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Time-Marching on the Slow Manifold: The Relationship between the Nonlinear Galerkin Method and Implicit Timestepping Algorithms*

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Abstract—Implicit timestepping schemes and the Nonlinear Galerkin (NG) algorithms are competitors for integrating on the slow manifold, that is, for computing solutions when the timestep-limiting high frequencies are physically unimportant and most of the energy is in low frequency flow. We show that implicit schemes are always at least as accurate as the lowest nontrivial NG method, and can be made superior to even high order NG schemes by shortening the time step.

1. INTRODUCTION: MULTIPLE TIME SCALES

The spectral discretization of a PDE reduces it to a system of coupled ODEs of the form

$$a_{k,t} + i\omega_k a_k = f_k, \tag{1.1}$$

where $a_k(t)$ is the k^{th} Fourier or normal mode coefficient, $\omega_k(t)$ is the frequency of the k^{th} mode, and $f_k(t)$ is the k^{th} component of the sum of the nonlinear terms and the external forcing [1], and subscript t denotes time differentiation. Examples, with the spectral basis $\{\exp(ikx)\}$, include the Korteweg-deVries (KdV) equation, for which $\omega = -k^3$, and Burgers' equation, for which $\omega = -ik^2$.

The numerical challenge is that (1.1) is often very "stiff." That is, the time scale for some wave modes, $1/\omega_k$, is very short. If the high frequency variability is physically important, then one needs a very small, $O(1/\omega_{\text{max}})$ time step to resolve it, where ω_{max} is the frequency of that mode which has the highest frequency in the truncated spectral basis.

Often, however, the high frequency variability is irrelevant. Numerical weather forecasting is a good example. The forecast equations allow high frequency gravity waves, but daily observations are too sparse to allow accurate prediction of these waves, which fortunately have very little energy. The low frequency motion, the Rossby waves, has most of the energy and can be forecast accurately. Therefore, the goal of forecasting is not to model flows in all possible points of the gravity-and-Rossby phase space, but only to predict motion on the so-called "slow manifold" which consists of pure Rossby motion [2].

In many other physics and engineering situations, one is also interested only in the "slow manifold." An explicit time-marching scheme then requires one to use a time step which is orders of magnitude shorter than needed to resolve the slow flow. The standard remedy is to use an *implicit* scheme, such as the Crank-Nicholson method, which is stable for an arbitrarily large time

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step τ. One can then tailor the time step to the slow flow, and not to the physically uninteresting high frequency motion.

In the last five years, there have been more than twenty papers written on an alternative family of algorithms collectively known as the "Nonlinear Galerkin" (NG) method. Although independently rediscovered in the late 80's [3,4], the method was first employed by Daley [5] eight years earlier. He compared the lowest order NG method with what it is now the standard scheme for weather forecasting—explicit treatment of the nonlinear terms, Crank-Nicholson for the linear terms—in integrations of a multilevel, spherical harmonics forecasting code initialized with operational meteorological data.

To his great surprise, Daley found that the Nonlinear Galerkin method, although just as accurate and efficient as the semi-implicit scheme, was no better. This surprised him because the semi-implicit method is just a brute-force timestepping algorithm, applicable to fast motions, slow motions and everything in between as long as the time step is sufficiently short to resolve the important time scales. In contrast, the NG method is *tuned* to the slow manifold; it has no meaning or accuracy for flows in which the timestep-limiting high frequencies are important.

To explain the surprising effectiveness of the semi-implicit scheme, we shall analytically compute the slow manifolds generated by Nonlinear Galerkin methods of various orders and by various implicit schemes for the simple problem

$$iu_t + i\omega u = f(\varepsilon t), \tag{1.2}$$

where in order to have both fast and slow time scales, we shall assume $\varepsilon \ll \omega$. This is simply one component of the general system (1.1). For the discretization of the Navier-Stokes equations and/or the primitive equations of forecasting, $f(\varepsilon t)$ depends on all the other spectral coefficients, but for the *fast* modes, this dependence is very *weak* and is in any event irrelevant to the derivation of the NG method.

