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On character amenability of Banach algebras

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ABSTRACT

We continue our work [E. Kaniuth, A.T. Lau, J. Pym, On φ-amenability of Banach algebras, Math. Proc. Cambridge Philos. Soc. 144 (2008) 85–96] in the study of amenability of a Banach algebra A defined with respect to a character φ of A. Various necessary and sufficient conditions of a global and a pointwise nature are found for a Banach algebra to possess a φ-mean of norm 1. We also completely determine the size of the set of φ-means for a separable weakly sequentially complete Banach algebra A with no φ-mean in A itself. A number of illustrative examples are discussed.

0. Introduction

The notion of an amenable Banach algebra was defined and studied in the seminal work of Johnson [10]. One of the fundamental results was that for a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if the group G is amenable. Since then amenability has become a major issue in Banach algebra theory and in harmonic analysis.

In this paper we continue our recent investigation [11] of a concept which might be referred to as amenability with respect to a character. Let A be an arbitrary Banach algebra and φ a character of A, that is, a homomorphism from A onto $\mathbb{C}$. We call A *φ-amenable* if there exists a bounded linear functional m on A^* satisfying $\langle m, \varphi\rangle = 1$ and $\langle m, f\cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$ and $f \in A^*$. Here $f \cdot a \in A^*$ is defined by $\langle f\cdot a, b\rangle = \langle f, ab\rangle$, $b \in A$. Any such m is called a *φ-mean*. This concept considerably generalizes the notion of left amenability for F-algebras which was introduced and studied by the second author in [14].

Note that in [16] a Banach algebra is called right character amenable if it is φ-amenable for each character φ and has a bounded right approximate identity. Note also that for a locally compact group G (respectively, a discrete semigroup S), the group algebra $L^1(G)$ (respectively, the semigroup algebra $l^1(S)$) is amenable with respect to the trivial character 1 precisely when G is amenable (respectively, S is left amenable). However, $l^1(\mathbb{N})$ is not amenable since it does not have a bounded approximate identity.

We briefly outline the contents of this paper. After introducing some notation, we give two characterizations (in terms of cohomology groups and a Hahn–Banach type extension property) of φ-amenability, which are close to results in [11]. In Section 2 we mainly focus on φ-means of norm 1. We establish various criteria for their existence (Corollary 2.3, Theorems 2.4 and 2.7). Pointwise conditions, in terms of elements $f \in A^*$ or $a \in \ker\varphi$, the kernel of φ, are given that ensure the existence of φ-means of norm 1 (Theorem 2.1, Proposition 2.2 and Theorem 2.4).

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In Section 3, we concentrate on weakly sequentially complete Banach algebras. We show that if there is no φ-mean in A itself, but there exists a so-called sequential bounded approximate φ-mean, then A admits at least $2^{\mathfrak{c}}$ φ-means, and there are no more if A is separable (Theorem 3.1). We also relate the existence of φ-means to Arens regularity of A (Theorem 3.9). A result of a flavour similar to that of Theorem 3.1 is obtained in Theorem 4.1. It implies that if A is a separable F-algebra and ϵ denotes the identity of the von Neumann algebra A^*, then there are $2^{\mathfrak{c}}$ ϵ-means of norm 1 with the additional property that $\|m-n\|=2$ for any two of them.

Finally, in Section 5 we present illustrative examples such as Lipschitz algebras and algebras $L^p(G)$, where G is a compact group.

1. Preliminaries and some basic results

In this paper, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product which is defined as follows (compare [2, Section 10]). For $a,b\in A$, $f\in A^*$ and $m,n\in A^{**}$, the elements $f\cdot a$ and $m\cdot f$ of A^* and $mn\in A^{**}$ are defined by

$$\langle f\cdot a,b\rangle=\langle f,ab\rangle,\qquad \langle m\cdot f,b\rangle=\langle m,f\cdot b\rangle \quad\text{and}\quad \langle mn,f\rangle=\langle m,n\cdot f\rangle,$$

respectively. With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**}. Alternatively, the multiplication on A^{**} can be defined by using iterated limits as follows. For $m,n\in A^{**}$, let

$$mn=w^*\text{-}\lim_{a\to m}\Big(w^*\text{-}\lim_{b\to n}ab\Big).$$

In general, the multiplication $(m,n)\to mn$ is not separately continuous with respect to the w^*-topology on A^{**}. But, for fixed $n\in A^{**}$, the mapping $m\to mn$ is w^*-continuous, and also for fixed $a\in A$, the mapping $m\to am$ is w^*-continuous. Moreover, for all $m,n\in A^{**}$ and $\varphi\in\Delta(A)$, the set of all homomorphisms from A onto $\mathbb{C}$, $\langle mn,\varphi\rangle=\langle m,\varphi\rangle\langle n,\varphi\rangle$. Consequently, each $\varphi\in\Delta(A)$ extends uniquely to some element φ^{**} of $\Delta(A^{**})$. The kernel of φ^{**}, $\ker\varphi^{**}$, contains $\ker\varphi$ in the same sense that A^{**} naturally contains A. Since each of these ideals has codimension 1, the theory of second polars shows that $\ker\varphi$ is w^*-dense in $\ker\varphi^{**}$ and that $\ker\varphi^{**}=(\ker\varphi)^{**}$.

The Banach algebra A is said to be *φ-amenable* if there exists $m\in A^{**}$ such that $\langle m,\varphi\rangle=1$ and $\langle m,f\cdot a\rangle=\varphi(a)\langle m,f\rangle$ for all $f\in A^*$ and $a\in A$, and any such m is called a *φ-mean*. By Theorem 1.4. of [11] the φ-means are nothing but the w^*-cluster points of bounded nets $(u_\gamma)_\gamma$ in A with $\varphi(u_\gamma)=1$ for all γ and $\|au_\gamma-\varphi(a)u_\gamma\|\to 0$ for all $a\in A$. Consequently, we call such a net $(u_\gamma)_\gamma$ a *bounded approximate φ-mean*. Given a φ-mean m, the net $(u_\gamma)_\gamma$ can be chosen so that $\|u_\gamma\|\to\|m\|$ (compare the proof of [11, Theorem 1.4]).

If X is a Banach A-module, then so is the dual X^* of X with the module actions given by

$$\langle a\cdot f,x\rangle=\langle f,x\cdot a\rangle\quad\text{and}\quad\langle f\cdot a,x\rangle=\langle f,a\cdot x\rangle,$$

$a\in A$, $x\in X$, $f\in X^*$. In the following theorem $H^1(A,X^*)$ denotes the first cohomology group of A with coefficients in X^*.

Theorem 1.1. *Let A be a Banach algebra and $\varphi\in\Delta(A)$. Then the following three conditions are equivalent.*

(i) *A is φ-amenable.*
(ii) *If X is a Banach A-bimodule such that $a\cdot x=\varphi(a)x$ for all $x\in X$ and $a\in A$, then $H^1(A,X^*)=\{0\}$.*
(iii) *Give $(\ker\varphi)^{**}$ a second A-bimodule structure by taking the left action to be $a\cdot m=\varphi(a)m$ for $m\in A^{**}$ and taking the right action to be the natural one. Then any continuous derivation $D:A\to(\ker\varphi)^{**}$ is inner.*

Proof. The equivalence of (i) and (ii) has been shown in [11, Theorem 1.1]. Trivially, (ii) implies (iii), and therefore we only have to show (iii) $\Rightarrow$ (i). Choose any $b\in A$ with $\varphi(b)=1$. Then $Da=ab-ba, a\in A$, defines a derivation from A into $(\ker\varphi)^{**}$. By (iii), D is inner, so there is $m\in(\ker\varphi)^{**}$ such that $Da=a(-m)-(-m)a$ for all $a\in A$. Then

$$a(b+m)=(b+m)a=\varphi(a)(b+m)$$

for all $a\in A$ and $\langle b+m,\varphi\rangle=\varphi(b)=1$. So $b+m$ is a φ-mean. □

The implication (iii) $\Rightarrow$ (ii) in the above shows that if $H^1(A,X^*)=\{0\}$ for the particular case in which $X=(\ker\varphi)^*$, then all such cohomology groups are zero. The following result should be compared with Theorem 1.5 of [11].

Theorem 1.2. *Let A be a Banach algebra and $\varphi\in\Delta(A)$. Then the following two conditions are equivalent.*

(i) *A is φ-amenable.*
(ii) *If X is any Banach A-module and Y is any Banach A-submodule of X and $g\in Y^*$ is such that the left action of A on g has the form $a\cdot g=\varphi(a)g$ for all $a\in A$, then g extends to some $f\in X^*$ such that $a\cdot f=\varphi(a)f$ for all $a\in A$.*

Proof. (i) ⇒ (ii) Let $\widetilde{g} \in X^*$ such that $\widetilde{g}$ extends g and $\|\widetilde{g}\| = \|g\|$. If $a \in A$ satisfies $\varphi(a) = 1$, then $a \cdot \widetilde{g}$ also extends g. Since A is φ-amenable, there exists a net $(u_\gamma)_\gamma$ in A such that, for all γ, $\varphi(u_\gamma) = 1$ and $\|u_\gamma\| \leqslant C$ for some constant $C > 0$ and $\|au_\gamma - \varphi(a)u_\gamma\| \to 0$ for all $a \in A$ [11, Theorem 1.4]. Then $u_\gamma \cdot \widetilde{g}$ extends g and we may assume that $\|u_\gamma \cdot \widetilde{g}\| \leqslant C\|g\| + 1$ for all γ. After passing to a subnet if necessary, we can also assume that $u_\gamma \cdot \widetilde{g} \to f$ in the w^*-topology for some $f \in X^*$. Clearly, f extends g. Taking w^*-limits, we obtain

$$a \cdot f = \lim_\gamma a \cdot (u_\gamma \cdot \widetilde{g}) = \lim_\gamma (au_\gamma) \cdot \widetilde{g} = \lim_\gamma \big[\big(au_\gamma - \varphi(a)u_\gamma\big) \cdot \widetilde{g} + \varphi(a)u_\gamma \cdot \widetilde{g}\big] = \varphi(a)f$$

for all $a \in A$. So (ii) holds.

