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Mutually *M*-intersecting Hermitian Varieties

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Let M be a set of integers. We consider a set of varieties in PG(n, q) such that each variety contains ρ points and the intersection of two distinct varieties contains μ points, where $\mu \in M$. Such a set is called a set of mutually M-intersecting varieties. In this paper, it is shown that there exist new sets of mutually M-intersecting varieties by using Hermitian varieties in $PG(2, q^2)$. © 1999 Academic Press

1. INTRODUCTION

Let f be a homogeneous polynomial. Then the set of points $\boldsymbol{x}$ of $PG(n\ q)$ satisfying $f(\boldsymbol{x}) = 0$ is called a *variety* and is denoted by $V(f)$. In a previous paper [4], we proposed a problem to find *mutually M-intersecting varieties.* This is a set of varieties $V(f_1)$, $V(f_2)$, ... , $V(f_s)$ which satisfies three conditions:

(i) M is a set of nonnegative integers.
(ii) $|V(f_i)| = \rho$ for $1 \le i \le s$.
(iii) $|V(f_i) \cap V(f_j)| \in M$ for $1 \le i, j \le s$, $i \ne j$.

We will use $\mathscr{V}(\rho, M)$ to denote such a set and when M is a singleton $\{\mu\}$, we simply write $\mathscr{V}(\rho, \mu)$. Note that $s = |\mathscr{V}(\rho, M)|$.

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Some results on mutually $\{\mu\}$-intersecting varieties are given in [4]. By using quadrics and a projective group on PG$(3, q)$, we obtained a $\mathscr{V}(q^2+1, q+1)$ consisting of q^2 varieties and a $\mathscr{V}((q+1)^2, 3q+1)$ of q^2 varieties. We are interested in finding as many as possible varieties which are mutually M-intersecting. This is not only an interesting geometrical problem but also a problem with combinatorial applications. When $M = \{\mu\}$, we can use a $\mathscr{V}(\rho, M)$ to construct combinatorial designs such as (r, λ)-designs and combinatorial arrays such as orthogonal, incomplete orthogonal or balanced arrays [2, 3]. When $|M| = 2$, a $\mathscr{V}(\rho, M)$ is related to a particular graph called a strong regular graph which is sometimes used in the design of experiments. In this paper, we will use results on intersections of Hermitian varieties given by Kestenband [8] to construct new sets of mutually M-intersecting Hermitian varieties in PG$(2, q^2)$ with $|M| = 1$ or 2.

2. HERMITIAN VARIETIES

An $(n+1) \times (n+1)$ square matrix $H = (h_{ij})$ with elements from GF(q^2) is called a *Hermitian matrix* if $h_{ij} = h_{ji}^q$ for all i, j. Let $A^{(q)}$ denote the matrix whose (i, j)-entry is a_{ij}^q if the (i, j)-entry of the matrix A is a_{ij}. A *Hermitian variety* (abbreviated to HV) in PG(n, q^2) is defined as $\{\boldsymbol{x} \in \text{PG}(n, q^2); f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} H \boldsymbol{x}^{(q)} = 0\}$, where H is a Hermitian matrix. Here we use $V(H)$, instead of $V(f)$ to denote the Hermitian variety. Two Hermitian matrices H and G are said to be *equivalent* if there exists a nonsingular matrix P over GF(q^2) such that $P^{\mathrm{T}} H P^{(q)} = G$. It is also known that any nonsingular Hermitian matrix is equivalent to the identity matrix I. This means that there exists a nonsingular matrix P such that $P^{\mathrm{T}} H P^{(q)} = I$ for any nonsingular Hermitian matrix H. When H is a Hermitian matrix of rank r, then $V(H)$ is called an HV of *rank* r. An HV of rank $n+1$ in PG(n, q^2) is said to be *nondegenerate*. The properties of an HV in PG$(2, q^2)$ have been studied in [1, 8]. An HV in PG$(2, q^2)$ contains q^2+1, q^3+q^2+1, or q^3+1 points, accordingly as the rank is 1, 2, or 3.

Kestenband [8] has determined the possible intersections of $V(H)$ with $V(I)$ in PG$(2, q^2)$ for any nonsingular Hermitian matrix H. Note that the minimal polynomial $m(x)$ of a matrix H satisfies $m(H) = \mathbf{0}$ and $m'(H) \neq \mathbf{0}$ for any polynomial $m'(x)$ with $\deg(m'(x)) < \deg(m(x))$.

