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The ANTI-order for caccc posets — Part I

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Abstract

This paper deals with a generalization of the following simple observation. Suppose there are distinct elements a, b of the chain complete poset $(P, \leqslant)$ such that $P(<a) \subseteq P(<b)$ and $P(>a) \subseteq P(>b)$; if $P(<a)$ and $P(>a)$ are both fixed point free (fpf), then P is also fpf (we say P is trivially fpf), otherwise, P has the fixed point property (fpp) if and only if $P \setminus \{a\}$ has this property. We introduce a new quasi-order on a poset $(P, \leqslant)$, called the ANTI-order denoted by $\lll$, where $x \lll y$ holds if and only if every element strictly comparable with x is also strictly comparable with y. A set $X \subseteq P$ is an ANTI-good subset of P, if X is maximal (with respect to inclusion) and its elements are $\lll$-maximal and pairwise $\lll$-incomparable. A poset $(P, \leqslant)$ is caccc if it is chain complete and every countably infinite antichain has a supremum (infimum) whenever the antichain is bounded above (below). The caccc property is preserved by retracts and the intersection of a decreasing chain of caccc subposets also has this property. We show that for a caccc poset $(P, \leqslant)$ an ANTI-good subset is a retract and it is uniquely determined up to isomorphism. Moreover, if P is not trivially fpf, then P has the fpp if and only if an ANTI-good subset has the fpp. A strictly decreasing sequence, $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$, of subsets of a caccc poset P is called an ANTI-perfect sequence of P, if $P = P_0$ and, for each $\xi < \lambda$, $P_{\xi+1}$ is a $\lll_\xi$-good subset of P_ξ, where $\lll_\xi$ is the ANTI-order on P_ξ, and $P_\xi = \bigcap\{P_\eta : \eta < \xi\}$ when ξ is a limit ordinal, and P_λ is a $\lll_\lambda$-good subset of itself. We call P_λ an ANTI-core of P. Our main result is that an ANTI-core of a caccc poset is a retract. The proof of this will be given separately in the second part of the paper [5]. In this part we establish the existence of ANTI-perfect sequences.

Keywords: Retract; Fixed point property; ANTI-order; Caccc poset

1. Basic definitions and background remarks

We shall always denote by P an arbitrary nonempty partially ordered set (poset) with a partial order $\leqslant$. For any $x, y \in P$, $x \| y$ denotes that x and y are *comparable*,

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i.e. either $x \leqslant y$ or $y \leqslant x$; otherwise, we say that x and y are *incomparable* and write $x \perp y$. As usual, $x < y$ means $x \leqslant y$ and $x \neq y$, and in this case we say that x and y are *strictly comparable*. If $a \in P$ and $X \subseteq P$, we define $X(<a) = \{x \in X : x < a\}$ and $X(>a) = \{x \in X : x > a\}$. When a subset of $X \subseteq P$ is considered as a poset, its order is always the *induced order*, $\leqslant \cap (X \times X)$. We say that a subset X of P is *dominating* in P if, for any $a \in P$, there is $x \in X$ such that $a \leqslant x$. $y \in P$ is called an *upper bound* (*lower bound*) of $X \subseteq P$ if $x \leqslant y (x \geqslant y)$ for all $x \in X$. For $X, Y \subseteq P$, an element $y \in Y$ is called the *supremum of* X *in* Y, denoted by $\sup_Y X$, if y is an upper bound of X and $y \leqslant z$ whenever $z \in Y$ and z is an upper bound of X. In this definition we do not require that X be a subset of Y. The *infimum of* X *in* Y is defined dually and is denoted by $\inf_Y X$. If $Y = P$, these are just the usual supremum and infimum and, in this case, we write $\sup X$ and $\inf X$ instead of $\sup_P X$ and $\inf_P X$. The poset $(P, \leqslant)$ is *complete* if every subset of P has both an infimum and a supremum. It is *chain complete* (cc) if every nonempty chain has an infimum and a supremum.

