathcal{T}_\alpha(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha})$ is bounded in $L^2_\rho(\mathcal{O}; L^2(Y;\mathbb{R}^n))$. We have:

$$\int_{\mathcal{O}\times Y} \left|\mathcal{T}_\alpha\left(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}\right)\right|^2 \left|\mathcal{T}_\alpha(\rho)\right|^2 = \int_{\mathcal{O}_\alpha} \left|\nabla v^\alpha\right|^2 \rho^2 \leqslant \left\|v^\alpha\right\|^2_\rho.$$

Function ρ is 1-Lipschitz, hence $|\mathcal{T}_\alpha(\rho)(x,y) - \rho(x)| \leqslant 2\sqrt{n}\alpha$ for all $(x,y) \in \mathcal{O}\times Y$. This yields:

$$\begin{aligned} \int_{\mathcal{O}\times Y} \left|\mathcal{T}_\alpha\left(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}\right)\right|^2 \rho^2 &\leqslant C \int_{\mathcal{O}\times Y} \left|\mathcal{T}_\alpha\left(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}\right)\right|^2 \left|\mathcal{T}_\alpha(\rho)\right|^2 + C\alpha^2 \int_{\mathcal{O}\times Y} \left|\mathcal{T}_\alpha\left(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}\right)\right|^2,\\ &\leqslant C\left\|v^\alpha\right\|^2_\rho + C\alpha^2 \int_{\mathcal{O}_\alpha} \left|\nabla v^\alpha\right|^2 \leqslant C\left\|v^\alpha\right\|^2_\rho. \end{aligned}$$

Therefore the sequence $\mathcal{T}_\alpha(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha})$ is bounded in $L^2_\rho(\mathcal{O}; L^2(Y;\mathbb{R}^n))$. From the sequences v^α, $\mathcal{T}_\alpha(v^\alpha)$ and $\mathcal{T}_\alpha(\nabla v^\alpha \mathbf{1}_{\mathcal{O}_\alpha})$, we can extract subsequences, still denoted in the same way, that weakly converge:

$$\begin{cases} v^\alpha \rightharpoonup v & \text{in } H^1_\rho(\mathcal{O}),\\ \mathcal{T}_\alpha(v^\alpha) \rightharpoonup \bar{v} & \text{in } L^2(\mathcal{O}\times Y),\\ \mathcal{T}_\alpha(\nabla_x v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}) \rightharpoonup \zeta & \text{in } L^2_\rho(\mathcal{O}; L^2(Y;\mathbb{R}^n)),\end{cases}$$

and such that for all $N \in \mathbb{N}^*$ and $\alpha \leqslant 1/N$, we have another set of weak convergences:

$$\begin{cases} v^\alpha \rightharpoonup v & \text{in } H^1(\mathcal{O}_{1/N}),\\ \mathcal{T}_\alpha(v^\alpha) \rightharpoonup v & \text{in } L^2(\mathcal{O}_{1/N}\times Y),\\ \mathcal{T}_\alpha(\nabla_x v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}) = \mathcal{T}_\alpha(\nabla_x v^\alpha) \rightharpoonup \nabla_x v + \nabla_y \bar{v}_N & \text{in } L^2(\mathcal{O}_{1/N}\times Y;\mathbb{R}^n),\end{cases}$$

where $\bar{v}_N$ belongs to $L^2(\mathcal{O}_{1/N}; H^1_{\mathrm{per}}(Y))$. Since the limit is unique, we get $\bar{v} = v$ in $L^2(\mathcal{O}\times Y)$, $\zeta = \nabla_x v + \nabla_y \bar{v}_N$ in $L^2(\mathcal{O}_{1/N}\times Y;\mathbb{R}^n)$, for all $N \in \mathbb{N}^*$. The sequence $\mathbf{1}_{\mathcal{O}_{1/N}}\bar{v}_N$ is bounded in $L^2_\rho(\mathcal{O}; H^1_{\mathrm{per}}(Y))$. It weakly converges to $\bar{v} \in L^2_\rho(\mathcal{O}; H^1_{\mathrm{per}}(Y))$ and $\bar{v} = \bar{v}_N$ in $L^2(\mathcal{O}_{1/N}; H^1_{\mathrm{per}}(Y))$ for all $N \in \mathbb{N}^*$. □

It is worth to point out the following result: Let the sequence v^α be bounded in $L^2(\mathcal{O})$, then there exists $v \in L^2(\mathcal{O}\times Y)$ such that, after extraction of a subsequence, one has the weak convergences:

$$\begin{cases} \mathcal{T}_\alpha(v^\alpha) \rightharpoonup v & \text{in } L^2(\mathcal{O}\times Y),\\ \mathcal{T}_\alpha(v^\alpha \mathbf{1}_{\mathcal{O}_\alpha}) \rightharpoonup v & \text{in } L^2(\mathcal{O}\times Y).\end{cases}$$

Lemma 2.2. *We have* $H(\mathrm{curl}, \mathrm{div}, \Omega) \subset H^1_\rho(\Omega;\mathbb{R}^3)$.

Proof. Let $\mathbf{v} \in H(\mathrm{curl}, \mathrm{div}, \Omega)$. The solution $\boldsymbol{w} \in H^1_0(\Omega;\mathbb{R}^3)$ to the variational problem,

$$\int_\Omega \nabla \boldsymbol{w} : \nabla \Phi = \int_\Omega \mathrm{curl}\, \boldsymbol{v}\cdot \mathrm{curl}\, \Phi + \mathrm{div}\, \boldsymbol{v}\, \mathrm{div}\, \Phi \quad \forall \Phi \in H^1_0(\Omega;\mathbb{R}^3),$$

satisfies $\|\nabla \boldsymbol{w}\|_{[L^2(\Omega;\mathbb{R}^3)]^3} \leqslant \|\mathrm{curl}\, \boldsymbol{v}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)}$. The vector field $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w} \in L^2(\Omega;\mathbb{R}^3)$ has a vanishing Laplacian in $H^{-1}(\Omega;\mathbb{R}^3)$ and it is in $\mathcal{C}^\infty(\Omega;\mathbb{R}^3)$. Let us denote by ρ_N the function defined by (here ρ represents the distance to $\partial\Omega$):

$$\rho_N(x) = \sup\left(0, \rho(x) - \frac{1}{N}\right), \quad x \in \Omega.$$

The test function $\boldsymbol{u}\rho_N^2 \in W^{1,\infty}_0(\Omega;\mathbb{R}^3)$ has a compact support included in Ω. Thus we have:

$$0 = \int_\Omega -\Delta \boldsymbol{u}\cdot \boldsymbol{u}\rho_N^2 = \int_\Omega |\nabla \boldsymbol{u}|^2 \rho_N^2 + 2\int_\Omega \rho_N \nabla \boldsymbol{u}\, \boldsymbol{u}\cdot \nabla \rho_N,$$

and by noting that ρ_N is 1-Lipschitz we get:

$$\int_\Omega |\nabla \boldsymbol{u}|^2\rho_N^2 \leqslant 2\left\{\int_\Omega |\nabla \boldsymbol{u}|^2\rho_N^2\right\}^{1/2}\left\{\int_\Omega |\boldsymbol{u}|^2\right\}^{1/2} \leqslant \frac{1}{2}\int_\Omega |\nabla \boldsymbol{u}|^2\rho_N^2 + 2\int_\Omega |\boldsymbol{u}|^2.$$