(ii) ⇒ (i) Take $X = A^*$ and $Y = \mathbb{C}\varphi$. Let $\varphi^* \in Y^*$ be defined by $\langle \varphi^*, \varphi\rangle = 1$. Then the left action of A on φ^* is given by $a \cdot \varphi^* = \varphi(a)\varphi^*$. By hypothesis, there exists $m \in A^{**}$ such that $m|_Y = \varphi^*$ and $a \cdot m = \varphi(a)m$ for all $a \in A$. Since $\langle m, \varphi\rangle = \langle \varphi^*, \varphi\rangle = 1$, m is a φ-mean. □

Using w^*-continuity, we easily see that an element $m \in A^{**}$ is a φ-mean for A if and only if for all $n \in A^{**}$ we have $nm = \varphi^{**}(n)m$. It is tempting to introduce a new general concept by saying that, when φ is a complex homomorphism on a complex algebra B, m is a *φ-right zero* if $nm = \varphi(n)m$ for all $n \in B$ (the term 'right zero' in this context comes from the measure algebra on a semigroup with a right zero). However, this is hardly worthwhile, as it would merely be giving a new name to a φ-mean which lies in B. But this viewpoint does reduce the idea of a φ-mean to a purely algebraic one, and sometimes it is easy to prove results in this context and then interpret them as applying to Banach algebras. It is trivial to notice that A has a φ-mean if and only if A^{**} has a φ^{**}-mean which lies in A^{**}. The next proposition and its corollary provide an example of this technique.

Proposition 1.3. *Let B be a complex algebra and $\varphi : B \to \mathbb{C}$ a homomorphism. Let J be an ideal in B with $J \subseteq \ker\varphi$ and let $\widetilde{\varphi} : B/J \to \mathbb{C}$ be the homomorphism induced by φ. If J has a right identity and B/J has a $\widetilde{\varphi}$-mean in B/J, then B has a φ-mean in B.*

Proof. Let $q : B \to B/J$, so that $\varphi = \widetilde{\varphi} \circ q$. Let e be a right identity for J and let $m \in B$ be such that $q(m)$ is a $\widetilde{\varphi}$-mean for B/J. Since $q(e) = 0$ we find for all $x \in B$,

$$q(x)q(m - me) = q(x)q(m) = \widetilde{\varphi}\big(q(x)\big)q(m) = \varphi(x)q(m - me).$$

This shows that $x(m - me) - \varphi(x)(m - me) \in J$. Since e is a right identity for J and $(m - me)e = 0$, we see that in fact $x(m - me) - \varphi(x)(m - me) = 0$, so that $m - me$ is a φ-mean for B. □

Corollary 1.4. *Let A be a Banach algebra, $\varphi \in \Delta(A)$ and I a closed ideal in A with $I \subseteq \ker\varphi$. Suppose that I has a bounded right approximate identity and that A/I is $\widetilde{\varphi}$-amenable, where $\widetilde{\varphi} \in \Delta(A/I)$ is the homomorphism induced by φ. Then A is φ-amenable.*

Proof. The statement follows from Proposition 1.3 on taking $B = A^{**}$ and $J = I^{**}$. In fact, since I has a bounded right approximate identity, I^{**} has a right identity, and since $B/J = A^{**}/I^{**} = (A/I)^{**}$ and A/I is $\widetilde{\varphi}$-amenable and $\widetilde{\varphi}^{**} = \widetilde{\varphi^{**}}$, B/J is $\widetilde{\varphi^{**}}$-amenable. Thus B/J has a $\widetilde{\varphi^{**}}$-mean and the proposition shows that B has a φ^{**}-mean. This says that A is φ-amenable. □

Note that this corollary generalizes Proposition 2.1 of [11].

2. φ-Means of norm 1

Let A be a Banach algebra and $\varphi \in \Delta(A)$. In this section we establish several criteria for A to possess a φ-mean of norm 1. We start by showing that the existence of such a mean is a pointwise property.

Theorem 2.1. *Let A be any Banach algebra and $\varphi \in \Delta(A)$. Suppose that for each $f \in A^*$ there exists $m_f \in A^{**}$ such that $\|m_f\| = \langle m_f, \varphi\rangle = 1$ and $\langle m_f, f \cdot a\rangle = \varphi(a)\langle m_f, f\rangle$ for all $a \in A$. Then A has a φ-mean of norm 1.*

Proof. Define a subset S of A^{**} by

$$S = \big\{m \in A^{**}\colon\ \|m\| = \langle m, \varphi\rangle = 1\big\} = \big\{m \in A^{**}\colon\ \|m\| \leqslant 1,\ \langle m, \varphi\rangle = 1\big\}.$$

Then S is w^*-compact and easily seen to be a semigroup for the first Arens product. Let $\mathcal{F}$ denote the collection of all finite subsets F of A^*, and for every $F \in \mathcal{F}$, let

$$S_F = \big\{m \in S\colon\ \langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle \text{ for all } f \in F \text{ and } a \in A\big\}.$$

Then S_F is closed in S and $S_{F_1} \supseteq S_{F_2}$ whenever $F_1 \subseteq F_2$. Clearly, every $m \in \bigcap\{S_F\colon\ F \in \mathcal{F}\}$ is a φ-mean with $\|m\| = 1$. It therefore suffices to show that $S_F \neq \emptyset$ for each $F \in \mathcal{F}$. We achieve this by induction on the number of elements in F.

So suppose that some $m_1 \in S_F$ exists and let $g \in A^* \setminus F$ and set $h = m_1 \cdot g \in A^*$. By hypothesis, there exists $m_2 \in S_{\{h\}}$. Let $m = m_2 m_1 \in A^{**}$. Then $m \in S$ since S is a semigroup. For $f \in F$ and $a, b \in A$, we have

$$\bigl\langle m_1 \cdot (f \cdot a), b\bigr\rangle = \bigl\langle m_1, f \cdot (ab)\bigr\rangle = \varphi(a)\langle m_1, f\rangle \varphi(b).$$

Hence $m_1 \cdot (f \cdot a) = \varphi(a)\langle m_1, f\rangle\varphi$, and similarly $m_1 \cdot f = \langle m_1, f\rangle\varphi$. It follows that, for $f \in F$ and all $a \in A$,

$$\langle m, f \cdot a\rangle = \bigl\langle m_2, m_1 \cdot (f \cdot a)\bigr\rangle = \varphi(a)\langle m_1, f\rangle\langle m_2, \varphi\rangle = \varphi(a)\bigl\langle m_2, \langle m_1, f\rangle\varphi\bigr\rangle = \varphi(a)\langle m_2, m_1 \cdot f\rangle = \varphi(a)\langle m, f\rangle.$$

Moreover, for all $a \in A$,

$$\langle m, g \cdot a\rangle = \bigl\langle m_2, (m_1 \cdot g) \cdot a\bigr\rangle = \varphi(a)\langle m_2, m_1 \cdot g\rangle = \varphi(a)\langle m, g\rangle.$$

So $m \in S_{F\cup\{g\}}$, and this finishes the proof. □

Let A be a Banach algebra and $\varphi \in \Delta(A)$. For $f \in A^*$ and $\epsilon > 0$, let

$$K_{f,\epsilon} = \overline{\bigl\{u \cdot f\colon\ u \in A,\ \varphi(u) = 1,\ \|u\| \leqslant 1 + \epsilon\bigr\}}^{w^*} \subseteq A^*.$$

Clearly, $K_{f,\epsilon}$ is convex and w^*-compact, and so is $K_f = \bigcap_{\epsilon>0} K_{f,\epsilon}$.

Proposition 2.2. *For $f \in A^*$, the following conditions are equivalent.*

(i) *There exists $m \in A^{**}$ such that $\|m\| = 1$, $\langle m, \varphi\rangle = 1$ and $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$.*
(ii) *K_f contains $\lambda\varphi$ for some $\lambda \in \mathbb{C}$.*

In fact, $\mathbb{C}\varphi \cap K_f$ equals the set of all $\langle m, f\rangle\varphi$, where m is as in (i)*.*

Proof. Let m be as in (i), and let $(u_\gamma)_\gamma$ be a net in A such that $\varphi(u_\gamma) = 1$ for all γ, $\|u_\gamma\| \to 1$ and $u_\gamma \to m$ in the w^*-topology. Then

$$\langle u_\gamma \cdot f, a\rangle = \langle u_\gamma, f \cdot a\rangle \to \langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$$

for all $a \in A$, and hence $\langle m, f\rangle\varphi \in K_{f,\epsilon}$ for every $\epsilon > 0$.

Conversely, assume that $\lambda\varphi \in K_f$ and let $\epsilon > 0$. There exists a net $(u_{\gamma,\epsilon})_\gamma$ in A such that $\varphi(u_{\gamma,\epsilon}) = 1$, $\|u_{\gamma,\epsilon}\| \leqslant 1 + \epsilon$ for all γ and $\lambda\varphi = w^*\text{-}\lim_\gamma(u_{\gamma,\epsilon} \cdot f)$. Let n_ϵ be a w^*-cluster point of the net $(u_{\gamma,\epsilon})_\gamma$ in A^{**}. Then $\|n_\epsilon\| \leqslant 1 + \epsilon$, $\langle n_\epsilon, \varphi\rangle = 1$ and $\langle n_\epsilon, f \cdot a\rangle = \lambda\varphi(a)$ for all $a \in A$ since

$$\langle u_{\gamma,\epsilon}, f \cdot a\rangle = \langle u_{\gamma,\epsilon} \cdot f, a\rangle \to \lambda\varphi(a).$$

Let n be a w^*-cluster point of the net $(n_\epsilon)_\epsilon$. Then $\|n\| = 1$, $\langle n, \varphi\rangle = 1$ and

$$\langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle = \lambda\varphi(a)$$

for all $a \in A$. Finally, let $m = n^2 \in A^{**}$. Then $\langle m, \varphi\rangle = \langle n, \varphi\rangle^2 = 1$ and $\|m\| = 1$. Moreover,

$$\langle m, f\rangle = \langle n, n \cdot f\rangle = \langle n, \lambda\varphi\rangle = \lambda\langle n, \varphi\rangle = \lambda,$$

and hence, for all $a \in A$,

$$\langle m, f \cdot a\rangle = \bigl\langle n, (n \cdot f) \cdot a\bigr\rangle = \bigl\langle n, (\lambda\varphi) \cdot a\bigr\rangle = \lambda\varphi(a)\langle n, \varphi\rangle = \varphi(a)\langle m, f\rangle.$$

So m satisfies all the requirements in (i).

