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Tri-quotient maps become inductively perfect with the aid of consonance and continuous selections [☆]

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Abstract

Generalizing the result of Arhangel'skii that each open map with Čech-complete domain is compact-covering, it is proved that every tri-quotient map with consonant domain is harmonious, thus compact-covering, and its range is consonant. The latter constitutes a strong answer to a question of Nogura and Shakhmatov. Conditions for harmonious maps to be inductively perfect, or countable compact-covering and for countable compact-covering maps to be harmonious are given. They extend theorems of Just and Wicke. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, all the maps $f : X \to Y$ are continuous and onto.

Tri-quotient maps have been introduced in [13] by Michael in a study of the preservation of Čech-completeness and sieve-completeness. Open maps and perfect maps are tri-quotient, and every tri-quotient map is bi-quotient.

The same year, in [18, Theorem 1], Ostrovsky, in the proof of this theorem, uses three properties which put together a second definition of tri-quotient maps.

In [20], answering a question of Michael [13], Uspenskij proves that tri-quotient maps are preserved by infinite products. To this end, he uses a characterization of tri-quotient maps that is essentially a rewriting of the three properties given by Ostrovsky in [18]. A map $f : X \to Y$ is tri-quotient if and only if there exists a continuous mapping $\mathcal{R}$ from Y to the set $\mathcal{D}(X)$ of compact families [1] (of open subsets of X) endowed with

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[1] The notion of compact families is introduced by Dolecki, Greco, Lechicki in [7].

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the Stone topology (similar to that usually considered on the set of all ultrafilters on X). Moreover, every compact families $\mathcal{R}(y)$ must be supported by $f^{-1}(y)$. Of special interest are compactly generated compact families that is, compact families generated by compact subsets. A tri-quotient map with a defining function $\mathcal{R}$ valued in compactly generated compact families is called harmonious by Ostrovsky.

Michael [13], Just and Wicke [10,11], Debs, Saint Raymond [5,6], Ostrovsky [19,17] and others have studied the relationship between tri-quotient maps and inductively perfect maps, compact-covering maps, countable compact-covering and harmonious maps. In these papers, essentially in [5,6], it turns that these four last notions can be characterized by the existence of certain (possibly multivalued) mappings from some spaces of compact subsets of Y to the set of compact subsets of X.

f **inductively perfect**		
$\Phi : Y \to \mathcal{K}_X$	$\Leftrightarrow$	$\Psi : \mathcal{K}_Y \to \mathcal{K}_X$
$f(\Phi(y)) = \{y\}$		$f(\Psi(K)) = K$
Φ continuous		Ψ continuous
$\Downarrow$		
f **compact-harmonious**	$\Rightarrow$	f **compact-covering**
$\Phi : \mathcal{K}_Y \rightrightarrows \mathcal{K}_X$		$\Phi : \mathcal{K}_Y \rightrightarrows \mathcal{K}_X$
$\forall C \in \Phi(K)\, f(C) = K$		$\forall C \in \Phi(K)\, f(C) = K$
Φ lower semicontinuous		
$\Downarrow$		$\Downarrow$
f **s-harmonious**	$\Rightarrow$	f **countable compact-covering**
$\Phi : \aleph_0\mathcal{K}_Y \rightrightarrows \mathcal{K}_X$		$\Phi : \aleph_0\mathcal{K}_Y \rightrightarrows \mathcal{K}_X$
$\forall C \in \Phi(K)\, f(C) = K$		$\forall C \in \Phi(K)\, f(C) = K$
Φ lower semicontinuous		
$\Downarrow$		
f **harmonious**		
$\Phi : Y \rightrightarrows \mathcal{K}_X$		
$f(\Phi(y)) = \{y\}$		
Φ lower semicontinuous		
$\Downarrow$		
f **tri-quotient**		
$\mathcal{R} : Y \to \mathfrak{D}(X)$		
$\mathcal{R}(y)$ supported by $f^{-}(y)$		
$\mathcal{R}(y)$ continuous		

The previous diagram, where f is a map from X to Y, gives such characterizations of the principal classes of maps studied in this paper. Denote $\mathcal{K}_X$ (respectively $\aleph_0\mathcal{K}_X$) the set of all nonempty compact subsets (respectively countable compact subsets) of X endowed with the upper Vietoris topology.

I have observed that in several theorems about the relationship between tri-quotient maps and inductively perfect maps, the hypothesis provide that tri-quotient maps are harmonious and therefore these results can be obtained with the aid of selection theorems for lower semicontinuous multivalued mappings of Michael [14].

On the other hand, consonant spaces are exactly the spaces where all the compact families are compactly generated, and it is known that Čech complete spaces are consonant (see [8] of Dolecki, Greco and Lechicki). Therefore consonance is an essential condition under which tri-quotient maps become harmonious or compact-harmonious.

In this paper I prove that tri-quotient maps with every consonant fiber are harmonious. After Arhangel'skii that have proved that every open map from a Čech-complete topology is compact-covering [1, Theorem 1.2] and Bouziad in [3] that have generalized this result to the effect that each open map from a consonant topology is compact-covering, I show that every tri-quotient map with consonant domain is compact-harmonious hence compact-covering. On the other hand, I prove that sieve complete spaces are consonant. Finally I give the following result:

Theorem 1.1. *Every harmonious map $f : X \to Y$ from a regular monotonic p-space X onto a countable regular space Y, is inductively perfect.*

Theorem 1.1 extends the following theorems:

Theorem 1.2 (Michael [15, Theorem 1.4]). *Every tri-quotient map $f : X \to Y$ from a metric space X onto a countable regular space Y, with each $f^{-1}(y)$ completely metrizable, is inductively perfect.*

Theorem 1.3 (Just and Wicke [10, Theorem 3.1]). *Every tri-quotient map $f : X \to Y$ from a regular monotonic p-space X onto a countable regular space Y, with each $f^{-1}(y)$ sieve complete subspace, is inductively perfect.*

2. Compact families

Throughout this paper, for every subset A of a topological space X, we denote

$$\mathcal{O}(A) = \{U \in \mathcal{O}_X \colon\ A \subset U\}$$

where $\mathcal{O}_X$ is the family of all open subsets of X, and

$$\mathcal{O}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathcal{O}(A)$$

for every family of subsets of X.[2] A family $\mathcal{S}$ is compact provided that:

- $\mathcal{O}(\mathcal{S}) = \mathcal{S}$,
- for every $\mathcal{Q} \subset \mathcal{O}_X$,

$$\bigcup \mathcal{Q} \in \mathcal{S} \Rightarrow \exists \mathcal{P} \subset \mathcal{Q} \quad (|\mathcal{P}| < \infty \text{ and } \bigcup \mathcal{P} \in \mathcal{A}). \tag{2.1}$$

I consider on $\mathfrak{D}(X)$, the set of all compact families X, the topology generated by the family $\{\widetilde{U}\colon U \in \mathcal{O}_X\}$, where $\widetilde{U} = \{\mathcal{S} \in \mathfrak{D}(X)\colon U \in \mathcal{S}\}$. I call it the *Stone topology* because its definition is similar to the classical one on the set of all ultrafilters on X. It is clear that a subset K of X is compact if and only if $\mathcal{O}(K)$ is compact. More generally, if $\mathcal{K}$ is a family of compact subsets of X, then $\mathcal{O}(\mathcal{K})$ is a compact family of X.