This yields $\int_\Omega |\nabla \boldsymbol{u}|^2\rho_N^2 \leqslant 4\int_\Omega |\boldsymbol{u}|^2$, the sequence $|\nabla \boldsymbol{u}|^2\rho_N^2$ simply converges to $|\nabla \boldsymbol{u}|^2\rho^2$ when $N\to\infty$. Lemma 2.2 follows from Fatou's lemma. □

Proof of Theorem 2. The direct sum

$$H(\mathrm{curl},\Omega) = \nabla H_0^1(\Omega)\oplus H(\mathrm{curl},\mathrm{div}\,0,\Omega)$$

is orthogonal for the scalar product in $L^2(\Omega;\mathbb{R}^3)$ (see [6]), hence any vector field $\boldsymbol{v}^\alpha$ can be uniquely decomposed as

$$\boldsymbol{v}^\alpha = \nabla_x v^\alpha + \boldsymbol{w}^\alpha,\quad v^\alpha\in H_0^1(\Omega),\ \boldsymbol{w}^\alpha\in H(\mathrm{curl},\mathrm{div}\,0,\Omega),$$

and we have:

$$\|v^\alpha\|_{H^1(\Omega)} + \|\boldsymbol{w}^\alpha\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant C\|\boldsymbol{v}^\alpha\|_{L^2(\Omega;\mathbb{R}^3)},\quad \mathrm{curl}\,\boldsymbol{w}^\alpha = \mathrm{curl}\,\boldsymbol{v}^\alpha,\ \mathrm{div}\,\boldsymbol{w}^\alpha = 0.$$

Let $\boldsymbol{v}^\alpha$ be a bounded sequence in $H(\mathrm{curl},\Omega)$. Then, by Lemma 2.2, the sequence $\boldsymbol{w}^\alpha$ is bounded in $H_\rho^1(\Omega;\mathbb{R}^3)$ and the sequence v^α is bounded in $H^1(\Omega)$. From these last two sequences we can extract subsequences, still denoted in the same way, that weakly converge,

$$\begin{cases} v^\alpha \rightharpoonup v & \text{in } H^1(\Omega),\\ \mathcal{T}_\alpha(v^\alpha)\rightharpoonup v & \text{in } L^2(\Omega;H^1(Y)),\\ \mathcal{T}_\alpha(\nabla_x v^\alpha)\rightharpoonup \nabla_x v + \nabla_y\bar v & \text{in } L^2(\Omega\times Y;\mathbb{R}^3),\\ \boldsymbol{w}^\alpha\rightharpoonup \boldsymbol{w} & \text{in } H_\rho^1(\Omega;\mathbb{R}^3)\cap H(\mathrm{curl},\Omega),\\ \mathcal{T}_\alpha(\boldsymbol{w}^\alpha)\rightharpoonup \boldsymbol{w} & \text{in } L^2(\Omega\times Y;\mathbb{R}^3),\\ \mathcal{T}_\alpha(\nabla_x \boldsymbol{w}^\alpha \mathbf{1}_{\mathcal{O}_\alpha})\rightharpoonup \nabla_x\boldsymbol{w} + \nabla_y\bar{\bar{\boldsymbol{v}}} & \text{in } L_\rho^2(\Omega;L^2(Y;\mathbb{R}^3)), \end{cases}$$

where $\bar v\in L^2(\Omega;H^1_{\mathrm{per}}(Y))$, $\bar{\bar{\boldsymbol{v}}}\in L^2_\rho(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))$ and $\mathrm{div}_y\,\bar{\bar{\boldsymbol{v}}} = 0$. Therefore we get the weak convergences:

$$\begin{cases} \boldsymbol{v}^\alpha\rightharpoonup \boldsymbol{v} = \boldsymbol{w} + \nabla_x v & \text{in } H(\mathrm{curl},\Omega),\\ \mathcal{T}_\alpha(\boldsymbol{v}^\alpha)\rightharpoonup \boldsymbol{v} + \nabla_y\bar v & \text{in } L^2(\Omega\times Y;\mathbb{R}^3),\\ \mathcal{T}_\alpha(\mathrm{curl}_x\,\boldsymbol{v}^\alpha\mathbf{1}_{\mathcal{O}_\alpha}) = \mathcal{T}_\alpha(\mathrm{curl}_x\,\boldsymbol{w}^\alpha\mathbf{1}_{\mathcal{O}_\alpha})\rightharpoonup \mathrm{curl}_x\,\boldsymbol{w} + \mathrm{curl}_y\,\bar{\bar{\boldsymbol{v}}} & \text{in } L^2_\rho(\Omega;L^2(Y;\mathbb{R}^3)). \end{cases}$$

The sequence $\mathcal{T}_\alpha(\mathrm{curl}_x\,\boldsymbol{v}^\alpha)$ is bounded in $L^2(\Omega\times Y;\mathbb{R}^3)$, it weakly converges to an element in this space. Hence $\bar{\bar{\boldsymbol{v}}}\in L^2(\Omega;H_{\mathrm{per}}(\mathrm{curl},\mathrm{div}\,0,Y))$, so that $\bar{\bar{\boldsymbol{v}}}\in L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))$ since $H_{\mathrm{per}}(\mathrm{curl},\mathrm{div}\,0,Y)\subset H^1_{\mathrm{per}}(Y;\mathbb{R}^3)$. □

A.3. Proof of Theorem 3

The proof is broken into 3 steps; first we state the weak convergence of the family $u^\alpha, \mathcal{T}_\alpha(u^\alpha), \mathcal{T}_\alpha(\operatorname{curl} u^\alpha)$, then we give the limit evolution problem and finally we establish the strong convergences.