Actually, the above proof shows that $\lambda\varphi$ belongs to K_f if and only if $\lambda = \langle m, f\rangle$ for some $m \in A^{**}$ as in (i). □

As an immediate consequence of Proposition 2.2 and Theorem 2.1 we obtain

Corollary 2.3. *For a Banach algebra A and $\varphi \in \Delta(A)$, the following are equivalent.*

(i) *A admits a φ-mean of norm* 1.
(ii) *For each $f \in A^*$, $\mathbb{C}\varphi \cap K_f \neq \emptyset$.*

The next theorem, which is one of the main results of the paper, in particular shows that the existence of a φ-mean of norm 1 is a pointwise property in the sense that it follows from the existence of a certain functional on A^* associated with each of the elements of the ideal $\ker\varphi$.

Theorem 2.4. *For a Banach algebra A and $\varphi \in \Delta(A)$, the following four conditions are equivalent.*

(i) *There exists a φ-mean m such that $\|m\| = 1$.*
(ii) *There exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1$ for all γ, $\|u_\gamma\| \to 1$ and $\|au_\gamma\| \to |\varphi(a)|$ for all $a \in A$.*
(iii) *For each $a \in \ker\varphi$, there exists $m_a \in A^{**}$ with $\|m_a\| \leqslant 1$, $\langle m_a, \varphi\rangle = 1$ and $am_a = 0$.*
(iv) *For each $a \in \ker\varphi$ and $\epsilon > 0$, there exists $u \in A$ such that $\|u\| \leqslant 1 + \epsilon$, $\|au\| \leqslant \epsilon$ and $\varphi(u) = 1$.*

Proof. (ii) $\Rightarrow$ (iv) is clear. Also, (i) $\Rightarrow$ (iii) is simple: If m is a φ-mean, we can choose $m_a = m$ for all $a \in A$. Therefore, in order to establish the theorem it suffices to show the implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) There exists a net $(u_\gamma)_\gamma$ in A with the following properties: $\varphi(u_\gamma) = 1$ for all γ, $\|u_\gamma\| \to 1$ and $\|au_\gamma - \varphi(a)u_\gamma\| \to 0$ for all $a \in A$. Thus,

$$\big|\|au_\gamma\| - |\varphi(a)|\big| \leqslant \big|\|au_\gamma\| - \|\varphi(a)u_\gamma\|\big| + \big|\|\varphi(a)u_\gamma\| - |\varphi(a)|\big| \leqslant \|au_\gamma - \varphi(a)u_\gamma\| + |\varphi(a)| \cdot \big|\|u_\gamma\| - 1\big|.$$

(iii) $\Rightarrow$ (iv) Fix $a \in \ker\varphi$ and take any net $(u_\gamma)_\gamma$ in A such that $\|u_\gamma\| \leqslant 1$ and $u_\gamma \to m_a$ in the w^*-topology. Then $\varphi(u_\gamma) \to 1$. By replacing each u_γ with a scalar multiple of itself and taking a cofinal subnet, we may arrange that $\|u_\gamma\| \leqslant 1 + \epsilon$ and $\varphi(u_\gamma) = 1$ for all γ. Since w^*-$\lim au_\gamma = am_a = 0$ and $au_\gamma \in A$, 0 is in the weak closure of the set $(au_\gamma)_\gamma$, and therefore 0 is in the norm closure of the convex hull of $(au_\gamma)_\gamma$. The set $(u_\gamma)_\gamma$ being contained in the closed hyperplane $\{x \in A\colon \varphi(x) = 1\}$, we easily reach our conclusion.

(iv) $\Rightarrow$ (i) We claim that for every finite subset F of A and $\epsilon > 0$, there exists $u_{F,\epsilon}$ such that $\varphi(u_{F,\epsilon}) = 1$, $\|u_{F,\epsilon}\| \leqslant 1 + \epsilon$ and

$$\|au_{F,\epsilon} - \varphi(a)u_{F,\epsilon}\| \leqslant \epsilon$$

for all $a \in F$. Let $F = \{a_1, \ldots, a_k\}$, say, and choose $\delta > 0$ such that $(1+\delta)^{k+1} \leqslant 1 + \epsilon$. By hypothesis, there exists $u_0 \in A$ such that $\varphi(u_0) = 1$ and $\|u_0\| \leqslant 1 + \delta$. Since $a_1u_0 - \varphi(a_1)u_0 \in \ker\varphi$, again by (iv) there exists $u_1 \in A$ such that

$$\varphi(u_1) = 1, \quad \|u_1\| \leqslant 1 + \delta \quad \text{and} \quad \|(a_1u_0 - \varphi(a_1)u_0)u_1\| \leqslant \delta.$$

Likewise, $a_2u_0u_1 - \varphi(a_2)u_0u_1 \in \ker\varphi$ and hence there exists $u_2 \in A$ such that

$$\varphi(u_2) = 1, \quad \|u_2\| \leqslant 1 + \delta \quad \text{and} \quad \|(a_2u_0u_1 - \varphi(a_2)u_0u_1)u_2\| \leqslant \delta.$$

For $j = 1, 2$, we have $\|u_j\| \leqslant 1 + \delta$, $\varphi(u_j) = 1$ and

$$\|a_ju_0u_1u_2 - \varphi(a_j)u_0u_1u_2\| \leqslant \delta(1+\delta).$$

Proceeding inductively, we see that there exist u_j, $1 \leqslant j \leqslant k$, such that $\varphi(u_j) = 1$, $\|u_j\| \leqslant 1 + \delta$ and, for $i = 1, \ldots, j$,

$$\|a_iu_0u_1 \cdot \cdots \cdot u_j - \varphi(a_i)u_0u_1 \cdot \cdots \cdot u_j\| \leqslant \delta(1+\delta)^{j-1} \leqslant \epsilon.$$

In particular, when $j = k$, setting $u_{F,\epsilon} = \prod_{j=0}^k u_j$ gives us $\varphi(u_{F,\epsilon}) = 1$, $\|u_{F,\epsilon}\| \leqslant 1 + \epsilon$ and $\|au_{F,\epsilon} - \varphi(a)u_{F,\epsilon}\| \leqslant \epsilon$ for all $a \in F$. This proves the above claim.

Now, order the pairs (F, ϵ), $F \subseteq A$ finite, $\epsilon > 0$, in the obvious manner, and let m be a w^*-cluster point of the net $(u_{F,\epsilon})_{F,\epsilon}$ in A^{**}. Then $\|m\| \leqslant 1$ and $\langle m, \varphi\rangle = 1$ (and hence $\|m\| = 1$) and $am = \varphi(a)m$ for all $a \in A$. So m is the required φ-mean. □

Remark 2.5. Using methods similar to those employed in the proof of Theorem 2.4, the following can be shown. Let A be a Banach algebra and $\varphi \in \Delta(A)$. For $C > 0$, the following statements are equivalent.

(i) A has a φ-mean of norm C.
(ii) A contains an approximate φ-mean with norm bound C.
(iii) For each $a \in \ker\varphi$, there exists $m_a \in A^{**}$ with $\|m_a\| = C$, $\langle m_a, \varphi\rangle = 1$ and $am_a = 0$.
(iv) There exists a net $(u_\gamma)_\gamma$ in A with $\varphi(u_\gamma) = 1$ for all γ, $\|u_\gamma\| \to C$ and $au_\gamma \to 0$ for every $a \in \ker\varphi$.

For a Banach algebra A and $\varphi \in \Delta(A)$, let $N(A, \varphi)$ denote the set of all $f \in A^*$ with the following property: for each $\delta > 0$, there exists a sequence $(a_n)_n$ in A such that $\varphi(a_n) = 1$, $\|a_n\| \leqslant 1 + \delta$ for all n and $\|f \cdot a_n\| \to 0$. We now aim at a criterion for the existence of a φ-mean of norm 1 involving the set $N(A, \varphi)$ (Theorem 2.8. below).

Lemma 2.6. *For a Banach algebra A and $\varphi \in \Delta(A)$, the following hold.*

(i) $\varphi \notin N(A, \varphi)$.
(ii) *$N(A, \varphi)$ is closed in A^* and closed under scalar multiplication.*
(iii) *If A is commutative, then $N(A, \varphi)$ is closed under addition.*

Proof. (i) is immediate since $\varphi \cdot a = \varphi$ for all $a \in A$ with $\varphi(a) = 1$.

(ii) Let $f_n \in A^*$, $n \in \mathbb{N}$, and $f \in A^*$ such that $f_n \to f$. For every n there exists $a_n \in A$ such that $\varphi(a_n) = 1$, $\|a_n\| \leqslant 1 + \frac{1}{n}$ and $\|f_n \cdot a_n\| \leqslant \frac{1}{n}$. Then $\|f \cdot a_n\| \leqslant \|f - f_n\| \cdot \|a_n\| + \frac{1}{n}$ for all n, whence $f \in N(A, \varphi)$.

(iii) Let $f_1, f_2 \in N(A, \varphi)$ and $\delta > 0$. If $a_j \in A$, $j = 1, 2$, are such that $\varphi(a_j) = 1$, $\|a_j\| \leqslant 1 + \delta$ and $\|f_j \cdot a_j\| \leqslant \delta$, then since A is commutative,

$$\big\|(f_1 + f_2) \cdot (a_1 a_2)\big\| \leqslant \|f_1 \cdot a_1\| \cdot \|a_2\| + \|f_2 \cdot a_2\| \cdot \|a_1\| \leqslant 2\delta(1 + \delta).$$

It follows that $f_1 + f_2 \in N(A, \varphi)$. □

Lemma 2.7. *Suppose that A admits a φ-mean of norm 1. Then $N(A, \varphi)$ is a subspace of A^*.*

Proof. Let $J = \{a \in A\colon\ \varphi(a) = 1\}$ and let $\epsilon > 0$. Since A has a φ-mean of norm 1, there exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1$ and $\|u_\gamma\| \leqslant 1 + \epsilon$ for all γ and $\|au_\gamma - u_\gamma\| \to 0$ for every $a \in J$.