We say that a compact family $\mathcal{A}$ is *compactly generated* if $\mathcal{A} = \mathcal{O}(\mathcal{H})$ for some $\mathcal{H} \subset \mathcal{K}_X$. We will denote $\mathfrak{D}_K(X)$ the set of all compactly generated compact families.

Throughout this paper, the set $\mathcal{K}_X$ of all nonempty compact subsets of X is endowed with the upper Vietoris topology which is generated by the family $\{\widetilde{U}\colon U \in \mathcal{O}_X\}$ where

$$\widetilde{U} = \{K \in \mathcal{K}_X\colon K \subset U\}. \tag{2.2}$$

Observe the following series of homeomorphic embeddings:

$$X \to \mathcal{K}_X \to \mathcal{K}_X \to 2^{\mathcal{K}_X} \to \mathcal{D}(X),$$

where $2^{\mathcal{K}_X}$ denotes the set of all subsets of $\mathcal{K}_X$ and is endowed with the lower Vietoris topology relative to the upper Vietoris topology on $\mathcal{K}_X$.[3] It is known that a multivalued mapping $\Phi : X \rightrightarrows Y$ between two topological spaces is lower semicontinuous if and only if it is continuous as a map from X to Y endowed with the lower Vietoris topology relative to X. Notice that $\mathfrak{D}_K(X)$ and $2^{\mathcal{K}_X}$, respectively endowed with the Stone topology and the lower Vietoris topology are homeomorphic.

In the following proposition, the space Y is considered as a subset of $\mathfrak{D}(Y)$ endowed with the Stone topology.

Proposition 2.1. *Let X and Y two topological spaces. Each continuous map $\mathcal{R} : Y \to \mathfrak{D}(X)$ has a continuous extension $\widetilde{\mathcal{R}} : \mathfrak{D}(Y) \to \mathfrak{D}(X)$ such that*

$$\widetilde{\mathcal{R}}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{R}(y).$$

Proof. Let $\mathcal{A}$ be a compact family of Y. Consider the family

$$\widetilde{\mathcal{R}}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{R}(y).$$

The family $\widetilde{\mathcal{R}}(\mathcal{A})$ is compact in X.

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[2] For every filter $\mathcal{F}$ on a topological space X, we call *adherence* of $\mathcal{F}$ the set $\operatorname{adh} \mathcal{F} = \bigcap\{\operatorname{cl} F\colon F \in \mathcal{F}\}$. A family $\mathcal{A}$ *meshes* a family $\mathcal{B}$ (in symbols $\mathcal{A}\#\mathcal{B}$) if and only if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. A family $\mathcal{S}$ is *compactoid* (respectively *compact*) if and only if for every filter $\mathcal{F}$ on X that meshes $\mathcal{S}$, one has $\operatorname{adh} \mathcal{F} \neq \emptyset$ (respectively $\operatorname{adh} \mathcal{F}$ meshes $\mathcal{S}$).

[3] Let $A^{\#} = \{B \subset X\colon B \cap A \neq \emptyset\}$. If X is a topological space, then lower Vietoris topology on 2^X relative to the topology of X is generated by the family $\{U^{\#}\colon U \in \mathcal{O}_X\}$.

Let $U \subset V$ two open subsets of X. If $U \in \widetilde{\mathcal{R}}(\mathcal{A})$ then there exists an $A \in \mathcal{A}$ such that $U \in \mathcal{R}(y)$ for every $y \in A$. Thus $V \in \mathcal{R}(y)$ for every $y \in A$ and $v \in \widetilde{\mathcal{R}}(\mathcal{A})$ and hence, $\mathcal{O}(\widetilde{\mathcal{R}}(\mathcal{A})) = \widetilde{\mathcal{R}}(\mathcal{A})$.

Let $\mathcal{E}$ be a family of open subsets of X. If $\bigcup \mathcal{E} \in \widetilde{\mathcal{R}}(\mathcal{A})$ then there exists an $A \in \mathcal{A}$ such that $\bigcup \mathcal{E} \in \mathcal{R}(y)$ for every $y \in A$. Let $y \in A$, by compactness of $\mathcal{R}(y)$, there exists a finite subfamily $\mathcal{E}_y \subset \mathcal{E}$ with $\bigcup \mathcal{E}_y \in \mathcal{R}(y)$. By continuity of $\mathcal{R}$, there exists an open neighborhood W_y of y such that $\bigcup \mathcal{E}_y \in \mathcal{R}(w)$ for each $w \in W_y$. Since $\mathcal{O}(\widetilde{\mathcal{R}}(\mathcal{A})) = \widetilde{\mathcal{R}}(\mathcal{A})$, the open set $\bigcup_{y \in A} W_y$ is in $\mathcal{A}$ and thus, by compactness of $\mathcal{A}$, there exists a finite subset A_0 of A such that $W = \bigcup_{y \in A_0} W_y \in \mathcal{A}$. The set $\widetilde{\mathcal{E}} = \{E \in \mathcal{E}_y\colon\ y \in A_0\}$ is a finite subfamily of $\mathcal{E}$. If $w \in W$, then there exists $y \in A_0$ such that $w \in W_y$ thus, $\bigcup \mathcal{E}_y \in \mathcal{R}(w)$. Hence $\bigcup \widetilde{\mathcal{E}} \in \mathcal{R}(w)$ for every $w \in W$ and since $W \in \mathcal{A}$, then $\bigcup \widetilde{\mathcal{E}} \in \widetilde{\mathcal{R}}(\mathcal{A})$.

The map $\widetilde{\mathcal{R}}$ is continuous. If $\widetilde{\mathcal{R}}(\mathcal{A}) \in \widetilde{U}$, that is $U \in \widetilde{\mathcal{R}}(\mathcal{A})$, then there exists $A \in \mathcal{A}$ such that $U \in \mathcal{R}(y)$ for every $y \in A$. By continuity of $\mathcal{R}$, for every $y \in A$, there exists $V_y \in \mathcal{O}(y)$ with $U \in \mathcal{R}(v)$ for every $v \in V_y$. The open set $V = \bigcup_{y \in A} V_y \in \mathcal{A}$, that is $\mathcal{A} \in \widetilde{V}$, and $\widetilde{\mathcal{R}}(\mathcal{S}) \in \widetilde{U}$ for every $\mathcal{S} \in \widetilde{V}$. □

3. Tri-quotient maps and others classes of maps

In [13] Michael introduces tri-quotient maps. The following definitions are written verbatim.

Definition 3.1. A continuous and onto map $f : X \to Y$ is *tri-quotient* if there exists a map t from $\mathcal{O}_X$ to $\mathcal{O}_Y$ such that:

(1) $t(U) \subset f(U)$;
(2) $t(X) = Y$;
(3) $U \subset V$ implies $t(U) \subset t(V)$;
(4) if $y \in t(U)$ and $\mathcal{W}$ is a cover of $f^{-1}(y) \bigcap U$ by open subsets of X, then there is a finite $\mathcal{F} \subset \mathcal{W}$ such that $y \in t(\bigcup \mathcal{F})$.

The same year, in [18, Theorem 1], Ostrovsky, in the proof of this theorem, uses three properties which put together a second definition of tri-quotient maps (see also [19]).

Definition 3.2. A continuous and onto map $f : X \to Y$ is tri-quotient if, for every $y \in Y$, there exists a family η_y of open subsets of X such that $X \in \eta_y$ and

(1) if $U \in \eta_y$ and γ is a cover of $U \cap f^{-1}(y)$ by open subsets of X, then there exists a finite number k of elements of γ such that $\bigcup_{i=1}^{k} U_i \in \eta_y$;
(2) if $U \in \eta_y$ then $y \in \operatorname{int} f(U)$;
(3) if $U \in \eta_y$ then there is a neighborhood $O(y)$ such that $U \in \eta_\xi$ for every $\xi \in O(y)$.