We shall introduce a variety of different orders (or quasi-orders) on subsets of the poset P. To avoid ambiguity, when we use a technical term, like lower bound or antichain or chain for these other orders we always indicate which particular order is intended by prefixing the term with the appropriate order relation. For example, we write $\lll_X$-antichain to indicate an antichain with respect to the ANTI-order $\lll_X$ of the subposet X, and similarly for other terms. If there is no such prefix, then it is to be understood that this always refers to the usual order $\leqslant$ on P.

A *sequence in* P is a mapping from a set A of ordinals to P which we write in the form of $\langle x_\eta : \eta \in A \rangle$. The sequence is *eventually bounded above* (*below*) *in* $Y \subseteq P$ if there is $\zeta \in A$ and $y \in Y$ so that $x_\eta \leqslant y$ $(y \leqslant x_\eta)$ hold for all $\eta \in A$ with $\zeta \leqslant \eta$. The sequence $\langle x_\eta : \eta \in A \rangle$ is *increasing* (*decreasing*) if $\eta \leqslant \zeta \Rightarrow x_\eta \leqslant x_\zeta (x_\eta \geqslant x_\zeta)$ for $\eta, \zeta \in A$.

A mapping $f : P \to P$ is *order preserving* if $x \leqslant y \Rightarrow f(x) \leqslant f(y)$ for any $x, y \in P$. A *retraction* on P is an order preserving mapping $f : P \to P$ which is *idempotent*, i.e. $f(f(x)) = f(x)$ for any $x \in P$. The image of a retraction on P is called a *retract* of P. If $f : P \to P$ and $f(x) = x$, we say that x is a *fixed point* of f. A subposet $X \subseteq P$ has the *fixed point property* (fpp) if every order preserving mapping on X has a fixed point; otherwise, X is *fixed point free* (fpf). In particular, the empty set is fpf. The following criterion due to Abian and Brown [1] is often useful when considering fixed points in a cc poset:

1.1. *An order preserving mapping f on a cc poset P has a fixed point iff there is $x \in P$ such that $x \| f(x)$.*

Retracts also play an important rôle since it is well known that

1.2. *If P has the fpp then every retract of P also has this property.*

A procedure which is frequently effective to determine if a finite poset P has the fpp is dismantling (see [2]). This procedure leads to a retract, called the core of P, which

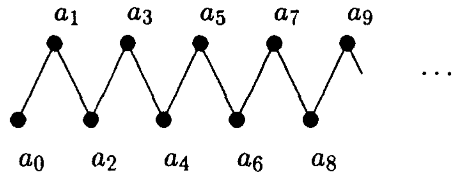

Fig. 1. F_ω.

is generally smaller and has the fpp iff P has this property. This method has been generalized to the infinite case by Li and Milner in [6,7]. We start with a new order $\trianglelefteq$ on P, called the *PT-order* of P, which is defined by writing $x \trianglelefteq y$ iff every maximal chain which passes through x also passes through y, or equivalently, $z \| x \Rightarrow z \| y$ for all $z \in P$. A *good subset* of P is a $\trianglelefteq$-dominating $\trianglelefteq$-antichain X. In [6] it was shown that

1.3. *If P is cc, then* (i) *there exists a good subset*; (ii) *any good subset is a retract of P*; (iii) *any two good subsets are order isomorphic*; (iv) *P has the fpp iff a good subset has this property.*

A retract of a cc poset P is also cc, and we showed in [6] that the intersection of a decreasing sequence of cc subposets in P is also cc. Therefore, we may iterate this construction repeatedly taking good subsets to make the poset smaller and smaller and at limit stages we take intersections. The formal definition for this procedure is the following. A *perfect sequence* $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ of a cc poset P is a strictly decreasing (with respect to containment) sequence of subsets of P so that $P_{\xi+1}$ is a $\trianglelefteq_\xi$-good subset of P_ξ for $\xi < \lambda$, where $\trianglelefteq_\xi$ is the PT-order generated by the induced order on P_ξ, $P_\xi = \bigcap\{P_\eta : \eta < \xi\}$ for a limit ordinal $\xi \leqslant \lambda$, and P_λ is *$\trianglelefteq_\lambda$-good*, i.e. P_λ is a $\trianglelefteq_\lambda$-good subset of itself. λ is called the *length* of the perfect sequence and P_λ is called the *core of P obtained by the perfect sequence*. Since chain completeness is preserved by retracts and the intersection of decreasing chain of cc posets, it follows that a perfect sequence Π and its core are well defined.