Now let $f_1, f_2 \in N(A, \varphi)$. Given $\epsilon > 0$, there exist $a_1, a_2 \in J$ such that $\|f_j \cdot a_j\| \leqslant \epsilon$ and $\|a_j\| \leqslant 1 + \epsilon$, $j = 1, 2$. By the first paragraph, there exists $u \in A$ with $\|u\| \leqslant 1 + \epsilon$, $\varphi(u) = 1$ and

$$\|a_1 u - a_2 u\| \leqslant \|a_1 u - u\| + \|u - a_2 u\| < \epsilon.$$

Then

$$\begin{aligned}\big\|(f_1 + f_2) \cdot (a_1 u)\big\| &\leqslant \big\|f_1 \cdot (a_1 u)\big\| + \big\|f_2 \cdot (a_1 u) - f_2 \cdot (a_2 u)\big\| + \big\|f_2 \cdot (a_2 u)\big\| \\ &\leqslant \|f_1 \cdot a_1\| \cdot \|u\| + \|f_2\| \cdot \|a_1 u - a_2 u\| + \|f_2 \cdot a_2\| \cdot \|u\| \\ &\leqslant \epsilon(1 + \epsilon) + \epsilon\|f_2\| + \epsilon(1 + \epsilon) \\ &= \epsilon\big(2 + 2\epsilon + \|f_2\|\big).\end{aligned}$$

Since $\varphi(a_1 u) = 1$ and $\|a_1 u\| \leqslant (1 + \epsilon)^2$ and $\epsilon > 0$ is arbitrary, it follows that $f_1 + f_2 \in N(A, \varphi)$. □

Theorem 2.8. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following two conditions are equivalent.*

(i) *There exists a φ-mean m with $\|m\| = 1$.*
(ii) *$N(A, \varphi)$ is a subspace of A^* and $f \cdot a - f \in N(A, \varphi)$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$.*

Proof. Let m be a φ-mean of norm 1. By Lemma 2.7, $N(A, \varphi)$ is a subspace of A^*. Let $f \in A^*$ and $a \in A$ with $\varphi(a) = 1$. There exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1$, $\|u_\gamma\| \to 1$ and $\|au_\gamma - u_\gamma\| \to 0$. Since $\|(f \cdot a - f) \cdot u_\gamma\| \leqslant \|f\| \cdot \|au_\gamma - u_\gamma\|$, it follows that $f \cdot a - f \in N(A, \varphi)$.

Conversely, suppose that $N(A, \varphi)$ is a subspace of A^* and that (ii) holds. Since $\varphi \notin N(A, \varphi)$ and $\|\varphi\| = 1$, by the Hahn–Banach theorem there exists $m \in A^{**}$ such that $\|m\| = \langle m, \varphi\rangle = 1$ and $m|_{N(A,\varphi)} = 0$. Then, by (ii), $\langle m, f \cdot a\rangle = \langle m, f\rangle$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$ and hence $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$. □

We shall see in Example 5.3 that if $\|m\| > 1$, it can even happen that $N(A, \varphi) = \{0\}$.

The following corollary is an immediate consequence of Lemma 2.6 and Theorem 2.8.

Corollary 2.9. *If A is a commutative Banach algebra and $\varphi \in \Delta(A)$, then A has a φ-mean of norm 1 if and only if $f \cdot a - f \in N(A, \varphi)$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$.*

Before proceeding, recall that an *F-algebra A* is a Banach algebra which is the predual of a von Neumann algebra M such that the identity ϵ of M is a multiplicative linear functional on A. In this case, the ϵ-means of norm 1 are nothing but the *topologically left invariant means* (TLIM) on A^* [14]. Examples of F-algebras include the group algebra, the Fourier algebra and the Fourier–Stieltjes algebra of a locally compact group. Other examples are the measure algebra of a locally compact semigroup and the predual of a Hopf–von Neumann algebra (see [14] for details and also [17]).

Let A be a Banach algebra and $\varphi \in \Delta(A)$. We say that an element a of A is *φ-maximal* if it satisfies $\|a\| = \varphi(a) = 1$. Let $P_1(A, \varphi)$ denote the collection of all φ-maximal elements of A. When A is an F-algebra and φ is the identity of the von Neumann algebra A^*, the φ-maximal elements are precisely the positive linear functionals of norm 1 on A^* and hence span A. However, in general $P_1(A, \varphi)$ can be quite small (see Examples 5.2 and 5.3).

Let $X(A, \varphi)$ denote the closed linear span of $P_1(A, \varphi)$. Then $X(A, \varphi)$ is a closed subalgebra of A.

Proposition 2.10. *Let A be a commutative Banach algebra and $\varphi \in \Delta(A)$. If $X(A, \varphi) = A$, then A has a φ-mean of norm 1.*

Proof. Let $K = \{m \in A^{**}\colon\ \|m\| = \langle m, \varphi\rangle = 1\}$. Then K is a w^*-compact convex subset of A^{**}. For each $a \in P_1(A, \varphi)$, let $T_a : K \to K$ denote the map $m \to am$. Then $a \to T_a$ is a representation of the commutative semigroup $P_1(A, \varphi)$ as w^*–w^*-continuous affine mappings from K into K. Therefore, by the Markov–Kakutani fixed point theorem (see [5]), there exists $m \in K$ with $am = m$ for all $a \in P_1(A, \varphi)$. For all $a \in A$, it then follows that $am = \varphi(a)m$, and hence m is a φ-mean. □

Remark 2.11. Let A be a Banach algebra such that A is a left or right ideal in A^{**}. Let $\varphi \in \Delta(A)$ and suppose that there exists a φ-mean m. Then there exists a φ-mean in A itself.

To see this, fix $a \in A$ with $\varphi(a) = 1$. If A is a right ideal in A^{**}, then $m = \varphi(a)m = am \in A$. If A is a left ideal in A^{**}, then

$$\langle ma, \varphi\rangle = \langle m, a \cdot \varphi\rangle = \langle m, \varphi\rangle = 1$$

and $b(ma) = \varphi(b)ma$ for all $b \in A$, whence $ma \in A$ is φ-mean.

3. Weakly sequentially complete Banach algebras

A φ-mean of a Banach algebra A is an element of the second dual of A. There are some aspects of the theory of second duals which are particularly striking for weakly sequentially complete algebras. In this section we offer some results which are relevant to φ-means.

A Banach algebra A is *weakly sequentially complete* if every sequence $(a_n)_n$ in A which is weakly Cauchy is weakly convergent in A. As is well-known, preduals of von Neumann algebras are weakly sequentially complete (see [20]). In particular, $L^1(G)$ and $A(G)$, the group algebra and the Fourier algebra of a locally compact group G, are weakly sequentially complete. The w^*-topology on A^{**} induces the weak topology on A, so an easy consequence of the definitions is that if a sequence $(a_n)_n$ in A converges to a w^*-limit $a \in A^{**}$, then in fact $a \in A$. Since bounded subsets in A^{**} are relatively w^*-compact, we see that if $(a_n)_n$ is a bounded sequence in A which has just one w^*-cluster point in A^{**}, then that cluster point is in A.

Theorem 3.1. *Let A be weakly sequentially complete with a sequential bounded approximate φ-mean, but with no φ-mean in A itself. Then A has at least $2^{\mathfrak{c}}$ φ-means. If A is separable, then it has precisely $2^{\mathfrak{c}}$ φ-means.*

Proof. Let $(u_n)_n$ be a sequential bounded approximate φ-mean, and let M denote the set of all w^*-cluster points of $(u_n)_n$ in A^{**}. Each element of M is a φ-mean, and M is w^*-compact. We claim that no element of M has a countable neighbourhood base in M. Indeed, suppose that for some $m \in M$, there is a decreasing countable base $(V_k)_k$ of closed neighbourhoods of m in M. Choose w^*-closed neighbourhoods W_k, $k \in \mathbb{N}$, of m in A^{**} with $W_k \cap M = V_k$. Then $M \cap (\bigcap_{k=1}^{\infty} W_k) = \{m\}$, and we can arrange for the sequence $(W_k)_k$ to be decreasing. For each k, select $u_{n_k} \in W_k$. Then every w^*-cluster point of the subsequence $(u_{n_k})_k$ lies in each W_k and in M, so must be equal to m. Since A is weakly sequentially complete, it follows that $m \in A$, which is impossible by hypothesis.

Thus no point of M has a countable neighbourhood base. This implies that M has at least $2^{\mathfrak{c}}$ elements (see Theorem 7.19 in Chapter I of [12]). Finally, if A is separable, then A^{**} has a countable w^*-dense subset and hence no more than $2^{\mathfrak{c}}$ elements. □

Example 3.2.

(1) If G is a locally compact group, then $\epsilon : f \to \int_G f(x)\,dx$ defines an element of $\Delta(L^1(G))$. In this case, the ϵ-means correspond to the set of topologically left invariant means on $L^\infty(G)$. Suppose that G is amenable, second countable and noncompact. Since then $L^1(G)$ is separable, it follows from Theorem 3.1 that $L^\infty(G)$ admits precisely $2^{\mathfrak{c}}$ topologically left invariant means, a fact which is known from [15].

(2) Let G be a locally compact group and $A(G)$ its Fourier algebra. Then $A(G)^* = VN(G)$, the von Neumann algebra generated by left translation operators on $L^2(G)$. The identity operator I on $L^2(G)$ defines an element ϵ of $\Delta(A(G))$ by $\epsilon(u) = \langle I, u\rangle = u(e)$, $u \in A(G)$. Then the set of ϵ-means coincides with the set of topologically invariant means studied in [18]. If G is second countable, then $L^2(G)$ is separable and hence $A(G)$ is separable and weakly sequentially complete (see [6, (3.1) and p. 218, Théorème]). If, in addition, G is not discrete, then no ϵ-mean can belong to $A(G)$. Thus the cardinality of the set of topologically invariant means on $VN(G)$ is exactly $2^{\mathfrak{c}}$ in this case, as was shown in [9] (see also [7]).