Result (B. C. Kestenband). Let H be a nonsingular Hermitian matrix. Let $m(x)$ and $g(x)$ be minimal and characteristic polynomial of H, respectively. Then $|V(H) \cap V(I)|$ is one of the following:

(1) $(q+1)^2$ points, if $m(x) = g(x) = (x-\alpha)(x-\beta)(x-\gamma)$, α, β, γ distinct elements of GF(q);

(2) q^2+q+1 points, if $m(x)=g(x)=(x-\alpha)(x-\beta)^2$, α, β, distinct elements of $\mathrm{GF}(q)$;

(3) $q+1$ collinear points if $m(x)=(x-\alpha)(x-\beta)$, α, β, distinct elements of $GF(q)$;

(4) q^2+1 points, if $m(x)=g(x)=(x-\alpha)p(x)$, $\alpha\in\mathrm{GF}(q)$, $p(x)$: irreducible over $\mathrm{GF}(q)$;

(5) q^2+1 points, if $m(x)=g(x)=(x-\lambda)^3$;

(6) one point if $m(x)=(x-\lambda)^2$;

(7) q^2-q+1 points, no three of which are collinear, if $g(x)$ is irreducible over $\mathrm{GF}(q)$.

Kestenband [7] also generated a set χ consisting of q^2+q+1 Hermitian matrices with irreducible characteristic polynomials over $\mathrm{GF}(q)$. The set of varieties from χ forms a $\mathscr{V}(q^3+1,\, q^2-q+1)$. Since χ is isomorphic to $\mathrm{PG}(2,q)$, the incidence matrix of the varieties $\mathscr{V}(q^3+1, q^2-q+1)$ and the points on $\mathrm{PG}(2,q^2)$ contains q^2-q+1 copies of $\mathrm{PG}(2,q)$. In the next section, we will use Hermitian matrices having the minimal polynomial $(x+1)^3$ over $\mathrm{GF}(q)$, where q is an even prime power, to construct new mutually M-intersecting varieties which are different from those found by Kestenband.

3. CONSTRUCTIONS

We assume in the rest of this paper that q is an even prime power. Let

$$H=\begin{pmatrix}1 & a & 0\\ a^q & 1 & b\\ 0 & b^q & 1\end{pmatrix},$$

where $a, b\in\mathrm{GF}(q^2)\backslash\{0\}$, and $a^{q+1}=b^{q+1}$. Note that a and b are not necessary distinct. Then the minimal polynomial of H is $(x+1)^3$ since the characteristic is two. We define a set of HV's by

$$\mathscr{H}=\{V(H_0), V(H_1), \ldots, V(H_q)\},$$

where

$$H_i=\begin{pmatrix}1 & au^{iq} & 0\\ a^q u^i & 1 & bu^{iq}\\ 0 & b^q u^i & 1\end{pmatrix},$$

$u \in \mathrm{GF}(q^2)$, and the order of u is $q+1$. Note that $H_0 = H$ and $V(H_i) \in \mathscr{H}$ is a nondegenerate HV containing q^3+1 points.

LEMMA 1. *For any two distinct HV's $V(H_i)$ and $V(H_j)$ of $\mathscr{H}$, there exists H_l of $\mathscr{H}$ such that $|V(H_i) \cap V(H_j)| = |V(H_l) \cap V(H_0)|$.*

Proof. Let $\boldsymbol{x}_1 = (x, u^k y, u^{k(1-q)} z)^{\mathrm{T}}$ be a point of $V(H_i) \cap V(H_j)$. Then

$$\boldsymbol{x}_1^{\mathrm{T}} H_i \boldsymbol{x}_1^{(q)} = (x, u^k y, u^{k(1-q)} z) \begin{pmatrix} 1 & au^{iq} & 0 \\ a^q u^i & 1 & bu^{iq} \\ 0 & b^q u^i & 1 \end{pmatrix} \begin{pmatrix} x^q \\ u^{kq} y^q \\ u^{k(1-q)q} z^q \end{pmatrix}$$