Step 1. Weak convergences

Up to subsequences, still denoted in the same way, we have the following weak∗ convergences:

$$\begin{cases} u^\alpha \rightharpoonup u & \text{in } L^\infty(0,T;V(\Omega)),\\ \mathcal{T}_\alpha(u^\alpha) \rightharpoonup u + \nabla_y \bar{u} & \text{in } L^\infty(0,T;L^2(\Omega, H(\operatorname{curl},Y))),\\ \mathcal{T}_\alpha(\frac{\mathrm{d}u^\alpha}{\mathrm{d}t}) \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} + \nabla_y \frac{\mathrm{d}\bar{u}}{\mathrm{d}t} & \text{in } L^\infty(0,T;L^2(\Omega\times Y;\mathbb{R}^6)),\\ \mathcal{T}_\alpha(\operatorname{curl}_x u^\alpha) \rightharpoonup \operatorname{curl}_x u + \operatorname{curl}_y \bar{\bar{u}} & \text{in } L^\infty(0,T;L^2(\Omega\times Y;\mathbb{R}^6)). \end{cases}$$

Proof. By using the uniform bounds $\|u^{\alpha,0}\|_{V(\Omega)} \leqslant c$, $\|j^\alpha\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} \leqslant c$, we show that the sequence u^α is uniformly bounded in $L^\infty(0,T;V(\Omega))$ and that the sequence $\frac{\mathrm{d}u^\alpha}{\mathrm{d}t}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))$. Then the proof follows from Theorem 2. □

Step 2. The evolution problem (6)

According to the property $\mathcal{T}_\alpha(vw) = \mathcal{T}_\alpha(v)\mathcal{T}_\alpha(w)$ for all $v, w \in L^2(\Omega)$, we deduce the weak∗ convergence $\mathcal{T}_\alpha(L^\alpha u^\alpha) \rightharpoonup L(u + \nabla_y \bar{u})$ in $L^\infty(0,T;L^2(\Omega\times Y))$, where

$$L^\alpha v^\alpha(t,x) = A^\alpha(t,x)v^\alpha(t,x) + \int_0^t G^\alpha(t,s,x)v^\alpha(t,x)\,\mathrm{d}s,$$

$$Lv(t,x,y) = A(t,y)v(t,x,y) + \int_0^t G(t,s,y)v(s,x,y)\,\mathrm{d}s.$$

The weak formulation of problem (3) is, for all $v \in L^2(\Omega;\mathbb{R}^6)$ and for all $t \in [0,T]$,

$$\int_\Omega L^\alpha u^\alpha(t)\cdot v = \int_\Omega \int_0^t \bigl(\operatorname{curl} u_2^\alpha(s)\cdot v_1 \operatorname{curl} u_1^\alpha(s)\cdot v_2\bigr)\,\mathrm{d}s - \int_\Omega\int_0^t j^\alpha(s)\cdot v\,\mathrm{d}s + \int_\Omega A^\alpha(0)u^{\alpha,0}\cdot v. \tag{A.2}$$

We recall (see [5]) the approximate integration formula:

$$\left|\int_\Omega v - \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\alpha(v)\right| \leqslant \|v\|_{L^1(\{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)<\sqrt{n}\varepsilon\})} \quad \text{for all } v \in L^1(\Omega).$$

We take in (A.2) the test function $v^\alpha(x) = \phi(x)\psi(\{\frac{x}{\alpha}\})$, $\phi \in \mathcal{D}(\Omega; \mathbb{R}^6)$, $\psi \in \mathcal{D}(Y; \mathbb{R}^6)$. We transform the equality by unfolding (hence $\mathcal{T}_\alpha(v^\alpha)(x, y) = \mathcal{T}_\alpha(\phi)(x, y)\psi(y)$) and we pass to the limit by using the strong convergence established in Theorem 2 and thanks to $\|v^\alpha\|_{L^2(\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)<\sqrt{n}\varepsilon\})} \to 0$. Thus we get the limit variational problem by the density of the tensor product $\mathcal{D}(\Omega; \mathbb{R}^6) \times \mathcal{D}(Y; \mathbb{R}^6)$ in $L^2(\Omega \times Y; \mathbb{R}^6)$,

$$\int_{\Omega\times Y} L(u + \nabla_y \bar{u})(t) \cdot v = \int_{\Omega\times Y} \int_0^t \Big(\big(\mathrm{curl}_x\, u_2(s) + \mathrm{curl}_y\, \bar{\bar{u}}_2(s)\big) \cdot v_1 - \big(\mathrm{curl}_x\, u_1(s) + \mathrm{curl}_y\, \bar{\bar{u}}_1(s)\big) \cdot v_2\Big)\,\mathrm{d}s - \int_{\Omega\times Y} \int_0^t j(s) \cdot v\,\mathrm{d}s + \int_{\Omega\times Y} A(0)u^0 \cdot v \quad \forall v \in L^2\big(\Omega \times Y; \mathbb{R}^6\big).$$

Step 3. Strong convergences

The proof of the strong convergences is broken into 4 intermediate steps.

3.1. A preliminary convergence result. Let $\mathcal{O}$ be a domain in $\mathbb{R}^3$ and K be a matrix in $L^1(0, T; L^\infty(\mathcal{O}; \mathbb{R}^{36}))$. We consider a sequence v^α that weakly $*$ converges: $v^\alpha \rightharpoonup v$ in $L^\infty(0, T, L^2(\mathcal{O}; \mathbb{R}^6))$. We denote by $\|.\|$ the norm in $L^\infty(\mathcal{O}; \mathbb{R}^{36})$ and by $\|.\|_2$ the norm in $L^2(\mathcal{O}; \mathbb{R}^6)$. We use the straightforward identity:

$$Kv^\alpha.v^\alpha - Kv.v = K(v^a - v).(v^\alpha - v) + K(v^\alpha - v).v + Kv.(v^\alpha - v)$$

and the estimate

$$\int_{\mathcal{O}} K(s)\big(v^\alpha(s) - v(s)\big).\big(v^\alpha(s) - v(s)\big) \leqslant \big\|K(s)\big\| \big\|v^\alpha(s) - v(s)\big\|_2^2 \quad \text{a.e. in } (0, T),$$

to obtain

$$\limsup_{\alpha\to 0} \int_{\mathcal{O}} K(s)\big(v^\alpha(s) - v(s)\big).\big(v^\alpha(s) - v(s)\big) \leqslant \big\|K(s)\big\| g^2(s),$$

where $g(s) = \limsup_{\alpha\to 0} \|v^\alpha(s) - v(s)\|_2$. Finally, with Fatou's lemma we get:

$$\limsup_{\alpha\to 0}\left(\int_0^t \int_{\mathcal{O}} K(s)v^\alpha(s).v^\alpha(s)\,\mathrm{d}s - \int_0^t \int_{\mathcal{O}} K(s)v(s).v(s)\,\mathrm{d}s\right) \leqslant \int_0^t \big\|K(s)\big\| g^2(s)\,\mathrm{d}s.$$