In [13] Michael shows that tri-quotient maps are preserved by finite products and asks whether tri-quotient maps are preserved by infinite products. In [20, Proposition 1], Uspenskij answers in the affirmative. In that paper, he rewrites the characterization of tri-quotient maps of Ostrovsky [18,19] in terms of $\mathcal{Q}$-systems. These $\mathcal{Q}$-systems are precisely compact families in our definition [8]. A family $\mathcal{S}$ of open subsets of X is *supported* by a family $\mathcal{E}$ of subsets of X whenever $U \in \mathcal{S}$, $U \cap E = V \cap E$ for some $E \in \mathcal{E}$, then $V \in \mathcal{S}$. This amounts to $U \in \mathcal{S}$ if and only if there exists $E \in \mathcal{E}$ such that $U \cap E \in \mathcal{S}|_E$, where $\mathcal{S}|_E = \{U \cap E\colon\ U \in \mathcal{S}\}$.

Definition 3.3 [20, Proposition 1]. A map $f : X \to Y$ is tri-quotient if and only if there exists a continuous map $\mathcal{R} : Y \to \mathfrak{D}(X)$ such that for every $y \in Y$ the family $\mathcal{R}(y)$ is supported by $f^{-1}(y)$.

Actually, the assignment t and the mapping $\mathcal{R}$ are related by:

$$U \in \mathcal{R}(y) \Leftrightarrow y \in t(U). \tag{3.1}$$

If f is a map from X to Y and $\mathcal{A}$ a family of subsets of Y, I denote $f^{-}(\mathcal{A}) = \{f^{-1}(A)\colon\ A \in \mathcal{A}\}$. Here is a third characterization of tri-quotient maps.

Proposition 3.4. *A map $f : X \to Y$ is tri-quotient if and only if there exists a continuous map $\mathcal{R} : \mathfrak{D}(Y) \to \mathfrak{D}(X)$ such that for every $\mathcal{A} \in \mathfrak{D}(Y)$ the family $\mathcal{R}(\mathcal{A})$ is supported by $f^{-}(\mathcal{A})$.*

Proof. If there exists a continuous map $\mathcal{R} : \mathfrak{D}(Y) \to \mathfrak{D}(X)$ such that for every $\mathcal{A} \in \mathfrak{D}(Y)$ the family $\mathcal{R}(\mathcal{A})$ is supported by $f^{-}(\mathcal{A})$, its restriction to Y satisfies the condition of Proposition 3.3.

Let $f : X \to Y$ be a tri-quotient map and let $\mathcal{R} : Y \to \mathfrak{D}(X)$ be a corresponding map in the characterization of Proposition 3.3. By Proposition 2.1, the map $\mathcal{R}$ has a continuous extension to $\mathfrak{D}(Y)$.

Let U and V two open subsets of X. If there exists $A \in \mathcal{A}$ such that $U \cap f^{-1}(A) = V \cap f^{-1}(A)$ and $U \in \mathcal{R}(\mathcal{A})$, then for every $y \in A$, $U \cap f^{-1}(y) = V \cap f^{-1}(y)$ and $U \in \mathcal{R}(y)$. Therefore $V \in \mathcal{R}(y)$ for every $y \in A$ that is $V \in \mathcal{R}(\mathcal{A})$. Hence the family $\mathcal{R}(\mathcal{A})$ is supported by $f^{-}(\mathcal{A})$. □

Here are two examples of tri-quotient maps given by Michael in [13, Theorem 6.5] with their t-assignment and the map $\mathcal{R} : Y \to \mathfrak{D}(X)$.

Example 3.5.

(1) Each open map is tri-quotient. In this case, $t(U) = f(U)$ for every open subset U and for every $y \in Y$, $\mathcal{R}(y) = f^{-1}(y)^{\#} = \{U \in \mathcal{O}_X\colon\ U \cap f^{-1}(y) \neq \emptyset\}$.

(2) Each perfect map is tri-quotient. Put $t(U) = f(U^c)^c$ for every open subset U and $\mathcal{R}(y) = \{U \in \mathcal{O}_X\colon\ f^{-1}(y) \subset U\}$ for every $y \in Y$.

In these examples, each family $\mathcal{R}(y)$ is *compactly generated* [21] by compact subsets of $f^{-1}(y)$. As we have seen in Section 2, the existence of a continuous map $\mathcal{R} : Y \to \mathfrak{D}_K(X)$ amounts to the existence of a lower semicontinuous multivalued mapping $\Phi : Y \rightrightarrows \mathcal{K}_X$. As in [5], a multi-valued mapping $\Phi : Y \rightrightarrows 2^X$ is a *lifting* for the map $f : X \to Y$ if

$$\emptyset \neq \bigcup \Phi(y) \subset f^{-1}(y) \quad \text{for every } y \in Y.$$

In [17], Ostrovsky introduces harmonious maps.

Definition 3.6. A continuous and onto map $f : X \to Y$ is *harmonious* if, for every $y \in Y$, there exists a non-empty family μ_y of open subsets of X such that:

(1) if $U \in \mu_y$ then there is a non-empty compact set $B \subset f^{-1}(y) \cap U$ such that, for every $V \supset B$, we have $V \in \mu_y$;

(2) if $U \in \mu_y$ then there is an open set $O(y) \ni y$ such that $U \in \mu_w$ for every $w \in O(y)$.

When $\mathcal{K}_X$ is endowed with the upper Vietoris topology, we have the following characterization of harmonious maps.

Proposition 3.7. *A map $f : X \to Y$ is harmonious if and only if there is a lower semicontinuous lifting*[4] *$\Phi : Y \rightrightarrows \mathcal{K}_X$ for f.*

Proof. Let $f : X \to Y$ be an harmonious map. It is clear that $\mu_y = \mathcal{O}(\mu_y)$ for every $y \in Y$. The first part of condition (1) induces a multivalued mapping $\Phi : Y \rightrightarrows \mathcal{K}_X$ such that $B \subset f^{-1}(y)$ and $\mu_y \subset \mathcal{O}(\Phi(y))$ for every $y \in Y$. The condition $V \in \mu_y$ for every $V \supset B$ amounts to $\mathcal{O}(\Phi(y)) \subset \mu_y$ for every $y \in Y$ and hence, $\mathcal{O}(\Phi(y)) = \mu_y$.

Let U be an open subset of X. We have to prove that

$$V = \{y \in Y\colon\ \Phi(y) \cap \widetilde{U} \neq \emptyset\}$$

is an open subset of X. A point $y \in V$ if and only if there is $B \in \Phi(y)$ such that $B \in \widetilde{U}$ that is, $B \subset U$. This amounts to $U \in \mathcal{O}(\Phi(y))$. Hence $U \in \mu_y$ and by condition (2), there is an open set $O(y) \ni y$ such that $U \in \mu_w$ for every $w \in O(y)$ that is $U \in \mathcal{O}(\Phi(w))$ for every $w \in O(y)$. Therefore, $O(y) \subset V$.

Conversely, if $f : X \to Y$ is an harmonious map with a lower semicontinuous lifting $\Phi : Y \rightrightarrows \mathcal{K}_X$, the family $\mu_y = \mathcal{O}(\Phi(y))$ is suitable for the definition. □

This characterization amounts to Property A introduced by Just and Wicke in [11, Definition 3.0] for maps with a metric space as domain.