(3) Consider the convolution algebra $A = l^1(\mathbb{Z}_+)$. For $z \in \overline{\mathbb{D}}$, the closed unit disc, define $\varphi_z : A \to \mathbb{C}$ by $\varphi_z(a) = \sum_{n=0}^{\infty} a_n z^n$, $a = (a_n)_n \in A$. Then the map $z \to \varphi_z$ is a homeomorphism between $\overline{\mathbb{D}}$ and $\Delta(A)$. We already know that A is φ_z-amenable if and only if $|z| = 1$ [11, Example 2.5(2)]. Let $z \in \overline{\mathbb{D}}$ with $|z| = 1$. Since A is weakly sequentially complete and separable, by Theorem 3.1 there either exists a φ_z-mean in A itself or there are precisely $2^{\mathfrak{c}}$ φ_z-means. Now, suppose that $u = (u_n)_n \in A$ is a φ_z-mean. Then, for all $a \in A$ and $f = (f_n)_n \in l^\infty(\mathbb{Z}_+) = A^*$,

$$\sum_{n=0}^{\infty} f_n \left(\sum_{k=0}^{n} a_k u_{n-k} \right) = \langle f, a * u\rangle = \langle f \cdot a, u\rangle = \varphi_z(a)\langle f, u\rangle = \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} f_n u_n.$$

Taking $f = \delta_k$ and $a = \delta_l$, $l > k$, we obtain $z^l u_k = 0$. Thus $u = 0$ and hence there are exactly $2^{\mathfrak{c}}$ φ_z-means.

If m_1 and m_2 are two φ-means on A, then $m_1 m_2 = \varphi(m_1) m_2 = m_2$. One of the immediate consequences of Theorem 3.1 is therefore that if A satisfies its hypotheses, A^{**} is not commutative, even if A is. There is a formulation of this which makes sense for non-commutative algebras. Define a second multiplication on A^{**} by

$$m \diamond n = w^*\text{-}\lim_{b \to n} \Big(w^*\text{-}\lim_{a \to m} ab \Big)$$

(a similar formula to that which determines the multiplication in A^{**}, but with the limits taken in the other order). The product $m \diamond n$ is w^*-continuous in n for fixed m. A is called *Arens regular* if $m \diamond n = mn$ for all $m, n \in A^{**}$. A condition equivalent to Arens regularity is that mn should be w^*-continuous in n for fixed m. When A is commutative, so that $ba = ab$, we find that in A^{**} we have $m \diamond n = nm$. Thus we have shown that, under the hypotheses of Theorem 3.1, a commutative A is not Arens regular. We shall obtain a non-commutative result generalizing this.

We must introduce some additional concepts. We call $m \in A^{**}$ a *2-sided φ-mean* if $\langle m, \varphi\rangle = 1$ and for each $f \in A^*$ and $a \in A$ we have not only $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$, but also $\langle m, a \cdot f\rangle = \varphi(a)\langle m, f\rangle$. Of course, the latter two conditions are equivalent to $am = \varphi(a)m$ and $ma = \varphi(a)m$ for all $a \in A$, respectively. W^*-continuity then gives $nm = \langle n, \varphi\rangle m$ for all $n \in A^{**}$. However, we cannot conclude that $mn = \langle n, \varphi\rangle m$ unless A is Arens regular. Notice that if A is commutative, every φ-mean is automatically a 2-sided φ-mean.

A bounded net $(u_\gamma)_\gamma$ in A is called a *bounded approximate 2-sided φ-mean* if $\varphi(u_\gamma) = 1$ for all γ and for each $a \in A$,

$$\|au_\gamma - \varphi(a)u_\gamma\| \to 0 \quad \text{and} \quad \|u_\gamma a - \varphi(a)u_\gamma\| \to 0.$$

Proposition 3.3. *An element m of A^{**} is a 2-sided φ-mean for A if and only if m is a w^*-cluster point of a bounded approximate 2-sided φ-mean.*

Proof. If m is a w^*-cluster point of a bounded approximate 2-sided φ-mean $(u_\gamma)_\gamma$, then for each $a \in A$, am is a w^*-cluster point of $(au_\gamma)_\gamma$ and this implies that $am = \varphi(a)m$. Similarly, $ma = \varphi(a)m$. Since also $\langle m, \varphi\rangle = \lim_\gamma \varphi(u_\gamma) = 1$, we get that m is a 2-sided φ-mean.

Conversely, let m be a 2-sided φ-mean. Then m is the w^*-limit of some net $(v_\gamma)_\gamma$ in A with $\|v_\gamma\| \to \|m\|$. Then $\varphi(v_\gamma) - 1 = \langle v_\gamma - m, \varphi\rangle \to 0$, and w^*-continuity gives

$$av_\gamma - \varphi(a)v_\gamma \to am - \varphi(a)m = 0 \quad \text{and} \quad v_\gamma a - \varphi(a)v_\gamma \to ma - \varphi(a)m = 0$$

in the w^*-topology for each $a \in A$. So the nets

$$\big(av_\gamma - \varphi(a)v_\gamma\big)_\gamma \quad \text{and} \quad \big(v_\gamma a - \varphi(a)v_\gamma\big)_\gamma$$

in A both converge to 0 weakly for all $a \in A$.

Now take any finite subset $F = \{a_1, \dots, a_k\}$ of A and let

$$C = \Big\{\Big(\big(a_j v - \varphi(a_j)v\big)_{j=1}^k, \big(va_j - \varphi(a_j)v\big)_{j=1}^k, \varphi(v) - 1\Big):\ v \in A\Big\}.$$

Then in the Banach space $A^{2k} \times \mathbb{C}$, 0 is in the weak closure of C and hence in the norm closure since C is convex. Thus, given $\epsilon > 0$, we can find $v_{F,\epsilon} \in A$ such that $\|v_{F,\epsilon}\| \leqslant 2\|m\|$, say, $|\varphi(v_{F,\epsilon}) - 1| < \epsilon$ and for all $a \in F$,

$$\|av_{F,\epsilon} - \varphi(a)v_{F,\epsilon}\| < \epsilon \quad \text{and} \quad \|v_{F,\epsilon}a - \varphi(a)v_{F,\epsilon}\| < \epsilon.$$

Finally replace $v_{F,\epsilon}$ by a scalar multiple $u_{F,\epsilon} = \lambda_{F,\epsilon} v_{F,\epsilon}$ for which $\varphi(u_{F,\epsilon}) = 1$. Then $|\lambda_{F,\epsilon}| < \frac{1}{1-\epsilon}$ and

$$\|au_{F,\epsilon} - \varphi(a)u_{F,\epsilon}\| < \frac{\epsilon}{1-\epsilon} \quad \text{and} \quad \|u_{F,\epsilon}a - \varphi(a)u_{F,\epsilon}\| < \frac{\epsilon}{1-\epsilon}.$$

So the net $(u_{F,\epsilon})_{F,\epsilon}$ is a bounded approximate 2-sided φ-mean and m is the w^*-limit of $(u_{F,\epsilon})_{F,\epsilon}$. □

We shall prove

Theorem 3.4. *Let A be weakly sequentially complete. Suppose that A has a bounded approximate 2-sided φ-mean, but that there is no 2-sided φ-mean in A itself. Then A is not Arens regular.*

In proving Theorem 3.4, we will partly follow an idea of Ülger [23] (see also [21]), where he established a parallel result for bounded approximate identities.

Let I be a commutative idempotent semigroup, that is, $i^2 = i$ for all $i \in I$. Define an order on I by $i \leqslant j$ if $ij = j$. Then I is a directed set with $\max\{i, j\} = ij$.

Proposition 3.5. *Let A be a Banach algebra. Let I be as above and let $h : I \to A$ be a homomorphism into the multiplicative semigroup of A such that $h(I)$ is bounded and $0 \notin h(I)$. If the net $(h(i))_i$ has a weak cluster point in A, then $h(I)$ has a maximal element.*

Proof. Let e be a weak cluster point of $(h(i))_i$. Take J to be a cofinal subset of I with $w\text{-}\lim_{i \in J} h(i) = e$. For $i \leqslant j$ in J we have $h(i)h(j) = h(j)$. Taking the j-limit gives $h(i)e = e$ and then taking the i-limit gives $e^2 = e$. Since weak and norm closures of convex sets coincide, e is in the norm closure of the convex hull of $\{h(i):\ i \in J\}$. Thus given $\epsilon > 0$, we can find $j_1, \dots, j_n \in J$ and scalars $\lambda_1, \dots, \lambda_n \geqslant 0$ with $\sum_{k=1}^n \lambda_k = 1$ such that

$$\Big\|\sum_{k=1}^n \lambda_k h(j_k) - e\Big\| \leqslant \epsilon.$$

For $j \in J$ with $j \geqslant \max\{j_1, \dots, j_n\}$ we have

$$\left(\sum_{k=1}^{n} \lambda_k h(j_k)\right) h(j) = \sum_{k=1}^{n} \lambda_k h(j_k j) = h(j).$$

Because $h(I)$ is commutative, we see that $eh(j) = e$ for all j and therefore

$$\|h(j) - e\| = \left\| \sum_{k=1}^{n} \lambda_k h(j_k) h(j) - eh(j) \right\| \leqslant \left\| \sum_{k=1}^{n} \lambda_k h(j_k) - e \right\| \cdot \|h(j)\| \leqslant \epsilon \sup_{j \in J} \|h(j)\|.$$

But $h(j) - e$ is an idempotent, so either is zero or satisfies $\|h(j) - e\| \geqslant 1$. Since $\epsilon > 0$ is arbitrary, it follows that $h(j) = e$. This holds for a cofinal set of j's and consequently e is a maximal element in $h(I)$. □

Next we present a general construction which produces subalgebras which have sequential bounded approximate φ-means.