$$= (x, y, z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^k & 0 \\ 0 & 0 & u^{k(1-q)} \end{pmatrix} \begin{pmatrix} 1 & au^{iq} & 0 \\ a^q u^i & 1 & bu^{iq} \\ 0 & b^q u^i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^{kq} & 0 \\ 0 & 0 & u^{k(1-q)q} \end{pmatrix} \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix}$$

$$= (x, y, z) \begin{pmatrix} 1 & au^{iq} & 0 \\ a^q u^{i+k} & u^k & bu^{iq+k} \\ 0 & b^q u^{i+k(1-q)} & u^{k(1-q)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^{kq} & 0 \\ 0 & 0 & u^{k(1-q)q} \end{pmatrix} \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix}$$

$$= (x, y, z) \begin{pmatrix} 1 & au^{(i+k)q} & 0 \\ a^q u^{i+k} & u^{k(1+q)} & bu^{(i+k)q + k(1-q)(1+q)} \\ 0 & b^q u^{i+k} & u^{k(1-q)(1+q)} \end{pmatrix} \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix}$$

$$= (x, y, z) \begin{pmatrix} 1 & au^{(i+k)q} & 0 \\ a^q u^{i+k} & 1 & bu^{(i+k)q} \\ 0 & b^q u^{i+k} & 1 \end{pmatrix} \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix} = \boldsymbol{x}^{\mathrm{T}} H_{i+k} \boldsymbol{x}^{(q)}$$

since the order of u is $q+1$. That is, $\boldsymbol{x} = (x, y, z)^{\mathrm{T}}$ is a point of $V(H_{i+k}) \cap V(H_{j+k})$. Hence the change of variables $x_1 = x$, $y_1 = u^k y$, and $z_1 = u^{k(1-q)} z$ established a one-to-one correspondence between the points of $V(H_i) \cap V(H_j)$ and those of $V(H_{i+k}) \cap V(H_{j+k})$.

Moreover, k can be chosen so that $H_{j+k} = H_0$. Therefore, we have $|V(H_i) \cap V(H_j)| = |V(H_l) \cap V(H_0)|$, where $l = i + k$.

THEOREM 1. *Let q be an even prime power. Then $\mathscr{H}$ is a set of mutually M-intersecting varieties $\mathscr{V}(q^3+1,\ q^2+1)$, where $|\mathscr{V}(q^3+1, q^2+1)| = q+1$.*

Proof. From Lemma 1, we will show that the number of points of $V(H_i) \cap V(H_0)$ for any $V(H_i) \in \mathscr{H}$, $H_i \neq H_0$, is $q^2 + 1$. We have $|V(H_i) \cap V(H_0)| = |V(P^{\mathrm{T}} H_i P^{(q)}) \cap V(I)|$, where P is a nonsingular matrix P such that $P^{\mathrm{T}} H_0 P^{(q)} = I$. We consider the following two cases: (1) $a^{q+1} = 1$ and (2) $a^{q+1} \neq 1$ to show the minimal polynomial of $P^{\mathrm{T}} H_i P^{(q)}$ is $(x + 1)^3$.

(1) For $a^{q+1} = 1$, let

$$P = \begin{pmatrix} 1 & 0 & a^q \\ 0 & 0 & 1 \\ 0 & 1 & b \end{pmatrix}.$$

As q is an even prime power and $a^{q+1} = b^{q+1}$,

$$P^{\mathrm{T}} H_0 P^{(q)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a^q & 1 & b \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ a^q & 1 & b \\ 0 & b^q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & 1 \\ 0 & 1 & b^q \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a & 0 \\ 0 & b^q & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & 1 \\ 0 & 1 & b^q \end{pmatrix} = I.$$

Then the characteristic polynomial of $P^{\mathrm{T}} H_i P^{(q)}$ is

$$\begin{aligned} \det(xI - P^{\mathrm{T}} H_i P^{(q)}) &= \det(x P^{\mathrm{T}} H_0 P^{(q)} + P^{\mathrm{T}} H_i P^{(q)}) \\ &= \det(P^{\mathrm{T}}) \det(x H_0 + H_i) \det(P^{(q)}) \\ &= \det(x H_0 + H_i) \\ &= (x + 1)^3 + (x + 1)(x + u^{iq})(x + u^i)(a^{q+1} + b^{q+1}) \\ &= (x + 1)^3. \end{aligned}$$