A direct extension to matrix $K \in L^1(\mathcal{S}; L^\infty(\mathcal{O}; \mathbb{R}^{36}))$ yields,

$$\limsup_{\alpha\to 0}\int_0^t\int_0^{t_1}\int_{\mathcal{O}}\big(K(t_1,s)v^\alpha(s).v^\alpha(t_1)-K(t_1,s)v(s).v(t_1)\big)\,\mathrm{d}s\,\mathrm{d}t_1$$
$$\leqslant\int_0^t\int_0^{t_1}\big\|K(t_1,s)\big\|g(s)g(t_1)\,\mathrm{d}s\,\mathrm{d}t_1.$$

3.2. Convergence in $L^2(]0,T[\times\Omega\times Y;\mathbb{R}^6)$. We apply the unfolding operator to the conservation law satisfied by u^α,

$$\frac{1}{2}\int_\Omega A^\alpha(t)u^\alpha(t)\cdot u^\alpha(t)+\frac{1}{2}\int_0^t\int_\Omega\frac{\mathrm{d}A^\alpha}{\mathrm{d}t}(s)u^\alpha(s)\cdot u^\alpha(s)\,\mathrm{d}s$$
$$+\int_0^t\int_0^{t_1}\int_\Omega\frac{\partial G^\alpha}{\partial t}(t_1,s)u^\alpha(s)u^\alpha(t_1)\,\mathrm{d}s\,\mathrm{d}t_1+\int_0^t\int_\Omega G^\alpha(s,s)u^\alpha(s)\cdot u^\alpha(s)\,\mathrm{d}s$$
$$+\int_0^t\int_\Omega j^\alpha(s)\cdot u^\alpha(s)\,\mathrm{d}s=\frac{1}{2}\int_\Omega A^\alpha(0)u^{\alpha,0}\cdot u^{\alpha,0},$$

and make use of the previous steps where the domain $\mathcal{O}$ is replaced by $\Omega\times Y$, the sequence v^α by $\mathcal{T}_\alpha(u^\alpha)$ and the limit v by $u+\nabla_y\bar{u}$. From the coercivity of the matrix A we have the inequality:

$$\frac{c}{2}\big\|v^\alpha(t)-v(t)\big\|_2^2\leqslant\frac{1}{2}\int_{\mathcal{O}}A(t,y)\big(v^\alpha(t)-v(t)\big).\big(v^\alpha(t)-v(t)\big),$$
$$\leqslant\frac{1}{2}\int_{\mathcal{O}}A(t)v^\alpha(t).v^\alpha(t)-\int_{\mathcal{O}}A(t)v^\alpha(t).v(t)+\frac{1}{2}\int_{\mathcal{O}}A(t)v(t).v(t)$$

to obtain, for all $t\in[0,T]$,

$$\frac{c}{2}g^2(t)\leqslant\int_0^t\theta_1(s)g^2(s)\,\mathrm{d}s+\int_0^t\int_0^{t_1}\theta_2(t_1,s)g(t_1)g(s)\,\mathrm{d}s\,\mathrm{d}t_1,$$
$$\leqslant\int_0^t\theta_1(s)g^2(s)\,\mathrm{d}s+\frac{1}{2}\int_0^t\left\{\int_0^{t_1}\theta_2(t_1,s)\,\mathrm{d}s\right\}g^2(t_1)\,\mathrm{d}t_1$$
$$+\frac{1}{2}\int_0^t\left\{\int_s^t\theta_2(t_1,s)\,\mathrm{d}t_1\right\}g^2(s)\,\mathrm{d}s,$$

where $\theta_1(s) = \|\frac{\mathrm{d}A}{\mathrm{d}t}(s)\|_{L^\infty(Y;\mathbb{R}^{36})} + \|G(s,s)\|_{L^\infty(Y;\mathbb{R}^{36})}$, $\theta_2(t_1,s) = \|\frac{\partial G}{\partial t}(t_1,s)\|_{L^\infty(Y;\mathbb{R}^{36})}$. Hence by Gronwall lemma, we have $g(t) = 0$ for all $t \in [0,T]$, which in turns leads to the strong convergence for all $t \in [0,T]$,

$$\mathcal{T}_\alpha(u^\alpha)(t) \to u(t) + \nabla_y \bar{u}(t) \quad \text{in } L^2\big(\Omega \times Y; \mathbb{R}^6\big).$$

Then, the Lebesgue theorem of dominated convergence implies the strong convergence:

$$\mathcal{T}_\alpha(u^\alpha) \to u + \nabla_y \bar{u} \quad \text{in } L^2\big(]0,T[\times \Omega \times Y; \mathbb{R}^6\big).$$

3.3. Convergence in $H^1(]0,T[\times\Omega\times Y;\mathbb{R}^6)$. The derivative $\frac{\mathrm{d}u^\alpha}{\mathrm{d}t}$ is the weak solution of an evolution problem with $\frac{\mathrm{d}j^\alpha}{\mathrm{d}t}$ in the right-hand side and with initial condition:

$$A^\alpha(0)\frac{\mathrm{d}u^\alpha}{\mathrm{d}t}(0) + \frac{\mathrm{d}A^\alpha}{\mathrm{d}t}(0)u^\alpha(0) + G^\alpha(0,0)u^\alpha(0) = M\big(u^\alpha(0)\big) - j^\alpha(0).$$

Assumptions (4) on the data imply that the sequences $\mathcal{T}_\alpha(\frac{\mathrm{d}j^\alpha}{\mathrm{d}t})$ and $\mathcal{T}_\alpha(\frac{\mathrm{d}u^\alpha}{\mathrm{d}t}(0))$ strongly converge respectively, in $L^1(0,T;L^2(\Omega\times Y;\mathbb{R}^6))$ and $L^2(\Omega\times Y;\mathbb{R}^6)$. Then we show as before, the strong convergence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_\alpha(u^\alpha) \to \frac{\mathrm{d}u}{\mathrm{d}t} + \nabla_y \frac{\mathrm{d}\bar{u}}{\mathrm{d}t} \quad \text{in } L^2\big(]0,T[\times \Omega \times Y; \mathbb{R}^6\big).$$

3.4. Strong convergence of the curl fields. We apply the unfolding operator to (3) and get from the previous strong convergence:

$$\mathcal{T}_\alpha(\mathrm{curl}_x\, u^\alpha) \to \mathrm{curl}_x\, u + \mathrm{curl}_y\, \bar{\bar{u}} \quad \text{in } L^2\big(]0,T[\times \Omega \times Y; \mathbb{R}^6\big).$$

A.4. Proof of Theorem 4

We begin the proof with two preliminary lemmas, the first one is a generalization of Lemma 1.1 to more complex right-hand sides.