Proposition 3.6. *Let A be a Banach algebra with a bounded approximate 2-sided φ-mean (respectively, a bounded approximate φ-mean) $(u_\gamma)_\gamma$. Let $X = \{x_1, x_2, \dots\}$ be any countable subset of A. Then there is a closed separable subalgebra $A(X)$ of A which contains X and has a sequential bounded approximate 2-sided φ-mean (respectively, a sequential bounded approximate φ-mean) $(u_{\gamma_n})_{\gamma_n}$ chosen from $(u_\gamma)_\gamma$.*

Proof. We shall only prove the 2-sided φ-mean case (the other one being easier). If we replace each element of X by any non-zero scalar multiple of itself we do not change $A(X)$, and we may therefore arrange for X to be bounded. Thus let $C > 0$ be such that $\|x_n\|, \|u_\gamma\| \leqslant C$ for all n, γ. We choose u_{γ_n}, $n \in \mathbb{N}$, inductively to satisfy

$$\|x_i u_{\gamma_n} - \varphi(x_i) u_{\gamma_n}\| \leqslant \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} x_i - \varphi(x_i) u_{\gamma_n}\| \leqslant \frac{1}{n}$$

for $1 \leqslant i \leqslant n$, and

$$\|u_{\gamma_i} u_{\gamma_n} - u_{\gamma_n}\| \leqslant \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} u_{\gamma_i} - u_{\gamma_n}\| \leqslant \frac{1}{n}$$

for $1 \leqslant i < n$. We take $A(X)$ to be the closed linear span of $X \cup \{u_{\gamma_1}, u_{\gamma_2}, \dots\}$.

Now $(u_{\gamma_n})_{\gamma_n}$ is a bounded approximate 2-sided φ-mean for $A(X)$. This requires a little argument, whereas the corresponding conclusion when dealing with bounded approximate identities is just a simple observation (see [4, Proposition 2.9.17]). Take any k elements $a_1, \dots, a_k$ from $X \cup \{u_{\gamma_1}, u_{\gamma_2}, \dots\}$ and let $\epsilon > 0$. Choose N so large that $kC^{k-1}/N < \epsilon$ and that if $n > N$, then x_n and u_{γ_n} do not belong to $\{a_1, \dots, a_k\}$. Then, for $n > N$, we can estimate the norm

$$c_n := \|a_1 \dots a_k u_{\gamma_n} - \varphi(a_1 \dots a_k) u_{\gamma_n}\|$$

as follows:

$$\begin{aligned} c_n &\leqslant \sum_{j=1}^{k} \|a_1 \dots a_j \varphi(a_{j+1}) \dots \varphi(a_k) u_{\gamma_n} - a_1 \dots a_{j-1} \varphi(a_j) \dots \varphi(a_k) u_{\gamma_n}\| \\ &\leqslant \sum_{j=1}^{k} \|a_1\| \dots \|a_{j-1}\| \cdot |\varphi(a_{j+1})| \dots |\varphi(a_k)| \cdot \|a_j u_{\gamma_n} - \varphi(a_j) u_{\gamma_n}\| \\ &\leqslant \sum_{j=1}^{k} C^{j-1} C^{k-j} \frac{1}{n} = kC^{k-1} \frac{1}{n} \\ &< \epsilon. \end{aligned}$$

A parallel calculation deals with $u_{\gamma_n} a_1 \dots a_k$. Similar methods will allow us to treat finite linear combinations of products $a_1 \cdot \dots \cdot a_k$. We then have a bounded approximate 2-sided φ-mean for the algebra generated algebraically by $X \cup \{u_{\gamma_n}\colon n \in \mathbb{N}\}$. Standard arguments extend this to the norm closure, that is, $A(X)$. □

Corollary 3.7. *Let A be a separable Banach algebra and $\varphi \in \Delta(A)$. If A is φ-amenable, then there exists a sequential bounded approximate φ-mean.*

Proof. This follows from the preceding proposition simply by taking for X a countable dense subset of A. □

Remark 3.8. Note that even if A is an amenable Banach algebra, a closed subalgebra B of A need not be φ-amenable for all $\varphi \in \Delta(B)$. Indeed, Hochster [8] has constructed a 2-step solvable group G which contains a free subsemigroup S on two generators. Then $l^1(G)$ is amenable [10], but $l^1(S)$ is not φ-amenable, where φ is the homomorphism defined by $\varphi(f) = \sum_{s\in S} f(s)$, $f \in l^1(S)$.

The algebra $A(X)$ constructed in the proof of Proposition 3.6 is of course not unique, as it depends on the choice of $(u_{\gamma_n})_n$. We now prove Theorem 3.4 in the following form.

Theorem 3.9. *Let A be weakly sequentially complete and Arens regular, and suppose that A has a 2-sided φ-mean m. Then m is unique and contained in A.*

Proof. We first consider the case in which the bounded approximate 2-sided φ-mean is sequential, say $(u_n)_n$. If m_1 and m_2 are both 2-sided φ-means, we have $a \cdot m_2 = \varphi(a)m_2$ for all $a \in A$, and choosing a net in A converging w^* to m_1, we get $m_1m_2 = \langle m_1, \varphi\rangle m_2 = m_2$. In the same way, from $m_1 \cdot a = \varphi(a)m_1$, because A is Arens regular, we get that $m_1m_2 = m_1$. Thus $m_1 = m_2$, and in particular any two w^*-cluster points of $(u_n)_n$ are equal. Since A is weakly sequentially complete, there exists a cluster point in A itself, which then is the unique 2-sided φ-mean.

Now let A be arbitrary. Take any countable subset X_1 of A and form $A(X_1)$ as in Proposition 3.6. Then $A(X_1)$ is weakly sequentially complete and Arens regular and has a sequential bounded approximate 2-sided φ-mean. By the first part of the proof, $A(X_1)$ has a unique 2-sided φ-mean, m_1. If m_1 is a 2-sided φ-mean for A, we are finished. Otherwise we can find a countable subset X_2 of A with $m_1 \in X_2$ for which m_1 is not a 2-sided φ-mean. Then $A(X_2)$ contains a 2-sided φ-mean, m_2 say. In particular, $m_1m_2 = \langle m_1, \varphi\rangle m_2 = m_2$ and similarly $m_2m_1 = m_2$. Again, if m_2 is a 2-sided φ-mean for A, we are finished. Otherwise, take X_3 with $m_1, m_2 \in X_3$ in order to find m_3, and so on. If this process stops we have found a 2-sided φ-mean in A. If it does not stop, we find a bounded infinite sequence $(m_n)_n$ in A with the product $m_km_l = m_{\max\{k,l\}}$. This is impossible by Proposition 3.5. □

4. Invariant means on *F*-algebras

Let A be an F-algebra. We now use the term *topologically left invariant mean* (TLIM) rather than ϵ-mean of norm 1 (see Section 2). The purpose of this section is to prove the following theorem which was proved by Chou [3] for the Fourier algebra of a locally compact group.

Theorem 4.1. *Let A be a separable F-algebra which is ϵ-amenable. Suppose that A^* contains a C^*-subalgebra B such that B is w^*-dense in A^* and $m(B) = \{0\}$ for every ϵ-mean m. Then there is a linear isometry Θ from $l^1(\mathbb{N})$ into A with the property that each $m \in \Theta^{**}(\beta\mathbb{N} \setminus \mathbb{N})$ is an ϵ-mean. In particular, if $m_1, m_2 \in \Theta^{**}(\beta\mathbb{N} \setminus \mathbb{N})$ are distinct, then $\|m_1 - m_2\| = 2$.*

The proof of Theorem 4.1 will make substantial use of the following lemma.

Lemma 4.2. *Let A and B be as in Theorem* 4.1.

(i) *If m is a* TLIM *on A^*, then $\|m - a\| = 2$ for every $a \in P_1(A, \epsilon)$.*
(ii) *If a net $(u_\gamma)_\gamma$ is an approximate ϵ-mean with $\|u_\gamma\| = 1$ for all γ, then* $\lim_\gamma \|u_\gamma - a\| = 2$ *for each $a \in P_1(A, \epsilon)$.*

Proof. (i) Since B is w^*-dense in A^*, by the Kaplansky density theorem [20, Chapter II, Theorem 4.8] the unit ball of B is w^*-dense in the unit ball of A^*. Consequently, the map $r : A^{**} \to B^*$, $m \to m|_B$ is a linear isometry of A^{**} into B^*. Choose a bounded approximate identity $(e_\beta)_\beta$ in B such that $e_\beta \geqslant 0$, $\|e_\beta\| \leqslant 1$ and $e_\beta \leqslant e_{\beta'}$ if $\beta \leqslant \beta'$ (such a bounded approximate identity exists in every C^*-algebra). Let $a \in P_1(A, \epsilon)$. Then $\|a\| = \lim_\beta \langle e_\beta, a\rangle$ and hence, given any $\delta > 0$, there exists β such that $\langle e_\beta, a\rangle \geqslant \|a\| - \delta = 1 - \delta$. Let $g = 2e_\beta - \epsilon \in A^*$. Now, if m is any ϵ-mean, then

$$\langle a - m, g\rangle = \langle a - m, 2e_\beta - \epsilon\rangle = 2\langle a - m, e_\beta\rangle \geqslant 2(1 - \delta) - 2\langle m, e_\beta\rangle = 2 - 2\delta,$$

as $\langle m, e_\beta\rangle = 0$. So $\|a - m\| \geqslant 2 - 2\delta$, and since $\delta > 0$ was arbitrary, $\|a - m\| = 2$.

(ii) If $\|u_\gamma - a\|$ does not converge to 2, then by taking a subnet, we may assume that $\|u_\gamma - a\| \leqslant 2 - \delta$ for all γ and some $\delta > 0$. Then $\|m - a\| \leqslant 2 - \delta$ for every w^*-cluster point of $(u_\gamma)_\gamma$. This contradicts (i) since any such m is an ϵ-mean. □

We now turn to the proof of Theorem 4.1. For $a \in A$, let $s(a)$ denote the support of a in A^*, that is, the smallest projection p such that $\langle p, a\rangle = \epsilon(a) = \|a\|$.

If m is a positive linear functional of norm 1 on A^*, then there exists a net $(u_\gamma)_\gamma$ in A such that $u_\gamma \geqslant 0$, $\|u_\gamma\| = 1$ (equivalently, $u_\gamma \in P_1(A, \epsilon)$) and $u_\gamma \to m$ in the w^*-topology. Thus w^*-$\lim(au_\gamma - \epsilon(a)u_\gamma) = 0$ for every $a \in A$.