Proposition 3.8. *A map $f : X \to Y$ is harmonious if and only if f is tri-quotient and the defining $\mathcal{R}(y)$ is compactly generated by a family of subsets of $f^{-1}(y)$.*

Proof. If f is harmonious with $\Phi : Y \to \mathcal{K}_X$ a lower semicontinuous lifting, let

$$\mathcal{R}(y) = \{U \in \mathcal{O}_X\colon\ \exists K \in \Phi(y)\ K \subset U\}.$$

Then $\mathcal{R} : Y \to \mathfrak{D}(X)$ is continuous and supported by $f^{-1}(y)$ for every $y \in Y$.

Conversely, if f is tri-quotient and the family $\mathcal{R}(y)$ is compactly generated by a family of compact subsets of $f^{-1}(y)$, then $\Phi : Y \to \mathcal{K}_X$ where $\Phi(y)$ is the family of compact subsets of $f^{-1}(y)$ that generates $\mathcal{R}(y)$ is a lower semicontinuous lifting of f. □

[4] In [5], Debs and Saint Raymond say that the map has quasi-upper semicontinuous lifting.

Ostrovsky also introduces compact-harmonious maps and s-harmonious maps. Denote $\aleph_0\mathcal{K}_Y$ the set of all nonempty countable compact subsets of Y. A map $f:X \to Y$ is *compact-harmonious* (respectively *s-harmonious*) if there exists a multi-valued lower semicontinuous mapping $\Phi:\mathcal{K}_Y \rightrightarrows \mathcal{K}_X$ (respectively $\Phi:\aleph_0\mathcal{K}_Y \rightrightarrows \mathcal{K}_X$) with $f(B) = K$ for every $K \in \mathcal{K}_Y$ (respectively $K \in \aleph_0\mathcal{K}_Y$) and for every $B \in \Phi(K)$. Note that every compact-harmonious map (respectively s-harmonious map) is compact-covering (respectively countable compact-covering).[5] In [17, Lemma 1], Ostrovsky proves that every compact-covering map between two separable metrizable spaces is compact-harmonious.

4. Consonance

We have seen that harmonious maps are those tri-quotient maps for which a defining map $\mathcal{R}$ consists of compact families that are compactly generated. A topology is said to be *consonant* if every compact family is compactly generated [8].[6] It is known that regular Čech-complete topologies are consonant.

We have immediately that

Proposition 4.1. *Each tri-quotient map with consonant fibers, onto an Hausdorff space is harmonious.*

Proof. For every $y \in Y$, the family

$$\mathcal{R}(y)|_{f^{-1}(y)} = \{U \cap f^{-1}(y):\ U \in \mathcal{R}(y)\}$$

is a compact family of $f^{-1}(y)$. Since $\mathcal{R}(y)$ is supported by $f^{-1}(y)$, an open subset U of X is in $\mathcal{R}(y)$ if and only if $U \cap f^{-1}(y)$ is in $\mathcal{R}(y)|_{f^{-1}(y)}$. And since $f^{-1}(y)$ is consonant, there exists a family $\Phi(y)$ of compact subsets of $f^{-1}(y)$ such that $U \in \mathcal{R}(y)$ if and only if there exists $K \in \Phi(y)$ included in U. □

As consonance is hereditary for closed subsets, each tri-quotient map from a consonant space onto an Hausdorff space is harmonious.

Arhangel'skii proved that every open map from a Čech-complete topology is compact-covering [1, Theorem 1.2]. Bouziad in [3] generalizes this result to the effect that each open map from a consonant topology is compact-covering. Here I give a double reinforcement of the above result of Bouziad: by relaxing the hypothesis on maps and spaces, and by strengthening the conclusion.

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[5] Recall that a continuous map f from X onto Y is compact-covering (respectively countable compact-covering) if and only if every compact (respectively countable compact) subset of Y is the image of a compact subset of X.

[6] In [8], consonant spaces are introduced by Dolecki, Greco and Lechicki as spaces where the upper Kuratowski topology and the co-compact topology coincide on $\mathbb{F}(X)$, the family of all closed subsets of X. The *co-compact topology* on $\mathbb{F}(X)$ has the family $\{A \in \mathbb{F}(X):\ A \cap K = \emptyset\}$, where K ranges over compact subsets of X, for a base of open sets while a subset $\mathcal{A}$ of $\mathbb{F}(X)$ is open in the *upper Kuratowski topology* if and only if $\mathcal{A}_c$ is a compact family of X. For a given family $\mathcal{A}$ of subsets of X, denote $\mathcal{A}_c = \{A^c:\ A \in \mathcal{A}\}$.

Theorem 4.2. *If $f : X \to Y$ is tri-quotient, Y is Hausdorff and $f^{-1}(K)$ is consonant for every compact* (*respectively countable compact*) *subset K of Y, then f is compact-harmonious* (*respectively s-harmonious*).

Proof. Let K be a compact (respectively countable compact) subset of Y. By Proposition 3.4 the family $\mathcal{R}(K)$ is a compact family of X supported by $f^{-1}(K)$. Since $\mathcal{R}(K)$ is compact and $f^{-1}(K)$ consonant, there exists $\Phi(K)$ a family of compact subsets of $f^{-1}(K)$ such that $U \in \mathcal{R}(K)$ if and only if there exists $C \in \Phi(K)$ such that $C \subset U$. For every $C \in \Phi(K)$, K is a subset of $f(C)$. Suppose $y \notin f(C)$, then $C \subset f^{-1}(y)^c$ and then $f^{-1}(y)^c \in \mathcal{R}(K)$ and $K \subset t(f^{-1}(y)^c) \subset f(f^{-1}(y)^c)$. Hence, y is not in K.

At last, $\Phi : \mathcal{K}_Y \rightrightarrows \mathcal{K}_X$ (respectively $\Phi : \aleph_0\mathcal{K}_Y \rightrightarrows \mathcal{K}_X$) is lower semicontinuous. If $\Phi(K) \cap \widetilde{V} \neq \emptyset$, then there exists $C \in \Phi(K)$ such that $C \subset V$, that is, $V \in \mathcal{R}(y)$ for every $y \in K$. Hence for every $y \in K$, there exists a neighborhood U_y of y such that $V \in \mathcal{R}(z)$ for every $z \in U_y$. Let $U = \bigcup_{y\in K} U_y$. Then $\widetilde{U}$ is a neighborhood of K and if $L \in \widetilde{U}$, then $V \in \mathcal{R}(z)$ for every $z \in L$ thus $\Phi(L) \cap \widetilde{V} \neq \emptyset$. □

As consonance is hereditary for closed subsets, each tri-quotient map from a consonant space onto a Hausdorff space is compact-harmonious.

In [16, Theorem 8.2] Nogura and Shakhmatov show that continuous open maps are consonance-preserving and Bouziad in [2] proves that perfect maps also preserve consonance. The following result embraces these two cases.

Theorem 4.3. *Tri-quotient maps preserve consonance.*

Proof. As $X \in \mathcal{R}(y)$ and $\mathcal{R}(y)$ is supported by $f^{-1}(y)$ for every $y \in Y$, every open subset of X which includes $f^{-1}(y)$ is in $\mathcal{R}(y)$. Hence, for every open subset W of Y, the statement $y \in W$ amounts to $f^{-1}(W) \in \mathcal{R}(y)$.