Lemma 4.1. *Let $m \in \mathbb{N}^*$, $r \in \{1,2\}$ and $p \in [1,+\infty]$. Let matrix $\tilde{A} \in W^{r,p}(0,T;\mathbb{R}^{m^2})$ be uniformly coercive, let $\widetilde{G} \in \mathcal{W}^{r,p}(\mathcal{S};\mathbb{R}^{m^2})$ be a square matrix and $\tilde{k} \in \mathcal{W}^{r,p}(\mathcal{S};\mathbb{R}^m)$. There exists a unique function $g \in \mathcal{W}^{r,p}(\mathcal{S};\mathbb{R}^m)$ such that*

$$\tilde{A}(t)g(t,s) + \int_s^t \widetilde{G}(t,s_1)g(s_1,s)\,\mathrm{d}s_1 = \tilde{k}(t,s) \quad \textit{a.e. in } \mathcal{S}.$$

Same use of Fredholm theory gives the proof.

Lemma 4.2. *With the same notation as in Lemma* 4.1 *let $A \in W^{r,p}(0,T;L^\infty(Y;R^{36}))$ be a uniformly coercive matrix, let $G \in \mathcal{W}^{r,p}(\mathcal{S};L^\infty(Y;R^{36}))$ be a square matrix and*

$k \in W^{r,p}(0,T;L^\infty(Y;R^6))$. *There exists a unique function* $g \in \mathcal{W}^{r,p}(\mathcal{S};H^1_{\mathrm{per}}(Y;\mathbb{R}^2))$ *such that*

$$\int_Y \left(A(t,y)\nabla_y g(t,s,y) + \int_s^t G(t,s_1,y)\nabla_y g(s_1,s,y)\,\mathrm{d}s_1 \right)\cdot \nabla_y z(y)$$
$$= \int_Y k(t,s,y)\cdot \nabla_y z(y), \quad \forall z \in H^1_{\mathrm{per}}\big(Y;\mathbb{R}^2\big) \text{ a.e. in } \mathcal{S}. \tag{A.3}$$

The proof is broken into 2 steps according to the values of p and r.

First step. Existence of a solution in the case $r = 1$, $p \neq 1$. Let $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_m,\ldots$ be a Hilbert basis of $H^1_{\mathrm{per}}(Y;\mathbb{R}^2)$, $H_m = \mathrm{Vect}(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_m)$. We look for an approximation of the solution to problem (A.3) under the form $g_m(t,s,y) = \sum_{l=1}^m \alpha_l^m(t,s)\boldsymbol{h}_l(y)$ where the scalar functions α_l^m are solution to the $m\times m$ system:

$$\int_Y A(t,y)\nabla_y g_m(t,s,y)\cdot\nabla_y \boldsymbol{h}_l(y) + \int_s^t\int_Y G(t,s_1,y)\nabla_y g_m(s_1,s,y)\cdot\nabla_y\boldsymbol{h}_l(y)\,\mathrm{d}s_1$$
$$= \int_Y k(t,s,y)\cdot\nabla_y\boldsymbol{h}_l(y). \tag{A.4}$$

With Lemma 4.1, the function g_m is in $\mathcal{W}^{1,p}(\mathcal{S};H_m)$. With Gronwall lemma we get the estimate,

$$\|g_m\|_{1,p;H^1(Y;\mathbb{R}^2)} \leqslant c\|k\|_{1,p;L^\infty(Y;\mathbb{R}^6)}$$

(the constant c only depends on T, p, $\|A\|_{W^{1,p}(0,T;L^\infty(Y;\mathbb{R}^{36}))}$, $\|A^{-1}\|_{L^\infty(0,T;L^\infty(Y;\mathbb{R}^{36}))}$ and $\|G\|_{1,p;L^\infty(Y;\mathbb{R}^{36})}$).

After extraction of a subsequence from the sequence g_m, we have the weak convergences:

$$\begin{cases} g_m \rightharpoonup g & \text{in } L^p(\mathcal{S};H^1_{\mathrm{per}}(Y;\mathbb{R}^6)),\\ g_m|_{\mathcal{S}} \rightharpoonup g|_{\mathcal{S}} & \text{in } L^p(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^6)),\\ \frac{\partial g_m}{\partial t} \rightharpoonup \frac{\partial g}{\partial t} & \text{in } L^p(\mathcal{S};H^1_{\mathrm{per}}(Y;\mathbb{R}^6)). \end{cases}$$

Then we check that g is solution to (A.3), by passing to the limit in (A.4) and by the density of the linear combinations of functions of the Hilbert basis $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_m,\ldots$. The uniqueness of g is a consequence of the linearity of the problem.

Second step. Existence of a solution in the case $p = 1$. We suppose now that $A \in W^{1,1}(0,T; L^\infty(Y;\mathbb{R}^{36}))$, $G \in \mathcal{W}^{1,1}(\mathcal{S}; L^\infty(Y;\mathbb{R}^{36}))$ and $k \in \mathcal{W}^{1,1}(\mathcal{S}; L^\infty(Y;\mathbb{R}^{6}))$. There exist sequences $A_N \in \mathcal{C}^1([0,T]; L^\infty(Y;\mathbb{R}^{36}))$, $G_N \in \mathcal{C}^1(\mathcal{S}; L^\infty(Y;\mathbb{R}^{36}))$ and $k \in \mathcal{C}^1(\mathcal{S}; L^\infty(Y;\mathbb{R}^{6}))$ that strongly converge as

$$\begin{cases} A_N \to A & \text{in } W^{1,1}(0,T; L^\infty(Y;\mathbb{R}^{36})),\\ G_N \to G & \text{in } \mathcal{W}^{1,1}(\mathcal{S}; L^\infty(Y;\mathbb{R}^{36})),\\ k_N \to k & \text{in } \mathcal{W}^{1,1}(\mathcal{S}; L^\infty(Y;\mathbb{R}^{6})). \end{cases}$$

For any $(M,N) \in \mathbb{N}^2$ the function $g_N - g_M$ belongs to $\mathcal{W}^{1,1}(\mathcal{S}; H^1_{\text{per}}(Y))$ and is the solution of the problem:

$$\begin{aligned} &\int_Y A_N(t,y)\nabla_y(g_N - g_M)(t,s,y)\cdot\nabla_y z(y)\\ &\qquad + \int_s^t\int_Y G_N(t,s_1,y)\nabla_y(g_N-g_M)g(s_1,s,y)\cdot\nabla_y z(y)\,\mathrm{d}s_1\\ &\quad = \int_Y (A_M - A_N)(t,y)\nabla_y g_M(t,s,y)\cdot\nabla_y z(y)\\ &\qquad + \int_s^t\int_Y (G_M-G_N)(t,s_1,y)\nabla_y g_M g(s_1,s,y)\cdot\nabla_y z(y)\,\mathrm{d}s_1\\ &\qquad + \int_Y (k_N-k_M)(t,s,y)\cdot\nabla_y z(y) \quad \forall z \in H^1_{\text{per}}\big(Y;\mathbb{R}^2\big). \end{aligned}$$