2. THE METHOD OF MULTIPLE TIMES AND THE NONLINEAR GALERKIN ALGORITHMS

The NG schemes are derived by applying the singular perturbation scheme known as the "Method of Multiple Scales." The slow manifold of (1.2) is that particular solution which varies only on the same slow $O(1/\varepsilon)$ time scale as the forcing, $f(\varepsilon t)$. Therefore, on the slow manifold, the time derivative is $O(\varepsilon/\omega)$ smaller than the other two terms in the equation. Continuing this reasoning gives the Baer-Tribbia series [6]

$$u(t) \sim \frac{1}{i\omega} \sum_{n=0}^{\infty} \left(\frac{i\varepsilon}{\omega}\right)^n \frac{d^n f(T)}{dT^n}, \tag{2.1}$$

where T is the slow time variable, $T \equiv \varepsilon t$. This formally applies to the system (1.1), also, provided we subscript both u and f with mode numbers. Whenever the series converges or is summable, (2.1) may be taken as the definition of the slow manifold.

The NG schemes of various orders m are defined to be the result of replacing the ODEs for the *fast* modes by m terms of the multiple scales series (2.1). We shall denote the m^{th} order as NG(m). For completeness, NG(0) will indicate total disregard of the fast modes, or equivalently, a standard Galerkin approximation which is truncated to the slow modes only. (The cutoff between fast and slow modes is somewhat arbitrary except that the slow modes have $\omega \sim O(\varepsilon)$, whereas the fast modes have frequencies $\omega \gg \varepsilon$.) In the later NG papers [7], ω is always imaginary, but the algorithm is the same (and equally effective) whether ω is real or complex.

The lowest nontrivial approximation is

$$u \sim \frac{f(\varepsilon t)}{(i\omega)} \qquad \text{[NG(1); "diagnostic approximation"]}. \tag{2.2}$$

This has a special role because it is the only nontrivial NG scheme which does not require computing a time derivative of f. Computing $\frac{df}{dT}$ for the Navier-Stokes equations is messy because f is actually a nonlinear term, the projection of the nonlinearity onto the fast mode. Consquently, Daley [5] and most NG papers have used only NG(1).

All the NG schemes, however, convert the differential system (1.1) into a mixed differential-algebraic system: differential equations for the slow modes, and algebraic relations which are the m^{th} order truncation of (2.1) for the fast modes. Since all the surviving differential equations have $\omega \sim O(\varepsilon)$, it follows that a time step τ such that $\varepsilon\tau \sim O(1)$—that is, a time step just short enough to resolve the slow $O(1/\varepsilon)$ time scale of the slow manifold—is stable. It follows that NG(m) schemes are direct competitors for integrating (1.1) or (1.2), left as a purely differential equation system, by an implicit method with $\tau \sim O(1/\varepsilon)$.

3. SLOW MANIFOLDS FOR THE MODEL PROBLEM

For the special case of (1.2) that the forcing is sinusoidal, i.e.,

$$u_t + i\omega u = \exp(i\varepsilon t), \tag{3.1}$$

it is possible to compute the slow manifold explicitly both for the differential equation and its implicit time discretization, which has the form

$$-i\delta Du + u = \frac{f}{(i\omega)}, \tag{3.2}$$

where $\delta \equiv 1/(\omega\tau)$ and D is the difference operator [8]. For the Backwards-Euler (BE scheme), $D\,u \equiv u(t) - u(t-\tau)$, and for the Crank-Nicholson (CN) scheme, D is the BE operator multiplied by the inverse of the averaging operator. Assuming $\delta \ll 1$, as true of all practical applications of implicit methods, we can apply the method of multiple scales to (3.2) just as to the equivalent diferential equation (1.2). Exponential functions are eigenfunctions of both the time derivative and of the difference operators, making it possible to sum the Baer-Tribbia series in closed form to obtain:

$$u_{\text{slow}} = \frac{Q\exp(i\varepsilon t)}{i\omega}, \tag{3.3}$$

$$Q_{\text{exact}} = \frac{1}{\{1+\varepsilon/\omega\}}, \tag{3.4}$$

$$Q_{\text{BE}} = \frac{1}{\{1 - i[1-\exp(-i\varepsilon\tau)]/(\omega/\tau)\}}, \tag{3.5a}$$

$$Q_{\text{CN}} = \frac{1}{\{1 + 2\tan(\varepsilon\tau)/(\omega\tau)\}}, \tag{3.5b}$$

$$Q_{\text{NG}}(m) = \frac{\{1-(-\varepsilon/\omega)^m\}}{\{1+\varepsilon/\omega\}}, \tag{3.6}$$

where the NG(m) scheme is the result of taking the first m terms in the power series expansion of Q_{exact} in powers of ε.