By an argument similar to the one in the proof of Proposition 3.6, we can find a sequence $(u_{\gamma_n})_n$ such that $\|au_{\gamma_n} - \epsilon(a)u_{\gamma_n}\| \to 0$ for all $a \in A$. By Lemma 4.2, $\lim_\gamma \|u_{\gamma_n} - a\| = 2$ for all $a \in P_1(A, \epsilon)$. Using Theorem 2.4 of [3], we can find a subsequence $(u_{\gamma_{n_j}})_j$ of $(u_{\gamma_n})_n$ and a sequence $(v_j)_j$ in $P_1(A, \epsilon)$ such that

$$\|u_{\gamma_{n_j}} - v_j\| < \frac{1}{2^{j-1}}$$

for all j and $s(v_j)s(v_k) = 0$ if $j \neq k$. Clearly, $(v_j)_j$ is an approximate ϵ-mean.

Let $V = \{v_1, v_2, \ldots\}$. Since V is orthogonal, V is a linearly independent subset of A. Let

$$\Theta : \operatorname{span}\{\delta_v \colon\ v \in V\} \to \operatorname{span} V$$

be defined by

$$\Theta\left(\sum_{j=1}^{n} \lambda_j \delta_{v_j}\right) = \sum_{j=1}^{n} \lambda_j v_j.$$

Clearly, $\|\Theta(\sum_{j=1}^n \lambda_j v_j)\| \leqslant \sum_{j=1}^n |\lambda_j|$. On the other hand, if $p_j = s(v_j)$, then $(p_j)_j$ is a sequence of pairwise orthogonal projections in A^*. Let M be the w^*-closure of the span of the p_j, $j \in \mathbb{N}$. Then M is a commutative W^*-subalgebra of A^*. For each j, let $\mu_j \in \mathbb{C}$ such that $\mu_j \lambda_j = |\lambda_j|$, and let $q = \sum_{j=1}^n \mu_j p_j \in A^*$. Then $\|q\| = 1$, and

$$\left\langle \Theta\left(\sum_{j=1}^{n} \lambda_j \delta_{v_j}\right), q \right\rangle = \sum_{j=1}^{n} |\lambda_j|,$$

and therefore

$$\left\| \Theta\left(\sum_{j=1}^{n} \lambda_j \delta_{v_j}\right) \right\| = \left\| \sum_{j=1}^{n} \lambda_j \delta_{v_j} \right\|.$$

Consequently, Θ extends to a linear isometry, also denoted Θ, from $l^1(V)$ into A.

Finally, since each $\eta \in \beta\mathbb{N} \setminus \mathbb{N}$ is a w^*-cluster point of $(\delta_{v_j})_j$, and Θ^{**} is w^*-continuous, it follows that $m = \Theta^{**}(\eta)$ is a w^*-cluster point of the sequence $(v_j)_j$. So for each w^*-neighbourhood U of m, there exists $n_U \in \mathbb{N}$ such that $v_{n_U} \in U$. Let $\mathcal{U}$ denote the set of all w^*-neighbourhoods of m. Then $(v_{n_U})_U$ is a subnet of the sequence $(v_j)_j$ and $v_{n_U} \to m$ in the w^*-topology. Indeed, otherwise there exists $N \in \mathbb{N}$ such that $n_U \leqslant N$ for all U, which implies that m is an ϵ-mean in A. However, since $m|_B = 0$, this is impossible. Clearly, m is an ϵ-mean.

Remark 4.3. The above condition on the existence of B is satisfied when $A = A(G)$ of a non-discrete locally compact group or when $A = L^1(G)$ of a non-compact locally compact amenable group. In the first case, B can be taken to be the reduced group C^*-algebra $C_\rho(G)$ (see [13]), whereas in the second case one can take $B = C_0(G)$.

5. Examples

In this final section we present two illustrative examples: algebras of Lipschitz functions on compact metric spaces and convolution algebras $L^p(G)$ on a compact group G. In both cases, the relevant singletons in $\Delta(A)$ are open. We therefore start by looking at how openness of $\{\varphi\}$ and φ-amenability are related.

Remark 5.1. Let A be a Banach algebra and $\varphi \in \Delta(A)$ and suppose that A is φ-amenable. For every $\psi \in \Delta(A)$ such that $\psi \neq \varphi$, there exists $a_\psi \in \ker\psi$ with $\varphi(a_\psi) = 1$. So, if m is a φ-mean, then $\langle m, \varphi\rangle = 1$, whereas

$$\langle m, \psi\rangle = \langle m, \psi \cdot a_\psi\rangle = \big\langle m, \psi(a_\psi)\psi\big\rangle = 0$$

for all $\psi \neq \varphi$. Hence $\{\varphi\}$ is open in $(\Delta(A), \text{weak})$.

Lemma 5.2. *Let A be a semisimple commutative Banach algebra. Let $\varphi \in \Delta(A)$ and suppose that $\{\varphi\}$ is open in $\Delta(A)$. Let a be the unique element of A with $\varphi(a) = 1$ and $\psi(a) = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$.*

(i) *Then a is a φ-mean for A and it is the only one in A^{**}.*
(ii) *If $\|a\| = 1$, then $N(A, \varphi) = \{f \in A^*\colon\ \langle f, a\rangle = 0\}$.*

Proof. The existence of a follows from Shilov's idempotent theorem. However, in the present special situation it is easy to avoid such heavy machinery. To see this, let J be the closed ideal of A defined by

$$J = \big\{a \in A\colon\ \psi(a) = 0 \text{ for all } \psi \in \Delta(A) \setminus \{\varphi\}\big\}.$$

Since $\{\varphi\}$ is open in $(\Delta(A), w^*)$, $\Delta(J) = \{\varphi|_J\}$ and hence J is 1-dimensional as A is semisimple. Of course, $\ker\varphi + J = A$ and $\ker\varphi \cap J = \{0\}$ since A is semisimple and $\psi(\ker\varphi \cap J) = \{0\}$ for all $\psi \in \Delta(A)$. Thus $A = \ker\varphi \oplus J$ and $A^* = (\ker\varphi)^* \oplus \mathbb{C}\varphi$.

Let $a \in J$ such that $\varphi(a) = 1$. Then $\psi(a) = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$, and a is the only element of A with these properties since A is semisimple.

(i) For each $x \in A$, $\varphi(xa) = \varphi(\varphi(x)a)$ and, for $\psi \in \Delta(A) \setminus \{\varphi\}$, $\psi(xa) = 0 = \psi(\varphi(x)a)$. So $xa = \varphi(x)a$ by semisimplicity and hence a is a φ-mean. Now, let $m \in A^{**}$ be any φ-mean for A. Since A is commutative, every element of A commutes with every element of A^{**}. Thus

$$m = \varphi(a)m = am = ma = \varphi^{**}(m)a = \langle m, \varphi\rangle a = a.$$

So a is the only φ-mean for A in A^{**}.

(ii) Let $\|a\| = 1$. Since $N(A, \varphi)$ is a proper linear subspace of A^*, by definition of $N(A, \varphi)$ it suffices to show that for any $f \in A^*$, $\langle f, a\rangle = 0$ implies $f \cdot a = 0$. Now, every $x \in A$ has a decomposition $x = y + \lambda a$ with $y \in \ker\varphi$ and $\lambda \in \mathbb{C}$. Since $ay \in \ker\varphi \cap J = \{0\}$, for $f \in A^*$,

$$\langle f \cdot a, x\rangle = \langle f, ay\rangle + \lambda\langle f, a\rangle = \lambda\langle f, a\rangle.$$

So $f \cdot a = 0$ whenever $\langle f, a\rangle = 0$. □

Since an amenable Banach algebra is φ-amenable for each $\varphi \in \Delta(A)$, Remark 5.1 generalizes Theorem 2.1 of [22]. If A is commutative and semisimple and the weak and weak* topologies coincide on $\Delta(A)$, then by Remark 5.1 and Lemma 5.2(i), A is φ-amenable if and only if $\{\varphi\}$ is open in $\Delta(A)$. The condition that the two topologies coincide, however, is quite restrictive. It is for instance satisfied if A is an ideal in A^{**} [22, Theorem 3.1].

Example 5.3. Let X be a compact metric space with metric d and let $0 < \alpha \leqslant 1$. Then $\mathrm{Lip}_\alpha X$ is the space of all complex-valued functions u on X such that

$$p_\alpha(u) = \sup\left\{\frac{|u(x) - u(y)|}{d(x, y)^\alpha}:\ x, y \in X,\ x \neq y\right\}$$

is finite, and $\mathrm{lip}_\alpha X$ is the subspace of functions satisfying

$$\frac{|u(x) - u(y)|}{d(x, y)^\alpha} \to 0 \quad \text{as } d(x, y) \to 0.$$

With pointwise multiplication and the norm $\|u\| = \|u\|_\infty + p_\alpha(u)$, $\mathrm{Lip}_\alpha X$ is a unital commutative Banach algebra and $\mathrm{lip}_\alpha X$ is a closed subalgebra. These algebras were first studied by Sherbert [19] and later by Bade, Curtis and Dales [1]. All facts relevant to us can now be found in Section 4.4 of the monograph [4].

We first treat $\mathrm{Lip}_\alpha X$. The map $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in \mathrm{Lip}_\alpha X$, is a homeomorphism from X onto $\Delta(\mathrm{Lip}_\alpha X)$. If x is a non-isolated point of X, then there exist non-zero continuous point derivations at φ_x and hence $\mathrm{Lip}_\alpha X$ is not φ_x-amenable (compare [11, Remark 2.4]). Now, let x be an isolated point of X. Then, by Lemma 5.2(i), there exists a unique φ_x-mean, namely the Dirac function $\delta_x \in \mathrm{Lip}_\alpha X$. In view of Section 2 where the means are supposed to have norm 1, we point out that

$$\|\delta_x\| = 1 + p_\alpha(\delta_x) = 1 + \sup\left\{\frac{1}{d(x, y)^\alpha}:\ y \neq x\right\},$$

which, depending on d, can be arbitrarily large.