We deduce the estimate:

$$\begin{aligned} \|g_N - g_M\|_{1,1;H^1(Y;\mathbb{R}^2)} \leqslant c\big\{&\|A_N - A_M\|_{W^{1,1}(0,T;L^\infty(Y;\mathbb{R}^{36}))} + \|G_N - G_M\|_{1,1;L^\infty(Y;\mathbb{R}^{36})}\\ &+ \|k_N - k_M\|_{1,1;L^\infty(Y;\mathbb{R}^6)}\big\}, \end{aligned}$$

where the constant c only depends on T, $\|A\|_{W^{1,1}(0,T;L^\infty(Y;\mathbb{R}^{36}))}$, $\|A^{-1}\|_{L^\infty(0,T;L^\infty(Y;\mathbb{R}^{36}))}$ and $\|G\|_{1,1;L^\infty(Y;\mathbb{R}^{36})}$. The sequence $(g_N)_N$ is a Cauchy sequence in $\mathcal{W}^{1,1}(\mathcal{S}; H^1_{\text{per}}(Y))$ whose limit g is solution to (A.4).

The lemma is obvious for $p = +\infty$. For $r = 2$ we proceed in the same way, using for $p = 1$ the density of $\mathcal{C}^2(\mathcal{S}; X)$ in $\mathcal{W}^{2,1}(\mathcal{S}; X)$, where X is a Banach space.

Proof of Theorem 4. It relies on an appropriate choice of test functions in the variational form (6).

First step. Computation of $\bar{u}$. We choose in (6) test functions v of the form $v(x,y) = \phi(x)\nabla_y \bar{v}(y)$, $\phi \in L^2(\Omega)$, $\bar{v} \in H^1_{per}(Y;\mathbb{R}^2)$. Hence, for all $t \in (0,T)$ we get:

$$\int_Y L(u + \nabla_y \bar{u}) \cdot \nabla_y \bar{v} = \int_Y A(0)u^0 \cdot \nabla_y \bar{v} \quad \forall \bar{v} \in H^1_{per}\big(Y;\mathbb{R}^2\big),$$

where $Lv(t,x,y)$ has already been defined in Step 2 of Lemma 3. We consider the decompositions $u(t,x) = u_k(t,x)e_k$ and $u^0(x) = u^0_k(x)e_k$ where e_k is the canonical basis of $\mathbb{R}^6$ and introduce three families of elementary correctors $(\overline{w}^A, \overline{w}^0, \overline{w})$ (with value in $\mathbb{R}^2$) which are solution to different local *diffusion problems* posed in Y.

- Corrector $\overline{w}^A_k \in W^{2,1}(0,T; H^1_{per}(Y;\mathbb{R}^2))$, depends on operator A and solves:

$$\int_Y A(t,y)\big(e_k + \nabla_y \overline{w}^A_k(t,y)\big) \cdot \nabla_y \bar{v}(y)\,\mathrm{d}y = 0 \quad \forall \bar{v} \in H^1_{per}\big(Y;\mathbb{R}^2\big).$$

- Corrector $\overline{w}^0_k \in W^{2,1}(0,T; H^1_{per}(Y;\mathbb{R}^2))$, associated to the initial condition $u(0,.)$, solves (Lemma 4.2):

$$\int_Y A(t,y)\nabla_y \overline{w}^0_k(t,y) \cdot \nabla_y \bar{v}(y) + \int_Y \int_0^t G(t,s,y)\nabla_y \overline{w}^0_k(s,y)\,\mathrm{d}s \cdot \nabla_y \bar{v}(y)$$
$$= \int_Y A(0,y)e_k \cdot \nabla_y \bar{v}(y) \quad \forall \bar{v} \in H^1_{per}\big(Y;\mathbb{R}^2\big).$$

- The kernel $\overline{w}_k \in \mathcal{W}^{2,1}(\mathcal{S}; H^1_{per}(Y;\mathbb{R}^2))$ solves (Lemma 4.2):

$$\int_Y A(t,y)\nabla_y \overline{w}_k(t,s,y) \cdot \nabla_y \bar{v}(y) + \int_Y \int_s^t G(t,s_1,y)\nabla_y \overline{w}_k(s_1,s,y)\,\mathrm{d}s_1 \cdot \nabla_y \bar{v}(y)$$
$$= -\int_Y G(t,s,y)\big(e_k + \nabla_y \overline{w}^A_k(s,y)\big) \cdot \nabla_y \bar{v}(y) \quad \forall \bar{v} \in H^1_{per}\big(Y;\mathbb{R}^2\big), \text{ a.e. in } \mathcal{S}.$$

Therefore, there exists a corrector $\bar{u} \in W^{2,1}(0,T; H^1_{per}(Y;\mathbb{R}^2))$ that can be written as

$$\bar{u}(t,x,y) = \overline{w}^A_k(t,y)u_k(t,x) + \overline{w}^0_k(t,y)u^0_k(x) + \int_0^t \overline{w}_k(t,s,y)u_k(s,x)\,\mathrm{d}s \tag{A.5}$$

and $\overline{w}^A_k(0,y) = -\overline{w}^0_k(0,y)$ implies $\bar{u}(0) = 0$.

Second step. Computation of u. We choose in (6) test functions $v \in L^2(\Omega;\mathbb{R}^6)$, hence for all $t \in (0,T)$ and for all $v \in L^2(\Omega;\mathbb{R}^6)$, we get:

$$\int_{\Omega\times Y} L(u+\nabla_y\bar{u})(t)\cdot v = \int_{\Omega\times Y} A(0)u^0\cdot v + \int_0^t \int_{\Omega\times Y} \big(\mathrm{curl}_x\ u_2(s)\cdot v_1 - \mathrm{curl}_x\ u_1(s)\cdot v_2 - j(s)\cdot v\big)\,\mathrm{d}s.$$

We let $t=0$ and obtain for all $v \in L^2(\Omega;\mathbb{R}^6)$:

$$\int_{\Omega\times Y} A(0)\big(u(0)+\nabla_y\bar{u}(0)\big)\cdot v = \int_{\Omega\times Y} A(0)u^0\cdot v,$$

from which we deduce $u(0)=u^0$. Hence we can write $\int_Y L(u+\nabla_y\bar{u})\,\mathrm{d}y = \mathcal{L}u + \mathcal{L}^0u^0$. The homogenized operator $\mathcal{L}^0$ is associated to the initial condition u^0 and can be written as

$$\mathcal{L}^0u^0(t,x) = \left(\int_Y L\nabla_y\overline{w}_k^0(t,y)\,\mathrm{d}y\right)u_k^0(x) = \left(\int_Y \mathbb{L}^0(t,y)\,\mathrm{d}y\right)u^0(x),$$

where the matrix $\mathbb{L}^0$ is described by its columns:

$$\mathbb{L}_k^0(t,y) = A(t,y)\nabla_y\overline{w}_k^0(t,y) + \int_0^t G(t,s,y)\nabla_y\overline{w}_k^0(s,y)\,\mathrm{d}s.$$