Let $\mathcal{A}$ be a compact family of Y. By Proposition 3.4 the family $\mathcal{R}(\mathcal{A})$ is a compact family of X. Since X is consonant, there exists $\mathcal{C} \subset \mathcal{K}_X$ with $\mathcal{R}(\mathcal{A}) = \mathcal{O}(\mathcal{C})$. Note that $f(\mathcal{C}) = \{f(C);\ C \in \mathcal{C}\}$. Now we claim that $\mathcal{A} = \mathcal{O}(f(\mathcal{C}))$. Indeed, $W \in \mathcal{A}$ if and only if $f^{-1}(W) \in \mathcal{R}(\mathcal{A})$ if and only if $C \in f^{-1}(W)$ for some $C \in \mathcal{C}$ that amounts to $f(C) \subset W$. □

5. Monotonic spaces

Recall that a *sieve* on a topological space X is a sequence of open covers $\{U_\alpha\colon \alpha \in A_n\}_n$ of X (with disjoint A_n) together with functions $\pi_n : \mathcal{A}_{n+1} \to \mathcal{A}_n$ such that for all $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}_n$,

$$U_\alpha = \bigvee \{U_\beta\colon\ \beta \in \pi_n^{-1}(\alpha)\}.$$

A π*-chain* for such a sieve is a sequence $(\alpha_n)_n$ such that $\alpha_n \in \mathcal{A}_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for all n. The sieve is *complete* if $(\alpha_n)_n$ is compactoid for every π-chain $(\alpha_n)_n$.

For a sequence $(B_k)_k$, note that $\mathcal{B} = \{B_k:\ k \in \mathbb{N}\}$. In [4] Chaber, Čoban and Nagami called a sequence $(B_k)_k$ an

(1) *md-sequence* if for every $x \in \bigcap \mathcal{B}$, the family $\mathcal{B}$ is a base at x.

(2) *mp-sequence* if $\bigcap \mathcal{B} \neq \emptyset$ and if the family $\mathcal{B}$ is compactoid.

(3) *mc-sequence* if the family $\mathcal{B}$ is compactoid.

If m is any of the properties md, mp, mc, a sequence $(\mathcal{B}_n)_n$ of bases is called an *m-sequence* of bases if every closurewise decreasing [7] sequence $(B_n)_n$, where $B_n \in \mathcal{B}_n$ for every $n \in \mathbb{N}$, has the property m.

A space X is said to be a *monotonic p-space* if it has an mp-sequence of bases, and is said to have a *base of countable order* if it has an md-sequence of bases.

It was proved in [4, Lemma 1.1] that a space has an mc-sequence of bases if and only if it has a complete sieve.

Dolecki, Greco and Lechicki have shown that every Čech-complete topology is consonant [8, Theorem 4.1]. A slight modification of their proof yields

Theorem 5.1. *Every regular sieve-complete topology is consonant.*

Proof. Let $(\mathcal{A}_n)_n$ be a finitely additive complete sieve on X. Let $\mathcal{D}$ be a compact family and $Q_0 \in \mathcal{O}(\mathcal{D})$. Since the space is regular, the family

$$\mathcal{U}_0 = \left\{U \in \mathcal{O}_X:\ \exists A \in \mathcal{A}_1\ \operatorname{cl} U \subset A \cap Q_0\right\}$$

is an open cover of Q_0. Thus by compactness of $\mathcal{D}$, there exists a nonempty set $Q_1 \in \mathcal{O}(\mathcal{D})$ and $A_1 \in \mathcal{A}_1$, such that $\operatorname{cl} Q_1 \subset Q_0$ and $Q_1 \subset A_1$. Now, the family

$$\mathcal{U}_1 = \left\{U \in \mathcal{O}_X:\ \exists A \in \pi_1^{-1}(A_1)\ \operatorname{cl} U \subset A \cap Q_1\right\}$$

is an open cover of Q_1. Hence, there exists a nonempty open set $Q_2 \in \mathcal{O}(\mathcal{D})$ and $A_2 \in \pi_1^{-1}(A_1)$, such that $\operatorname{cl} Q_2 \subset Q_1$ and $Q_2 \subset A_2$. Similarly, we can construct a π-chain $(A_n)_n$ and a sequence $(Q_n)_n$ of nonempty open sets such that, for every n, $Q_n \in \mathcal{O}(\mathcal{D})$, $\operatorname{cl} Q_{n+1} \subset Q_n$ and $Q_n \subset A_n$. If $\mathcal{G}$ is the filter base $\{Q_n:\ n \in \mathbb{N}\}$, then $\operatorname{adh} \mathcal{G} \subset Q_0$. Every filter $\mathcal{F}$ that meshes $\mathcal{G}$, meshes the π-chain $(A_n)_n$. Hence, by sieve-completeness of X, $\operatorname{adh} \mathcal{F} \neq \emptyset$ and therefore, $\mathcal{G}$ is compactoid. By regularity, this implies that adherence of $\mathcal{G}$ is compact and every open set $O \supset \operatorname{adh} \mathcal{G}$ belongs to $\mathcal{G}$ [8, Proposition 2.1] and thus belongs to $\mathcal{O}(\mathcal{D})$. The consonance is proved. □

In fact, we have a shorter indirect proof of Theorem 5.1. From [4, Theorem 3.7], it is known that a regular space is sieve-complete if and only if it is an open image of a paracompact Čech-complete space. Since every Čech-complete space is consonant [8] and open maps preserve consonance, it follows that every regular sieve-complete space is consonant. [8]

[7] A sequence $(B_n)_n$ is *closurewise decreasing* if $\operatorname{cl} B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$.

[8] Uspenskij has brought to my attention that, in January 1993, thinking about Michael's problem on infinite products of tri-quotient maps, quite independently of other people, he came to the definition of consonant spaces (he called them for himself "Q-complete") and proved that regular sieve-complete spaces are consonant.

As a consequence of Proposition 4.1, we have

Corollary 5.2. *Each tri-quotient map with sieve-complete fibers onto an Hausdorff space is harmonious.*

On the other hand, Proposition 4.2 implies that

Corollary 5.3. *Each tri-quotient map from a regular sieve-complete space to a Hausdorff space is compact-harmonious.*

6. Harmonious maps, countable compact-covering maps and inductively perfect maps

Recall that a continuous and onto map $f : X \to Y$ is *perfect* if and only if f is closed and each fiber is compact. Note that f is perfect if and only if each fiber is compact and the map $\Phi : Y \to \mathcal{K}_X$ with $\Phi(y) = f^{-1}(y)$ is continuous. This amounts to the compactness of $f^{-1}(K)$ for every compact subset K of Y together with the continuity of the map $\Psi : \mathcal{K}_Y \to \mathcal{K}_X$ such that $\Psi(K) = f^{-1}(K)$.

A map $f : X \to Y$ is called *inductively perfect* if there exists $Z \subset X$ such that $f(Z) = Y$ and $f|Z$ is perfect. Thus f is inductively perfect if and only if there exists a continuous map $\Phi : Y \to \mathcal{K}_X$ such that $\Phi(y)$ is a compact subset of $f^{-1}(y)$ for every $y \in Y$. This amounts to the existence of a continuous map $\Psi : \mathcal{K}_Y \to \mathcal{K}_X$ such that $f(\Psi(K)) = K$. Therefore it appears that the relation between compact-harmonious or harmonious maps and inductively perfect maps is essentially a problem of selection for a lower semicontinuous multivalued mapping.