A perfect sequence is a generalization of dismantling to the case of infinite posets. However, in general, nice properties associated with dismantling are no longer preserved for perfect sequences. The *one-way infinite fence* F_ω is a typical example to show what can go wrong. F_ω is the poset on $\langle a_n : n < \omega \rangle$ shown in Fig. 1, in which $a_n < a_{n+1}$ for even n and $a_n > a_{n+1}$ for odd n and there are no other comparabilities. Trivially, the core of F_ω is the empty set, and so is not a retract. However, under certain conditions perfect sequences do behave well as in dismantling. In [6,7] we showed that

1.4. *If P is a cc poset with no infinite antichain then*

(a) *any two cores of P are isomorphic*;

(b) *the core of P is a finite retract of P*;

(c) *P has the fpp iff the core has the fpp;*
(d) *any two perfect sequences of P have the same length and the length is finite.*

Perfect sequences also keep some of nice dismantling properties under certain weaker conditions. In [4], we proved

1.5. *Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be a perfect sequence of a cc poset P. If P contains no tower* (see [4] for the definition) *and no one-way infinite fence then: each P_ξ ($\xi \leqslant \lambda$) is a retract and P has the fpp iff P_ξ has the fpp; in particular P has the fpp iff the core P_λ has the fpp.*

2. Introduction

In this paper we shall use a different quasi-order on P, the ANTI-order, and we use this to introduce the concepts of ANTI-good subsets, ANTI-perfect sequences and ANTI-cores. These concepts look formally similar to the corresponding notions associated with the PT-order. But they are essientially different, and the ANTI-order is more difficult to handle than the PT-order. In this section we give the definitions and establish the existence of ANTI-perfect sequences for caccc posets. The rather long proof of the main result (Theorem 2.1) will be given separately in Part II [5].

The ANTI-order concept was motivated by the following consideration. Dismantling has no effect on a *ramified* finite poset P, that is a poset in which every non-maximal (non-minimal) element has at least two upper (lower) covers. The core of a ramified poset P is just P itself, and so is not particularly useful in deciding whether or not P has the fpp. However, we may be able to use dismantling again, if we can change the poset P in some way without changing its fpp status. Schröder [9] observed (it is a special case of Theorem 3.4(3)) that:

> *If there is a pair of distinct elements $\{x, y\}$ in a finite poset P such that $P(<x) \subseteq P(<y)$ and $P(>x) \subseteq P(>y)$, then* Either (a) *$P(<x)$ and $P(>x)$ are both fpf, in which case P is also fpf,* Or (b) *P has the fpp or not according as the subposet $P \backslash \{x\}$ has this property or not.*

This means that, if there is such a pair $\{x, y\}$ in P, whether or not P has the fpp can be answered if we know how to answer the question for smaller posets. The point is that the ordinary dismantling might again be effective on these smaller posets. If there is a pair of distinct elements $\{x, y\}$ such that every element strictly comparable with x is strictly comparable with y and if condition (a) above is satisfied, then we say that P is *trivially fixed point free.*

This idea works for 6 of the 10 ramified posets listed in Rutkowski [8] (see Fig. 2). In Q_1, we take $x = 2$, $y = 1$, since both are minimal elements and all elements strictly greater than 2 are strictly greater than 1. $Q_1(>2)$ is dismantable since $\langle 6, 4, 8, 5, 7 \rangle$ is