The homogenized operator $\mathcal{L}$ is given by $\mathcal{L}u(t,x) = \mathcal{A}(t)u(t,x) + \int_0^t \mathcal{G}(t,s)u(s,x)\,\mathrm{d}s$ with the new homogenized matrices $\mathcal{A}$, $\mathcal{G}$:

$$\mathcal{A}(t) = \int_Y \mathbb{A}(t,y)\,\mathrm{d}y, \qquad \mathcal{G}(t,s) = \int_Y \mathbb{G}(t,s,y)\,\mathrm{d}y, \tag{A.6}$$

where the matrices $\mathbb{A}$, $\mathbb{G}$ are described by their columns:

$$\begin{cases} \mathbb{A}_k(t,y) = A(t,y)(e_k + \nabla_y\overline{w}_k^A(t,y)), \\ \mathbb{G}_k(t,s,y) = G(t,s,y)(e_k+\nabla_y\overline{w}_k^A(s,y)) + A(t,y)\nabla_y\overline{w}_k(t,s,y) \\ \quad + \int_s^t G(t,s_1,y)\nabla_y\overline{w}_k(s_1,s,y)\,\mathrm{d}s_1. \end{cases}$$

This yields the limit evolution problem (7):

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}u\cdot v = \int_\Omega \big(\mathrm{curl}\,u_2\cdot v_1 - \mathrm{curl}\,u_1\cdot v_2\big) - \int_\Omega \left(j + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^0u^0\right)\cdot v \quad \forall v \in L^2\big(\Omega;\mathbb{R}^6\big).$$

Since the term $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^0u^0$ plays the same role as a source term, let us denote $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^0u^0 = \mathcal{J}^0$. As in the general case of diffusion problems, we can show that the homogenized matrix $\mathcal{A}$ is *symmetric and coercive*. It is easy to check the following regularity of the *effective* matrices and extra source:

$$\mathcal{A} \in W^{2,1}\big(0,T;\mathbb{R}^{36}\big), \quad \mathcal{G} \in \mathcal{W}^{2,1}\big(\mathcal{S};\mathbb{R}^{36}\big), \quad \mathcal{J}^0 \in W^{1,1}\big(0,T;L^\infty\big(\Omega;\mathbb{R}^6\big)\big). \tag{A.7}$$

The uniqueness of the solution to (7) results from Proposition 1.

Third step. Computation of $\bar{\bar{u}}$. We choose in (6) test functions v as $v(x,y) = \phi(x)\bar{\bar{v}}(y)$, $\phi \in L^2(\Omega)$, $\bar{\bar{v}} \in L^2(Y;\mathbb{R}^6)$, this leads to

$$\begin{pmatrix} \mathrm{curl}_y\, \bar{\bar{u}}_2(t,x,y) \\ -\mathrm{curl}_y\, \bar{\bar{u}}_1(t,x,y) \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\big(L(u+\nabla_y\bar{u})(t,x,y)\big) - A(0,y)u^0(x).$$

It is easy to check that $\mathrm{div}_y\, \frac{\mathrm{d}}{\mathrm{d}t}(L(u+\nabla_y\bar{u})) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}(L(u+\nabla_y\bar{u})) \in L^\infty(0,T;L^2(\Omega \times Y))$; hence, once u and $\bar{u}$ are computed, we get the expression of $\bar{\bar{u}}$ by using the relation $\mathrm{curl}_y(\mathrm{curl}_y\, \bar{\bar{u}}) = -\Delta_y\bar{\bar{u}}$ since $\mathrm{div}_y\, \bar{\bar{u}} = 0$. More precisely

$$\begin{pmatrix} -\Delta_y\bar{\bar{u}}_2(t,x,y) \\ \Delta_y\bar{\bar{u}}_1(t,x,y) \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{curl}_y\, L(u+\nabla_y\bar{u})(t,x,y) - \mathrm{curl}_y\, A(0,y)u^0(x), \tag{A.8}$$

with

$$L(u+\nabla_y\bar{u})(t,x,y) = \mathbb{A}(t,y)u(t,x) + \int_0^t \mathbb{G}(t,s,y)u(s,x)\,\mathrm{d}s + \mathbb{L}^0(t,y)\,u^0(x).$$

As we did for the first corrector, we may compute the second corrector $\bar{\bar{u}}$ as a linear combination of u. □

Remarks. (i) When $G(t,s) = B(s) + C(t-s)$ with $B \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ and $C \in W^{2,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$, we immediately find the definitions and regularities of the effective matrices as follows:

$$\begin{aligned}
&\mathcal{A}(t) = \int_Y \mathbb{A}(t,y)\,\mathrm{d}y, \qquad \mathcal{B}(t) = \int_Y \mathbb{B}(t,y)\,\mathrm{d}y,\\
&\mathcal{C}(t,s) = \int_Y \mathbb{C}(t,s,y)\,\mathrm{d}y, \qquad \mathcal{J}^0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_Y \mathbb{L}^0(t,y)\,\mathrm{d}y\,u^0(x),\\
&\mathcal{A} \in W^{2,1}\big(0,T;\mathbb{R}^{36}\big), \qquad \mathcal{B} \in W^{1,1}\big(0,T;\mathbb{R}^{36}\big),\\
&\mathcal{C} \in \mathcal{W}^{2,1}\big(\mathcal{S};\mathbb{R}^{36}\big), \qquad \mathcal{J}^0 \in W^{1,1}\big(0,T;L^\infty\big(\Omega;\mathbb{R}^6\big)\big),
\end{aligned} \tag{A.9}$$

with

$$\begin{cases} \mathbb{A}_k(t,y) = A(t,y)(e_k + \nabla_y \overline{w}_k^A(t,y)), \quad \mathbb{B}_k(t,y) = B(t,y)(e_k + \nabla_y \overline{w}_k^A(t,y)), \\ \mathbb{C}_k(t,s,y) = C(t-s)(e_k + \nabla_y \overline{w}_k^A(s,y) + A(t,y)\nabla_y \overline{w}_k(t,s,y) \\ \quad + \int_s^t (B(s_1,y) + C(t-s_1,y))\nabla_y \overline{w}_k(s_1,s,y))\,\mathrm{d}s_1, \\ \mathbb{L}_k^0(t,y) = A(t,y)\nabla_y \overline{w}_k^0(t,y) + \int_0^t B(s,y)\nabla_y \overline{w}_k^0(s,y)\,\mathrm{d}s \\ \quad + \int_0^t C(t-s,y)\nabla_y \overline{w}_k^0(s,y)\,\mathrm{d}s. \end{cases}$$