In [15, Theorem 1.4] Michael has proved that every tri-quotient map $f : X \to Y$, with completely metrizable fibers, from a metrizable space onto a countable regular space Y is inductively perfect. Just and Wicke extended this result to tri-quotient maps, with sieve-complete fibers, from regular monotonic p-spaces to countable regular spaces [10, Theorem 3.1]. Note that in the two cases the hypothesis imply that a map f has consonant fibers thus is harmonious. The proofs of these two theorems use the following result of Michael:

Theorem 6.1 [14, Theorem 1.1]. *If $\Phi : Y \to 2^Z$ is lower semicontinuous, with Y a countable regular space and Z first-countable, then there exists a continuous map $g : Y \to Z$ such that $g(y) \in \Phi(y)$ for all $y \in Y$.*

Using this theorem with $Z = \mathcal{K}_X$ which is first-countable if X is metric, as f is harmonious, the proof of [15, Theorem 1.4] of Michael is immediate.

I show that the result of Just and Wicke [10, Theorem 3.1] remains true if we assume that f is only harmonious.

At first, I give two propositions that generalize Lemma 2.0 and Theorem 3.0 of Just and Wicke in [10].

Proposition 6.2.

(1) *If X is a regular monotonic p-space, then for every compact subset K of X and for every neighborhood $\widetilde{U}$ of K, there exists a compact set C having a countable base of neighborhoods such that $K \subset C$ and $C \in \widetilde{U}$.*

(2) *Moreover, if F is a G_δ-subset of X and $K \subset F$, then C can be chosen to also satisfy $C \subset F$.*

Proof. Let K be a compact subset of X and U be an open subset of X. Recall that $K \in \widetilde{U}$ amounts to $K \subset U$. Consider $(\mathcal{B}_n)_n$ an mp-sequence of bases and $U = \bigcup \mathcal{B}'_0$ for a $\mathcal{B}'_0 \subset \mathcal{B}_0$. Then there is a finite $\mathcal{U}_0 \subset \mathcal{B}'_0$ such that $K \subset \bigcup \mathcal{U}_0$. Denote $U_0 = \bigcup \mathcal{U}_0$. If F is a G_δ-subset of X such that $K \subset F$, then $F = \bigcap_{n\in\mathbb{N}} H_n$ else $H_n = Y$ for every $n \in \mathbb{N}$. The family $B'_1 = \{B \in \mathcal{B}_1\colon\ B\#K, \exists D \in \mathcal{U}_0 \ \mathrm{cl}\, B \subset D \cap H_1\}$ is an open cover of K, then there is a finite $\mathcal{U}_1 \subset \mathcal{B}'_1$ such that $K \subset \bigcup \mathcal{U}_1$. Denote $U_1 = \bigcup \mathcal{U}_1$. Similarly, we construct a sequence $(\mathcal{U}_n)_n$ of finite families of $\mathcal{B}_n$ for every $n \in \mathbb{N}$, a sequence $(U_n)_n$ of open sets that include K such that

$$U_n \subset \mathrm{cl}\, U_n \subset \bigcap_{i \leqslant n} H_i \quad \text{and} \quad U_n \subset \mathrm{cl}\, U_n \subset U_{n-1},$$

for every $n \in \mathbb{N}$. Now, let $\mathcal{U}$ the filter composed on the filter base $(U_n)_n$ and $\mathcal{G}$ a filter that meshes $\mathcal{U}$. Then $\mathcal{M}_n = \{B \in \mathcal{U}_n\colon\ B\#\mathcal{G}\}$ is nonempty and, for every $n \in \mathbb{N}$ if $B \in \mathcal{M}_n$, then there is $C \in \mathcal{M}_{n-1}$ such that $\mathrm{cl}\, B \subset C$. Since each $\mathcal{M}_n$ is finite, by Kunig's Lemma [9] there exists a closurewise decreasing sequence $(B_n)_n$ such that $B_n \in \mathcal{M}_n$ for every $n \in \mathbb{N}$. Since $(B_n)_n$ meshes K,

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \mathrm{cl}\, B_n \neq \emptyset$$

and thus $\mathcal{G}$ clusters in X and therefore $\mathcal{U}$ is compactoid. By [8, Proposition 2.1], $\mathrm{adh}\,\mathcal{U} = \bigcap_{n\in\mathbb{N}} U_n$ is a compact subset C of F and for every open set $U \supset C$ there exists $U_n \subset U$. □

Corollary 6.3. *If X is a regular monotonic p-space, then the set of compact subsets with a countable base of neighborhoods is dense in $\mathcal{K}_X$.*

Proposition 6.4. *Let $f : X \to Y$ be a harmonious map with a lower semicontinuous lifting Φ and let X be a regular monotonic p-space.*

(1) *If every point of Y is a G_δ-subset, then there exists a lower semicontinuous lifting Ψ such that $\mathcal{O}(\Psi(y)) = \mathcal{O}(\Phi(y))$ for every $y \in Y$ and every $L \in \Psi(y)$ has a countable base of neighborhoods in $\mathcal{K}_X$.*

(2) *If Y is regular, then Y is of pointwise countable type.*

(3) *Moreover if X has a base of countable order, then Y is first countable.*

Proof. (1) A compact set $L \in \Psi(y)$ if and only if L is a compact subset of $f^{-1}(y)$ with a countable base of neighborhoods and if there exists $K \in \Phi(y)$ such that $K \subset$

[9] Every tree of height ω with finite levels has an infinite branch.

L. By Proposition 6.2.1, $\Psi(y)$ is non-empty for every $y \in Y$. We have to show that $V = \{y \in Y;\ \Psi(y) \cap \widetilde{U} \neq \emptyset\}$ is an open subset of Y, for every open subset U of X. If $\Psi(y) \cap \widetilde{U} \neq \emptyset$ then there is $L \in \Psi(y)$ such that $L \subset U$. Since $L \in \Psi(y)$, there is $K \in \Phi(y)$ such that $K \subset L \subset U$ that is, $\Phi(y) \cap \widetilde{U} \neq \emptyset$. Since Φ is lower semicontinuous, there exists an open set $O(y) \ni y$ such that $\Phi(w) \cap \widetilde{U} \neq \emptyset$, for every $w \in O(y)$. Hence, for every $w \in O(y)$, there is $K_w \in \Phi(w)$ such that $K_w \subset U$. By Proposition 6.2.1, for every $w \in O(y)$, there exists a compact set C_w having a countable base of neighborhoods such that $K_w \subset C_w \subset U$. The set $C_w \in \Psi(w) \cap \widetilde{U}$ for every $w \in O(y)$, hence $O(y) \subset W$ and therefore, Ψ is lower semicontinuous.

Since every point of Y is a G_δ-subset, $f^{-1}(y)$ is a G_δ-subset of X and by Proposition 6.2.2 where $F = f^{-1}(y)$, the mapping Ψ is a lifting for f.