When the first row of $P^{\mathrm{T}} H_i P^{(q)}$ is expressed by $\boldsymbol{p}^{\mathrm{T}} = (1, 0, a(1 + u^{iq}))$, the $(1,1)$-entry of $(P^{\mathrm{T}} H_i P^{(q)} + I)^2$ is

$$\boldsymbol{p}^{\mathrm{T}} \boldsymbol{p}^{(q)} + 1 = 1 + a^{q+1}(1 + u^{iq})^{q+1} + 1 = (1 + u^i)^{q+1}.$$

As $u^i \neq 1$, we have $(1 + u^i)^{q+1} \neq 0$; that is, $(P^{\mathrm{T}} H_i P^{(q)} + I)^2 \neq \mathbf{0}$ for $1 \leq i \leq q$. Hence, the minimal polynomial of $P^{\mathrm{T}} H_i P^{(q)}$ is $(x + 1)^3$.

(2) For $a^{q+1} \neq 1$, let

$$P = \begin{pmatrix} 1 & a^q t & a^q b^q t \\ 0 & t & b^q t \\ 0 & 0 & t^{-1} \end{pmatrix}.$$

where $t \in \mathrm{GF}(q)$ and $t^2(a^{q+1}+1) = 1$. Then

$$P^{\mathrm{T}} H_0 P^{(q)} = \begin{pmatrix} 1 & 0 & 0 \\ a^q t & t & 0 \\ a^q b^q t & b^q t & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ a^q & 1 & b \\ 0 & b^q & 1 \end{pmatrix} \begin{pmatrix} 1 & at & abt \\ 0 & t & bt \\ 0 & 0 & t^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a & 0 \\ 0 & (a^{q+1}+1)t & bt \\ 0 & (a^{q+1}+1)b^q t + b^q t^{-1} & b^{q+1} t + t^{-1} \end{pmatrix} \begin{pmatrix} 1 & at & abt \\ 0 & t & bt \\ 0 & 0 & t^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (a^{q+1}+1)t^2 & (a^{q+1}+1)bt^2 + b \\ 0 & (a^{q+1}+1)b^q t^2 + b^q & (a^{q+1}+1)b^{q+1}t^2 + b^{q+1} + b^{q+1} + t^{-2} \end{pmatrix} = I$$

since $t^{-2} = a^{q+1} + 1$. The characteristic polynomial of $P^{\mathrm{T}} H_i P^{(q)}$ is also $(x+1)^3$. When the first row of $P^{\mathrm{T}} H_i P^{(q)}$ is expressed by $\boldsymbol{p}^{\mathrm{T}} = (1, at(1+u^{iq}), abt(1+u^{iq}))$, the $(1,1)$-entry of $(P^{\mathrm{T}} H_i P^{(q)} + I)^2$ is

$$\begin{aligned} \boldsymbol{p}^{\mathrm{T}} \boldsymbol{p}^{(q)} + 1 &= 1 + a^{q+1} t^2 (1+u^i)^{q+1} + a^{q+1} b^{q+1} t^2 (1+u^i)^{q+1} + 1 \\ &= a^{q+1}(1+u^i)^{q+1} t^2 (1+b^{q+1}) \\ &= a^{q+1}(1+u^i)^{q+1} \neq 0 \end{aligned}$$

by $t^2(1+b^{q+1}) = 1$. Since $(P^{\mathrm{T}} H_i P^{(q)} + I)^2 \neq \mathbf{0}$, the minimal polynomial of $P^{\mathrm{T}} H_i P^{(q)}$ is $(x+1)^3$ for $1 \leq i \leq q$.

Therefore, in both cases, we have $|V(H_i) \cap V(H_0)| = q^2 + 1$ for $1 \leq i \leq q$ from the result given in the previous section. ■

Next we consider two nonsingular Hermitian matrices H and H' both having the minimal polynomial $m(x) = (x+1)^3$. As we mentioned before, we can define two sets as

$$\mathscr{H}_{a,b} = \{V(H_0), V(H_1), \ldots, V(H_q)\}$$

and

$$\mathscr{H}_{c,d} = \{V(H'_0), V(H'_1), \dots, V(H'_q)\},$$

where

$$H_i = \begin{pmatrix} 1 & au^{iq} & 0 \\ a^q u^i & 1 & bu^{iq} \\ 0 & b^q u^i & 1 \end{pmatrix}, \quad H'_j = \begin{pmatrix} 1 & cu^{jq} & 0 \\ c^q u^j & 1 & du^{jq} \\ 0 & d^q u^j & 1 \end{pmatrix}.$$