In light of Lemma 5.2(ii) which shows that $N(A, \varphi)$ is a linear subspace of codimension 1 if there exists a φ-mean of norm 1, it is interesting to note that $N(\mathrm{Lip}_\alpha X, \varphi_x) = \{0\}$ for any isolated point x of X. To see this, let $f \in N(\mathrm{Lip}_\alpha X, \varphi_x)$. There exists a sequence $(u_n)_n$ in $\mathrm{Lip}_\alpha X$ with $u_n(x) = 1$ for all n, $\|u_n\| \to 1$ and $f \cdot u_n \to 0$ in norm. It suffices to show that $u_n \to 1$ in $\mathrm{Lip}_\alpha X$ because then $f = f \cdot 1 = \lim_{n\to\infty} f \cdot u_n = 0$. Since

$$1 + p_\alpha(u_n) = \left|u_n(x)\right| + p_\alpha(u_n) \leqslant \|u_n\| \to 1,$$

it follows that $p_\alpha(u_n - 1) = p_\alpha(u_n) \to 0$. Therefore it remains to verify that $u_n \to 1$ uniformly on X. Since X is compact, there exists $C > 0$ such that $d(y, x)^\alpha \leqslant C$ for all $y \in X$. For $y \neq x$ it follows that

$$\left|u_n(y) - 1\right| = \left|u_n(y) - u_n(x)\right| \leqslant C \cdot \frac{|u_n(y) - u_n(x)|}{d(y, x)^\alpha} \leqslant C p_\alpha(u_n),$$

which tends to zero. So $u_n \to 1$ uniformly on $X \setminus \{x\}$ and hence on all of X.

We now turn to $\mathrm{lip}_\alpha X$. Note that $\mathrm{lip}_1 X$ can be very small since for X a compact interval it consists only of the constant functions. In fact, if $d(x, y) = |x - y|$, then each $u \in \mathrm{lip}_1 X$ is differentiable with $u' = 0$ on X. Thus, let $0 < \alpha < 1$. Then $\mathrm{Lip}_1 X$ is dense in $\mathrm{lip}_\alpha X$ and $\Delta(\mathrm{lip}_\alpha X)$ can be identified with X in the same manner as above. However, in contrast to $\mathrm{Lip}_\alpha X$, all continuous point derivations on $\mathrm{lip}_\alpha X$ are zero. Nevertheless, for $x \in X$, $\mathrm{lip}_\alpha X$ is also φ_x-amenable if and only if x is an isolated point of X. This follows from Lemma 5.1 and the remarkable result that $(\mathrm{lip}_\alpha X)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_\alpha X$. Indeed, this latter fact implies that the weak* and the weak topologies coincide on $\Delta(\mathrm{lip}_\alpha X)$ since X is homeomorphic to both $\Delta(\mathrm{lip}_\alpha X)$ and $\Delta(\mathrm{Lip}_\alpha X)$.

Example 5.4. Let G be a compact group with normalized Haar measure and consider the convolution algebra $L^p(G)$, $1 \leqslant p < \infty$. Let $\widehat{G}$ denote the set of all continuous homomorphisms from G into the circle group $\mathbb{T}$, equipped with the topology of uniform convergence. For $\chi \in \widehat{G}$, define $\varphi_\chi : L^p(G) \to \mathbb{C}$ by $\varphi_\chi(f) = \int_G f(x)\overline{\chi(x)}\,dx$. It is routine to show that the map $\chi \to \varphi_\chi$ is a homeomorphism from $\widehat{G}$ onto $\Delta(L^p(G))$.

Let $q = \frac{p}{p-1}$. Fix $\chi \in \widehat{G}$ and define m_χ on $L^q(G) = L^p(G)^*$ by

$$\langle m_\chi, g\rangle = \int_G g(x)\chi(x)\,dx, \quad g \in L^q(G).$$

Then $\langle m_\chi, \varphi_\chi\rangle = \int_G |\chi(x)|^2\,dx = 1$ and

$$\langle m_\chi, g\cdot f\rangle = \langle m_\chi, g * \check{f}\rangle = \int_G\int_G g(xy)f(y)\chi(x)\,dy\,dx = \int_G\int_G g(x)f(y)\chi\big(xy^{-1}\big)\,dx\,dy = \varphi_\chi(f)\langle m_\chi, g\rangle$$

for all $g \in L^q(G)$ and $f \in L^p(G)$. Thus m_χ is a φ_χ-mean, and we claim that it is the only one. In this context, note that $L^p(G)$ does not have a bounded approximate identity and hence Lemma 5.2(i) does not apply. So let m be a φ-mean and let

$$L = \big\{g - \widehat{g}(\overline{\chi})\overline{\chi}:\ g \in L^q(G)\big\}.$$

Then $L = \ker m_\chi$ and, since $g * \check{\chi} = \widehat{g}(\overline{\chi})\overline{\chi}$, we also have

$$\big\langle m, g - \widehat{g}(\overline{\chi})\overline{\chi}\big\rangle = \big\langle m, \langle m, \varphi_\chi\rangle g - \widehat{g}(\overline{\chi})\overline{\chi}\big\rangle = \big\langle m, g * \check{\chi} - \widehat{g}(\overline{\chi})\overline{\chi}\big\rangle = 0$$

for all $g \in L^q(G)$. So $m|_L = 0$ and since $\langle m, \varphi_\chi\rangle = \langle m_\chi, \varphi_\chi\rangle$, it follows that $m = m_\chi$.

We now determine $P_1(L^p(G), \varphi_\chi)$. If $p = 1$, then

$$P_1\big(L^1(G), \varphi_\chi\big) = \big\{f \in L^1(G):\ f\overline{\chi} \geqslant 0,\ \|f\overline{\chi}\| = 1\big\}.$$

For every $h \in L^1(G)$ and $\chi \in \widehat{G}$, $h\overline{\chi}$ can be written as a linear combination $h\overline{\chi} = \sum_{j=1}^4 c_j h_j$, where $c_j \in \mathbb{C}$, $h_j \geqslant 0$, and $\|h_j\| = 1$, $1 \leqslant j \leqslant 4$. Hence $h = \sum_{j=1}^4 c_j h_j \chi$ and $h_j\chi \in P_1(L^1(G), \varphi_\chi)$. So $P_1(L^1(G), \varphi_\chi)$ spans $L^1(G)$. Alternatively, we could appeal to the fact that $L^1(G)$ is an F-algebra.

We claim that $P_1(L^p(G), \varphi_\chi) = \{\chi\}$ whenever $p > 1$, so that $P_1(L^p(G), \varphi_\chi)$ is as small as it can be in this case.

Suppose first that $p \geqslant 2$. Then $L^p(G) \subseteq L^2(G)$ and hence, for $f \in P_1(L^p(G), \varphi_\chi)$,

$$1 = \|f\|_p^2 \geqslant \|f\|_2^2 = \sum_{\eta\in\widehat{G}} \big|\widehat{f}(\eta)\big|^2 = 1 + \sum_{\eta\neq\chi} \big|\widehat{f}(\eta)\big|^2.$$

So $\widehat{f}(\eta) = 0$ for $\eta \neq \chi$ and hence $f = \sum_{\eta\in\widehat{G}} \widehat{f}(\eta)\eta = \chi$ in $L^2(G)$.

Finally, let $1 < p < 2$ and $f \in L^p(G) \subseteq L^1(G)$. Then, by the Hausdorff–Young inequality, $\widehat{f} \in l^q(\widehat{G})$ and $\|\widehat{f}\|_q \leqslant \|f\|_p$. Thus, if $f \in P_1(L^p(G), \varphi_\chi)$, then

$$1 = \|f\|_p \geqslant \bigg(\sum_{\eta\in\widehat{G}} \big|\widehat{f}(\eta)\big|^q\bigg)^{1/q} = \bigg(1 + \sum_{\eta\neq\chi} \big|\widehat{f}(\eta)\big|^q\bigg)^{1/q}.$$

Again, $\widehat{f}(\eta) = 0$ for $\eta \neq \chi$ and hence, since $L^p(G) \subseteq L^1(G)$, $f \in k(\widehat{G} \setminus \{\chi\}) = \mathbb{C}\chi$. Then $f = \chi$ since $\widehat{f}(\chi) = 1$.

We now determine $N(L^p(G), \varphi_\chi)$. If G is abelian, it follows easily from Lemma 5.2(ii) that

$$N\big(L^p(G), \varphi_\chi\big) = \big\{f \in L^q(G):\ \widehat{f}(\overline{\chi}) = 0\big\}.$$

We show that the same description of $N(L^p(G), \varphi_\chi)$ is true when G is an arbitrary compact group.

Observe first that, for $f \in L^q(G)$ and $\eta \in \widehat{G}$, we have

$$\widehat{f\cdot\chi}(\eta) = \widehat{f * \overline{\chi}}(\eta) = \int_G \overline{\eta(x)}\bigg(\int_G f(xy)\chi(y)\,dy\bigg)dx = \int_G\int_G f(x)\overline{\eta(x)}\chi(y)\eta(y)\,dx\,dy = \widehat{f}(\eta)\int_G \chi(y)\eta(y)\,dy.$$

The orthogonality relations now imply that $\widehat{f\cdot\chi} = 0$ whenever $\widehat{f}(\overline{\chi}) = 0$. Thus $f\cdot\chi = 0$, and since $\varphi_\chi(\chi) = 1$ and $\|\chi\|_p = 1$, this shows that

$$\big\{f \in L^q(G):\ \widehat{f}(\overline{\chi}) = 0\big\} \subseteq N\big(L^p(G), \varphi_\chi\big).$$

Conversely, let $f \in N(L^p(G), \varphi_\chi)$ and let $(g_n)_n$ be a sequence in $L^p(G)$ with $\|f \cdot g_n\| \to 0$ and $\varphi_\chi(g_n) = 1$ for all n. Since

$$\big|\widehat{f}(\overline{\chi})\big| = \left|\int_G f(x)\chi(x)\,dx \cdot \int_G g_n(y)\overline{\chi(y)}\,dy\right| = \left|\int_G\int_G f(xy)\check{g}_n\big(y^{-1}\big)\chi(x)\,dy\,dx\right| \leqslant \left(\int_G\left|\int_G f(xy)\check{g}_n\big(y^{-1}\big)\,dy\right|^q dx\right)^{1/q}$$
$$= \|f \cdot g_n\|_q,$$

which tends to 0. It follows that $\widehat{f}(\overline{\chi}) = 0$ and hence

$$N\big(L^p(G), \varphi_\chi\big) \subseteq \big\{f \in L^q(G)\colon\ \widehat{f}(\overline{\chi}) = 0\big\},$$

as required.

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