(2) Let $y \in Y$ and $K \in \Phi(y)$. By Proposition 6.2.1, there exists a compact subset $L \supset K$ of X with a countable base $(\widetilde{U}_n)_{n\in\mathbb{N}}$ of neighborhoods. Since $K \in \Phi(y)$, we have that $y \in t(U_0)$ and since Y is regular, there exists an open set $V_0 \subset Y$ such that $y \in V_0$ and $\operatorname{cl} V_0 \subset t(U_0)$. If we denote $W_1 = f^{-1}(V_0) \cap U_1$, then $W_1 \cap f^{-1}(y) = U_1 \cap f^{-1}(y)$, thus $y \in t(W_1)$ and by the isotonicity of t, we have $\operatorname{cl}(t(W_1)) \subset t(U_0)$. Similarly, we construct a sequence $(W_n)_{n\in\mathbb{N}}$ such that $\operatorname{cl}(t(W_{n+1})) \subset t(W_n)$ and $y \in \bigcap_{n\in\mathbb{N}} t(W_n)$. Let $\mathcal{G}$ be the filter generated by $(t(W_n))_{n\in\mathbb{N}}$. If a filter $\mathcal{F}$ meshes $\mathcal{G}$, since $t(W_n) \subset f(W_n)$, the filter $f^-(\mathcal{F})$ meshes $(W_n)_{n\in\mathbb{N}}$ thus it meshes $(U_n)_{n\in\mathbb{N}}$. Therefore $\operatorname{adh} f^-\mathcal{F}$ is nonempty and by the continuity of f, the filter $\mathcal{F}$ clusters in Y. Thus the filter $\mathcal{G}$ is compactoid and by [8, Proposition 2.1], $\operatorname{adh}\mathcal{G} = \bigcap_{n\in\mathbb{N}} t(W_n)$ is a compact subset of Y that has a countable base of neighborhoods in $\mathcal{K}_Y$ and such that $y \in \operatorname{adh}\mathcal{G}$.

If X has a base of countable order, the proof goes through verbatim the proof of [10, Theorem 3.0(b)]. □

Now, I am in position to give the following result.

Theorem 6.5. *Let $f : X \to Y$ be a harmonious map. If X is a regular monotonic p-space and Y is a countable regular space, then f is inductively perfect.*

Proof. If Y is countable and regular then every point of Y is a G_δ-subset. Since f is an harmonious map, there exists a lower semicontinuous lifting for f, $\Phi : Y \rightrightarrows \mathcal{K}_X$. Since X is a regular monotonic p-space and $\{y\}$ is a G_δ-subset of Y for each $y \in Y$, by Proposition 6.4.1, there exists a lower semicontinuous lifting for f, $\Psi : Y \rightrightarrows \mathcal{K}_X$ such that, for every $y \in Y$, all compact sets of $\Psi(y)$ have a countable base of neighborhoods in $\mathcal{K}_X$. Now, if in Theorem 6.1, the space Z is the subspace of $\mathcal{K}_X$ of all elements which admit a countable base of neighborhoods, then Ψ has a continuous selection and therefore, the map f is inductively perfect. □

Corollary 6.6. *Every tri-quotient map with consonant fibers from a regular monotonic p-space onto a countable regular space Y is inductively perfect.*

Corollary 6.7. *Every tri-quotient map from a regular sieve-complete space onto a countable regular space Y is inductively perfect.*

Proof. Every sieve-complete space is a monotonic p-space and by Theorem 5.1, X is consonant. Since $f^{-1}(y)$ is closed, $f^{-1}(y)$ is consonant for every $y \in Y$. Now the conclusion follows from Corollary 6.6. □

Corollary 6.8. *Let $f : X \to Y$ be a harmonious map. If X is a regular monotonic p-space and Y is Hausdorff, then f is countable compact-covering.*

The proof is the same one given in [10, Theorem 3.2]. I give it verbatim. Just before, note that it is easy to see that harmonious map are hereditar on closed subsets.

Proof. Let f, X and Y be as in assumptions, and let Z be a compact, countable subspace of Y. We show that there exists a compact cover of Z. Let $W = f^{-1}(Z)$. The space W is a closed subspace of X, and thus a regular, monotonic p-space. Therefore, the restriction of f to W satisfies the assumptions of Theorem 6.5. It follows that there is a subspace $W' \subset W$ that is mapped by f onto Z and such that $f|W'$ is perfect. Since inverse images of compact spaces under perfect maps are compact, W' is a compact cover of Z.

Corollary 6.9. *If $f : X \to Y$ is a tri-quotient map with consonant fibers from a regular monotonic p-space X onto a Hausdorff space Y, then f is countable compact-covering.*

In [11] Just and Wicke have shown that

Theorem 6.10. *If X is a metric space, Y a first-countable zero-dimensional space, and f a countable compact-covering surjection such that every fiber is separable, then f is harmonious.*

I am in a position to refine Theorem 6.10.

Theorem 6.11. *Let X be a regular monotonic p-space. Let Y be a Hausdorff first-countable space. If $f : X \to Y$ is a countable compact-covering map with every fiber second countable, then f is harmonious.*

7. Proof of Theorem 6.11

Proof. Note that if $f^{-1}(y)$ is second countable, then $\mathcal{K}_{f^{-1}(y)}$ is also second countable, and therefore by [9, Theorem 1.1.14], every open cover of every subspace of $\mathcal{K}_{f^{-1}(y)}$ has a countable subcover.

For every $y \in Y$, let $\Phi(y)$ be the family of all $K \in \mathcal{K}_X$ such that

(1) $K \subset f^{-1}(y)$,

(2) K has a first countable base of neighborhoods in $\mathcal{K}_X$,

(3) for every countable compact subset E of Y there exists a compact $C \subset X$ such that $C \cap f^{-1}(y) \subset K$ and $f(C) = E$.

Denote $\omega\mathcal{K}_{f^{-1}(y)}$ the family of all compact subsets of $f^{-1}(y)$ which have a countable base of neighborhoods in $\mathcal{K}_X$.

Let E be a compact, countable subset of Y, let $y \in Y$, and let $K \in \omega\mathcal{K}_{f^{-1}(y)}$. We say that E *eliminates* K if there is no compact $C \subset X$ with $f(C) = E$ and $C \cap f^{-1}(y) \subset K$.

Note that $\Phi(y)$ is the family of all $K \in \omega\mathcal{K}_{f^{-1}(y)}$ such that no countable, compact $E \subset Y$ eliminates K. □

Proposition 7.1.

(1) *If E eliminates $K \in \omega\mathcal{K}_{f^{-1}(y)}$, then $y \in E$.*

(2) *If E eliminates $K \in \omega\mathcal{K}_{f^{-1}(y)}$, and U is a neighborhood of y in Y, then $E \cap \operatorname{cl} U$ also eliminates K.*

(3) *If $E \subset F$ are countable, compact subsets of Y, and if E eliminates K, then so does F.*

(4) *If E eliminates some $K \in \omega\mathcal{K}_{f^{-1}(y)}$, then there is a neighborhood $\widetilde{V}$ of K such that E eliminates every $L \in \widetilde{V} \cap \omega\mathcal{K}_{f^{-1}(y)}$.*

Proof. (1) Let E be a countable compact subset of Y, since f is countable compact-covering, then there exists a compact subset of X such that $f(C) = E$. If y is not in E, then $C \cap f^{-1}(y) = \emptyset$. Hence E does not eliminate K.

(2) $E \cap \operatorname{cl} U$ and $E \cap U^c$, closed if U is open, are countable compact subsets of Y. If C and D are compact subsets of X such that $f(C) = E \cap \operatorname{cl} U$ and $f(D) = E \cap U^c$, then $C \cup D$ is a compact set of X with $f(C \cup D) = E$, and $(C \cup D) \cap f^{-1}(y) = C \cap f^{-1}(y)$.

(3) If C is compact with $f(C) = F$, then $D = C \cap f^{-1}(E)$ is also compact with $f(D) = E$ and $D \cap f^{-1}(y) = C \cap f^{-1}(y)$.