(ii) When, in addition the matrices A and B are independent of t, we can take matrix C in $W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ and the correctors take a simpler form; $\overline{w}^A \in H^1_{\mathrm{per}}(Y;\mathbb{R}^2)$ is independent of t, $\overline{w}^0 \in W^{2,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^2))$, and $\overline{w} \in W^{1,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^2))$ depends on only one variable. They solve the following variational problems satisfied for all $\bar{v} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^2)$:

- $$\int_Y A(y)\big(e_k + \nabla_y \overline{w}_k^A(y)\big)\cdot \nabla_y \bar{v}(y)\,\mathrm{d}y = 0,$$
- $$\int_Y A(y)\nabla_y \overline{w}_k^0(s,y)\cdot\nabla_y \bar{v}(y) + \int_Y\int_0^t B(y)\nabla_y \overline{w}_k^0(s,y)\,\mathrm{d}s\cdot\nabla_y \bar{v}(y) + \int_Y\int_0^t C(t-s,y)\nabla_y \overline{w}_k^0(s,y)\,\mathrm{d}s\cdot\nabla_y\bar{v}(y) = \int_Y A(y)e_k\cdot\nabla_y \bar{v}(y),$$
- $$\int_Y \left(A(y)\nabla_y \overline{w}_k(t,y) + \int_0^t B(y)\nabla_y \overline{w}_k(s,y)\,\mathrm{d}s + \int_0^t C(t-s,y)\nabla_y \overline{w}_k(s,y)\,\mathrm{d}s\right)\cdot\nabla_y \bar{v}(y) = -\int_Y \big(B(y)\big(e_k + \nabla_y \overline{w}_k^A(y)\big) + C(t,y)\big(e_k + \nabla_y \overline{w}_k^A(y)\big)\big)\cdot\nabla_y \bar{v}(y),$$
  a.e. in $(0,T)$.

The difference with previous situation (i) is that now $\mathcal{C} \in W^{1,1}(0,T;\mathbb{R}^{36})$ reads:

$$\mathcal{C}_k(t,y) = \int_Y C(t)\big(e_k + \nabla_y \overline{w}_k^A(y)\big) + \int_Y A(y)\nabla_y \overline{w}_k(t,y) + \int_Y\int_0^t \big(B(y) + C(t-s,y)\big)\nabla_y \overline{w}_k(s,y)\,\mathrm{d}s. \tag{A.10}$$

This case, with weaker regularity on matrix C, is the one we investigate by a frequency approach.

A.5. Proof of Proposition 5

First we show that the sequence $u^\alpha(p,x)$ is uniformly bounded in $L^2(\Omega)$, for all $p > p_0$, and then by writing:

$$Mu^\alpha(p,x) = \big(pA^\alpha(x) + B^\alpha(x) + p\widehat{C}^\alpha(p,x)\big)\hat{u}^\alpha(p,x) - \hat{j}^\alpha(p,x) - A^\alpha(x)u^{\alpha,0}(x),$$

we deduce that there exists $c > 0$ such that $\|\operatorname{curl}\hat{u}^\alpha(p)\|_{L^2(\Omega)} < c$ for all $p > p_0$.

A.6. Proof of Lemma 6

The proof follows point by point that of Lemma 3.

A.7. Proof of Theorem 7

The first corrector is given by:

$$\underline{u}(p,x,y) = \underline{w}_k^A(y)\hat{u}_k(p,x) + \underline{w}_k^0(p,y)u_k^0(x) + \underline{w}_k(p,y)\hat{u}_k(p,x), \tag{A.11}$$

where $\underline{w}_k^A \in H^1_{\text{per}}(Y;\mathbb{R}^2)$, $\underline{w}_k \in L^\infty(p_0,\infty;H^1_{\text{per}}(Y;\mathbb{R}^2))$, $\underline{w}_k^0 \in L^\infty(p_0,\infty;H^1_{\text{per}}(Y;\mathbb{R}^2))$ are respectively the unique solution to the following variational problems posed for all $\bar{v} \in H^1_{\text{per}}(Y;\mathbb{R}^2)$:

- $\displaystyle\int_Y A\big(e_k + \nabla_y \underline{w}_k^A\big)\cdot\nabla_y\bar{v} = 0,$
- $\displaystyle\int_Y \big(pA + B + p\widehat{C}\big)\nabla_y\underline{w}_k\cdot\nabla_y\bar{v} = -\int_Y \big(B + p\widehat{C}\big)\big(e_k + \nabla_y\underline{w}_k^A\big)\cdot\nabla_y\bar{v},$
- $\displaystyle\int_Y \big(pA + B + p\widehat{C}\big)\nabla_y\underline{w}_k^0\cdot\nabla_y\bar{v} = \int_Y Ae_k\cdot\nabla_y.$

Hence we get the following expressions of the *effective* matrices and extra source:

$$\begin{cases} \mathcal{A}_k = \int_Y A(y)(e_k + \nabla_y\underline{w}_k^A(y)), \quad \mathcal{B}_k = \int_Y B(y)(e_k + \nabla_y\underline{w}_k^A(y)), \\ \widehat{\mathcal{C}}_k(p) = \int_Y \widehat{C}(p,y)(e_k + \nabla_y\underline{w}_k^A(y)) \\ \quad + \int_Y (A(y) + \frac{1}{p}B(y) + \widehat{C}(p,y))\nabla_y\underline{w}_k(p,y), \\ \widehat{\mathcal{J}}^0(p,x) = \int_Y (pA(y) + B(y) + p\widehat{C}(p,y))\nabla_y\underline{w}_k^0(p,y)u_k^0(x), \end{cases} \tag{A.12}$$

with the regularity $\widehat{\mathcal{C}} \in W^{1,1}(p_0,\infty;\mathbb{R}^{36})$, $\widehat{\mathcal{J}}^0 \in W^{1,1}(p_0,\infty;L^2(\Omega;\mathbb{R}^6))$.

To establish the strong convergences, we first note that matrix $A^\alpha(x) + \widehat{C}^\alpha(p, x)$ is uniformly coercive for almost all $(p, y) \in]p_0, +\infty[\times Y$ and next we follow the same steps as in the time-formulation.

With assumptions of Theorem 7 it is easy to check that $\overline{w}^{\,A} = \underline{w}^A$ and that $\underline{w}^0, \underline{w}, \widehat{\mathcal{C}}, \widehat{\mathcal{J}}^0$ are the Laplace transforms of respectively $\overline{w}^{\,0}, \overline{w}, \mathcal{C}, \mathcal{J}^0$.

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