Note that $a^{q+1} = b^{q+1}$, $c^{q+1} = d^{q+1}$, and the order of u is $q+1$. In order that $\mathscr{H}_{a,b}$ and $\mathscr{H}_{c,d}$ be disjoint, a,b,c, and d should have some restrictions. Let w be a primitive element of the multiplicative group $\mathrm{GF}(q^2)\backslash\{0\}$ of order q^2-1. Let $K = \{1, w^{q-1}, \dots, w^{q(q-1)}\}$ be a multiplicative subgroup of order $q+1$ and $K_k = K \cdot w^k$ for k cosets of K, $0 \le k \le q-2$. Suppose $a \in K_l$, $0 \le l \le q-2$. Then the $(1,2)$-entry au^{iq} of H_i is also an element of K_l since u is included in K. So for $0 \le i \le q$, au^{iq} runs over all elements of K_l. As $a^{q+1} = b^{q+1}$, b must be contained in K_l. Hence we must choose c and d from a coset K_m, $m \ne l$, so that $\mathscr{H}_{a,b} \cap \mathscr{H}_{c,d} = \phi$.

THEOREM 2. *Let q be an even prime power. If a and c belong to different cosets K_l and K_m respectively, then $\mathscr{H}_{a,b} \cup \mathscr{H}_{c,d}$ is a set of mutually M-intersecting varieties $\mathscr{V}(q^3+1, M)$, where $M \subseteq \{q^2+1, (q+1)^2\}$ and $|\mathscr{V}(q^3+1, M)| = 2(q+1)$.*

Proof. From Theorem 1, $\mathscr{H}_{a,b}$ and $\mathscr{H}_{c,d}$ are both $\mathscr{V}(q^3+1, q^2+1)$. So we need only consider the number of points in the intersection of $V(H_i)$ and $V(H'_j)$ for $H_i \in \mathscr{H}_{a,b}$ and $H'_j \in \mathscr{H}_{c,d}$. It is easily seen that $|V(H_i) \cap V(H'_j)| = |V(H_0) \cap V(H'_{j+k})|$ for some k such that $H_{i+k} = H_0$ similar to Lemma 1. And we have $|V(H_0) \cap V(H'_j)| = |V(I) \cap V(P^T H'_j P^{(q)})|$, where P is a nonsingular matrix such that $P^{\mathrm{T}} H_0 P^{(q)} = I$. The characteristic polynomial $g(x)$ of $P^{\mathrm{T}} H'_j P^{(q)}$ is

$$\begin{aligned}
\det(xI - P^{\mathrm{T}} H'_j P^{(q)}) &= \det(xH_0 + H'_j) \\
&= (x+1)^3 + (x+1)\{(xa + cu^{jq})(xa^q + c^q u^j) + (xb + du^{jq})(xb^q + d^q u^j)\} \\
&= (x+1)^3 + (x+1)\{(ac^q + bd^q)u^j + (a^q c + b^q d)u^{jq}\} \\
&= (x+1)(x^2 + \delta x + 1),
\end{aligned}$$

where $\delta = (ac^q + bd^q)u^j + (a^q c + b^q d)u^{jq}$. The quadratic equation $x^2 + \delta x + 1 = 0$ has one solution over $\mathrm{GF}(q)$ if $\delta = 0$. In this case we have $g(x) = (x+1)^3$. We will show that $(P^{\mathrm{T}} H'_j P^{(q)} + xI)^2 \ne \mathbf{0}$ similar to Theorem 1.

(1) For $a^{q+1} = 1$, the first row of $P^{\mathrm{T}} H'_j P^{(q)}$ is expressed by $\boldsymbol{p}^{\mathrm{T}} = (1, 0, a + cu^{jq})$. Then the $(1,1)$-entry of $(P^{\mathrm{T}} H'_j P^{(q)} + I)^2$ is

$$\boldsymbol{p}^{\mathrm{T}} \boldsymbol{p}^{(q)} + 1 = 1 + (a + cu^{jq})^{q+1} + 1 = (a + cu^{jq})^{q+1} \neq 0,$$

since a and c belong to different cosets.