(4) Suppose (4) is false, and let $K \in \omega\mathcal{K}_{f^{-1}(y)}$ and E that eliminates K. Let $(\widetilde{V}_n)_{n\in\mathbb{N}}$ a base of neighborhoods of K in $\mathcal{K}_X$. For every $n \in \mathbb{N}$, there exists a compact subset $L_n \in \widetilde{V}_n$ of $f^{-1}(y)$ such that E does not eliminate L_n. Hence for every $n \in \mathbb{N}$ there exists a compact subset C_n of X such that $f(C_n) = E$ and $C_n \cap f^{-1}(y) \subset L_n$. Therefore, for every $n \in \mathbb{N}$, there exists a compact subset C_n of X such that $C_n \cap f^{-1}(y) \in \widetilde{V}_n$.

Lemma 7.2. *Let $(U_k)_{k\in\mathbb{N}}$ be a base of neighborhoods of y. For each $n \in \mathbb{N}$, there exists U_{k_n} such that $C_n \cap f^{-}(\operatorname{cl} U_{k_n}) \in \widetilde{V}_n$.*

Proof. Assume it is false. If there exists $n \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $C_n \cap f^{-}(\operatorname{cl} U_k) \notin \widetilde{V}_n$, that amounts to $C_n \cap f^{-}(\operatorname{cl} U_k) \not\subset V_n$, then there is w_k in $C_n \cap f^{-}(\operatorname{cl} U_k)$ such that $w_k \notin V_n$. C_n is compact, then the sequence $(w_k)_{k\in\mathbb{N}}$ clusters in C_n at some point w. Since f is continuous and $\bigcap_{k\in\mathbb{N}} \operatorname{cl} U_k = \{y\}$, the point w is in $f^{-1}(y)$. But w is not in V_n, hence $C_n \cap f^{-1}(y) \notin \widetilde{V}_n$. □

Continuation of the proof of Proposition 7.1. By Lemma 7.2, we find a base $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of y such that $C_n \cap f^{-}(\operatorname{cl} U_n) \in \widetilde{V}_n$ for every $n \in \mathbb{N}$. Let

$$C = \bigcup_{n\in\mathbb{N}} \bigl(C_n \cap f^{-1}\bigl((\operatorname{cl} U_n) \setminus U_{n+1}\bigr)\bigr) \cup K.$$

C is compact [11, Claim 1.4] and $f(C) = E \cap \operatorname{cl} U_0 = E'$. By Proposition 7.1.2, $E \cap \operatorname{cl} U_0 = E'$ eliminates $K \in \omega\mathcal{K}_{f^{-1}(y)}$ hence $y \in E'$. If $x \in C_n \cap f^{-1}((\operatorname{cl} U_n) \setminus U_{n+1})$, then $f(x) \in \operatorname{cl} U_0$ and since $f(C_n) = E$ for every n, then $f(x) \in E$. If $z \in E' \setminus \{y\}$, then for every $n \in \mathbb{N}$, there exists $x_n \in C_n$ such that $f(x_n) = z$ and since $z \in \operatorname{cl} U_0$, there is $n_z \in \mathbb{N}$ such that $z \in (\operatorname{cl} U_{n_z}) \setminus U_{n_z+1}$ and $x_{n_z} \in C_{n_z} \cap f^{-1}(\operatorname{cl} U_{n_z}) \setminus U_{n_z+1})$. □

Note that by Proposition 7.1.4, the family $\Phi(y)$ is closed.

Proposition 7.3. *If $y \in Y$ and $G \subset \Phi(y)^c$, then there is a countable compact set that eliminates all the elements of $G \cap \omega\mathcal{K}_{f^{-1}(y)}$.*

Proof. Let $K \in G \cap \omega\mathcal{K}_{f^{-1}(y)}$ and $K \notin \Phi(y)$, then there exists a countable compact set E_K and a neighborhood V_K of K such that E_K eliminates all the compact subsets $L \in V_K \cap \omega\mathcal{K}_{f^{-1}(y)}$.

The family $(V_k)_k$ is an open cover of $G \cap \omega\mathcal{K}_{f^{-1}(y)}$ and by [9, Theorem 1.1.14], there is a sequence $(K_n)_{n\in\mathbb{N}}$ such that for every compact subset $L \in G$ of $f^{-1}(y)$, there exists E_{K_n} that eliminates L. Let $(U_n)_{n\in\mathbb{N}}$ be a base of neighborhoods of y. $E = \bigcup_{n\in\mathbb{N}}(E_{K_n} \cap \operatorname{cl} U_n)$ is a countable compact set that eliminates all compact subsets $L \in G \cap \omega\mathcal{K}_{f^{-1}(y)}$. □

Proposition 7.4. *For every $y \in Y$, the family $\Phi(y)$ is nonempty.*

Proof. If $\Phi(y)$ is empty, then $\omega\mathcal{K}_{f^{-1}(y)} \subset \Phi(y)^c$ and by Proposition 7.3, there exists a countable compact set that eliminates all elements of $\omega\mathcal{K}_{f^{-1}(y)}$. But this is impossible because for every compact subset C of X such that $f(C) = E$, E does not eliminate the compact sets $L \in \omega\mathcal{K}_{f^{-1}(y)}$ such that $C \cap f^{-1}(y) \subset L$. □

Now we are in a position to prove that Φ is lower semicontinuous. Suppose that $\Phi : Y \to \omega\mathcal{K}_X$ is not lower semi-continuous.Then

$$\exists K \in \Phi(y)\ \exists \widetilde{V} \in \mathcal{N}(K)\ \forall U \in \mathcal{N}(y)\ \exists y' \in U\ \Phi(y') \cap \widetilde{V} = \emptyset.$$

Let $(U_n)_{n\in\mathbb{N}}$ be a base of neighborhoods of y. Then we can find a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in U_n$ for every $n \in \mathbb{N}$ and $\Phi(y_n) \cap \widetilde{V} = \emptyset$. Passing to a subsequence if necessary, assume that $y_n \in U_n \setminus U_{n+1}$. By the Proposition 7.3, for every $n \in \mathbb{N}$ we may choose a countable compact subset E_n of Y that eliminates all elements of $\widetilde{V} \cap \omega\mathcal{K}_{f^{-1}(y_n)}$. Hence $G_n = E_n \cap \operatorname{cl} U_n$ eliminates all compact subsets $L \in \widetilde{V} \cap \omega\mathcal{K}_{f^{-1}(y_n)}$. The set $E = \bigcup_{n\in\mathbb{N}} G_n \cup \{y\}$ is countable and compact. Let C be a compact subset of X with $f(C) = E$. For every $n \in \mathbb{N}$, the set E eliminates all elements of $\widetilde{V} \cap \omega\mathcal{K}_{f^{-1}(y_n)}$, hence by Proposition 6.2, $C \cap f^{-1}(y_n) \notin \widetilde{V}$. Therefore for every $n \in \mathbb{N}$, there is $w_n \in C \cap f^{-1}(y_n)$ such that $\{w_n\} \notin \widetilde{V}$. Since C is compact the sequence $(w_n)_{n\in\mathbb{N}}$ clusters at some point $w \in C$. By the continuity of f and $\{y\} = \bigcap_{n\in\mathbb{N}} U_n$, the point w is in $f^{-1}(y)$. Since $w_n \notin V$ for every $n \in \mathbb{N}$, the point w is not in K. Therefore E eliminates K. □

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