4. ANALYSIS

The errors in the NG and implicit schemes are

$$E = \frac{R\exp(i\varepsilon t)\varepsilon}{i\omega^2}, \tag{4.1}$$

$$R_{\text{NG}}(m) = \left(\frac{\varepsilon}{\omega}\right)^{m-1}, \tag{4.2}$$

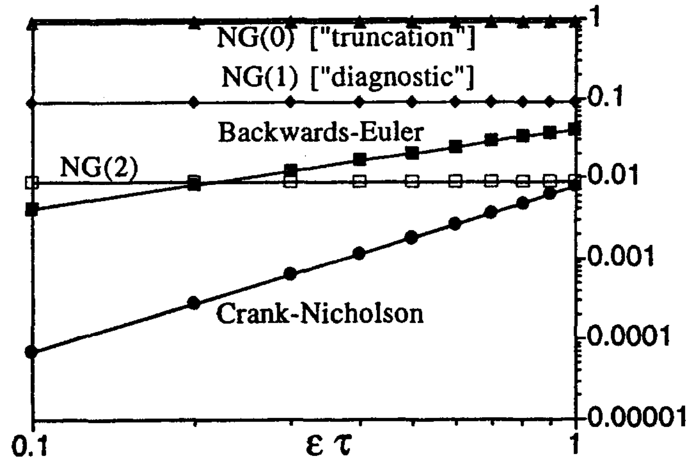

Figure 1. Maximum pointwise errors in approximating the slow manifold of $\frac{du}{dt} + i\omega u = \exp(i\varepsilon\tau)$, plotted versus the product of ε with τ. The errors in the NG schemes are independent of τ, and are thus shown as horizontal lines.

$$R_{\mathrm{BE}} = \frac{\varepsilon\tau}{2}, \tag{4.3a}$$

$$R_{\mathrm{CN}} = \frac{(\varepsilon\tau)^2}{12}. \tag{4.3b}$$

In words, the implicit schemes are all as accurate as $O(\varepsilon/\omega)$, that is, as accurate as Daley's "diagnostic scheme," NG(1), with proportionality constants that become arbitrarily small as $\tau \Rightarrow 0$.

Figure 1 compares the errors for various timesteps τ. The NG(m) schemes, which are independent of τ, are horizontal lines. For this value of ε/ω $(= 1/10)$, increasing m by 1 reduces the error by a factor of 10. Nevertheless, even for $\varepsilon\tau = 1$—that is, just 2π timesteps to resolve one full period of the slow manifold—both implicit schemes are much better than NG(1), and the Crank-Nicholson scheme is in fact as good as NG(2).

Why is this so? A partial answer lies in the time difference form (3.2). When $\varepsilon\tau$ is $O(1)$, the approximation of the time derivative by the difference may be crude, but even so, $D\,u$ is no larger than $O(u)$. We must divide this difference by τ to obtain the complete difference approximation, but $1/\tau$ is $O(\varepsilon)$. Thus, the time derivative term is $O(\varepsilon/\omega)$ in comparison to the other two terms. Neglecting it gives back the "diagnostic," NG(1) approximation (2.2), but derived from the *difference* equation. It is then obvious that both schemes must be at least as accurate as NG(1). In the limit that $\varepsilon\tau \Rightarrow 0$—not a limit one would take in practice since the whole point of implicit and NG schemes is to allow a *long* time step—the errors in the implicit schemes must go to zero since these are general time-integration methods, exact in the limit of vanishing time step. It follows that the errors in the BE and CN algorithms must be proportional to the NG(1) error with a proportionality constant that tends to zero as the appropriate power of $\varepsilon\tau$: first power for the first order Backwards-Euler method and quadratic for the second order Crank-Nicholson scheme.

5. CONCLUSIONS

Since the high order $(m > 1)$ NG schemes require computing time derivatives of $f(\varepsilon\tau)$, and therefore grow rapidly in expense and programming complexity with m, it follows that our main conclusion is: hurray for implicit methods! As we found analytically in our simple model, as Daley found numerically for a very complicated, three-dimensional forecasting model, the Nonlinear Galerkin schemes are *not* superior to implicit and semi-implicit time-marching algorithms.

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