(2) For $a^{q+1} \neq 1$, the first row of $P^{\mathrm{T}} H'_j P^{(q)}$ is expressed by $\boldsymbol{p}^{\mathrm{T}} = (1, (a + cu^{jq})t, (a + cu^{jq})bt)$. Then the $(1,1)$-entry of $(P^{\mathrm{T}} H'_j P^{(q)} + I)^2$ is

$$\begin{aligned} \boldsymbol{p}^{\mathrm{T}} \boldsymbol{p}^{(q)} + 1 &= 1 + (a + cu^{jq})^{q+1} t^2 + (a + cu^{jq})^{q+1} b^{q+1} t^2 + 1 \\ &= (a + cu^{jq})^{q+1} \neq 0. \end{aligned}$$

Hence, the minimal polynomial $m(x)$ of $P^{\mathrm{T}} H'_j P^{(q)}$ is $m(x) = (x+1)^3$.

If the equation $x^2 + \delta x + 1 = 0$ has two solutions, then $m(x) = g(x) = (x+1)(x+\beta)(x+\gamma)$, where $1 \neq \beta \neq \gamma \neq 1 \in \mathrm{GF}(q)$. If the equation has no solutions, then $m(x) = g(x) = (x+1)(x^2 + \delta x + 1)$; that is, $x^2 + \delta x + 1$ is irreducible over $\mathrm{GF}(q)$. Therefore, $V(P^{\mathrm{T}} H'_j P^{(q)})$ and $V(I)$ intersect on $q^2 + 1$ points or $(q+1)^2$ points. ■

In the proof of Theorem 2, if $\delta = 0$, the minimal polynomial $m(x)$ of $P^{\mathrm{T}} H'_j P^{(q)}$ is $(x+1)^3$. When $a = b$ and $c = d$, we always have $\delta = 0$. Since $|V(H_i) \cap V(H'_j)| = q^2 + 1$ for $H_i \in \mathscr{H}_{a,b}$ and $H'_j \in \mathscr{H}_{c,d}$, we can obtain the next corollary.

COROLLARY 1. *Let q be an even prime power. If a and c belong to different cosets K_l and K_m, respectively, then $\mathscr{H}_{a,a} \cup \mathscr{H}_{c,c}$ is a set of mutually M-intersecting varieties $\mathscr{V}(q^3 + 1, q^2 + 1)$ consisting of $2(q+1)$ varieties.*

THEOREM 3. *Let q be an even prime power. Then there exists a set of mutually M-intersecting varieties $\mathscr{V}(q^3 + 1, q^2 + 1)$ consisting of $q^2 - 1$ varieties.*

Proof. In virtue of Theorem 1 and Corollary 1, what we need to do is to consider the set of matrices

$$H = \begin{pmatrix} 1 & a & 0 \\ a^q & 1 & a \\ 0 & a^q & 1 \end{pmatrix},$$

as a ranges through $\mathrm{GF}(q^2) \setminus \{0\}$. ■

THEOREM 4. *Let q be an even prime power. Then there exists a set of mutually M-intersecting varieties $\mathscr{V}(q^3 + 1, \{q^2 + 1, (q+1)^2\})$ consisting of $(q+1)^2(q-1)$ varieties.*

Proof. Let $J = \{1, w, \dots, w^{q-2}\}$ be a set of representatives of the cosets K_k for $0 \le k \le q-2$. Let $L = \{(a,b): a^{q+1} = b^{q+1}, a \in J, b \in GF(q^2)\}$. Then L consists of $(q-1)(q+1)$ elements and we have $\mathscr{H}_{a,b} \cap \mathscr{H}_{c,d} = \phi$ for $(a,b), (c,d) \in L$, $(a,b) \ne (c,d)$. Therefore, $\bigcup_{(a,b) \in L} \mathscr{H}_{a,b}$ is $\mathscr{V}(q^3+1, \{q^2+1, (q+1)^2\})$ by Theorem 2. ■

We remark that we can add $V(I)$ to $\mathscr{V}(\rho, M)$ in all theorems because we can show $|V(H_i) \cap V(I)| = q^2 + 1$ for any $V(H_i) \in \mathscr{H}$.

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