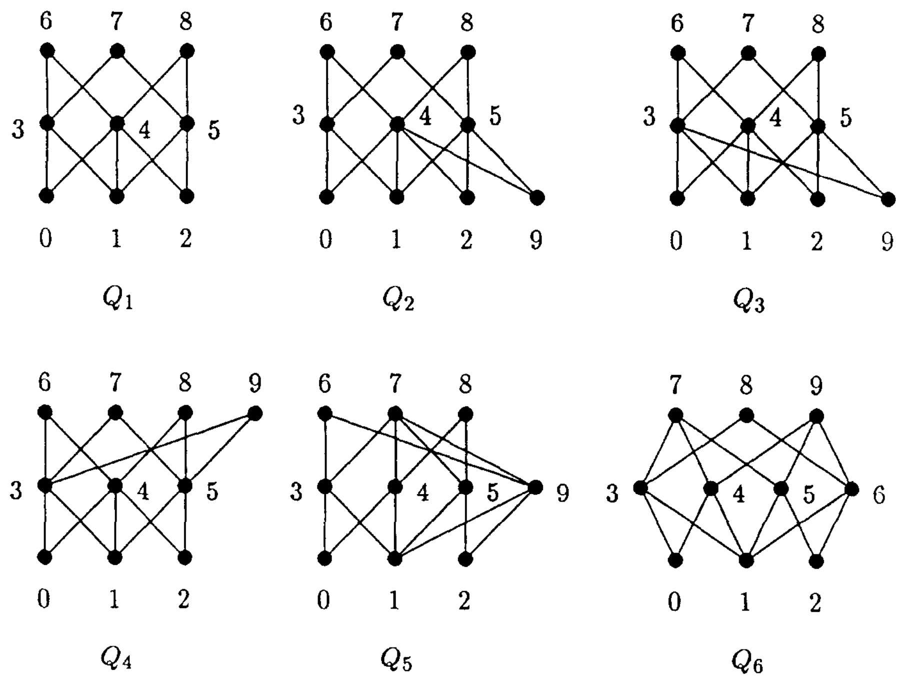

Fig. 2.

a dismantling procedure. Hence $Q_1(>2)$ has the fpp. Then, Q_1 has the fpp iff $Q_1\backslash\{2\}$ has fpp. But $Q_1-\{2\}$ does have this property since $\langle 5,7,8,4,6,0,3,1\rangle$ is a dismantling procedure. Therefore, Q_1 has the fpp. A similar argument works for the other 5 posets in Fig. 2 (choose $x=9$ and $y=1$ for Q_2 and Q_3; $x=9$ and $y=7$ for Q_4; $x=2$ and $y=1$ for Q_5 and Q_6).

The *ANTI-order* $\lessapprox_Q$ on a subset Q of P is a quasi-order generated by the induced order $\leqslant \bigcap(Q\times Q)$ in the following way: for any $x,y\in Q$, $x\lessapprox_Q y$ holds if and only if any element of Q strictly comparable with x is also stricly comparable with y. We write $x \ll_Q y$ when $x\lessapprox_Q y$ but $y\not\lessapprox_Q x$. If $Q=P$, we shall omit the subscript and simply write $\lessapprox$ or$\ll$.

We illustrate with an example, let $Q_7=\{x_n : n<\omega\}\bigcup\{y_n : n<\omega\}$ and define an order $\leqslant$ on Q_7 such that $\{x_n : n<\omega\}$ and $\{y_n : n<\omega\}$ are antichains, $y_n<x_m$ iff $n\geqslant m$. Fig. 3 shows the order $\leqslant$ and the corresponding ANTI-order $\lessapprox$. In the order $(Q_7,\lessapprox)$, $\{x_n : n<\omega\}$ is a $\lessapprox$-decreasing chain and $\{y_n : n<\omega\}$ is an $\lessapprox$-increasing chain and there are no $\lessapprox$-comparabilities between x's and y's.

The main difference between the PT-order $\trianglelefteq$ and the ANTI-order $\lessapprox$ on P is that, for $x,y\in P$, $x\trianglelefteq y$ means that every element comparable with x is comparable with y while $x\lessapprox y$ requires that every element strictly comparable with x is strictly comparable with y. Although they look similar, they behave quite differently. For instance, when $x,y\in P$, $x\trianglelefteq y$ implies $x\| y$, but $x\lessapprox y$ implies $x\perp y$ if $x\neq y$.

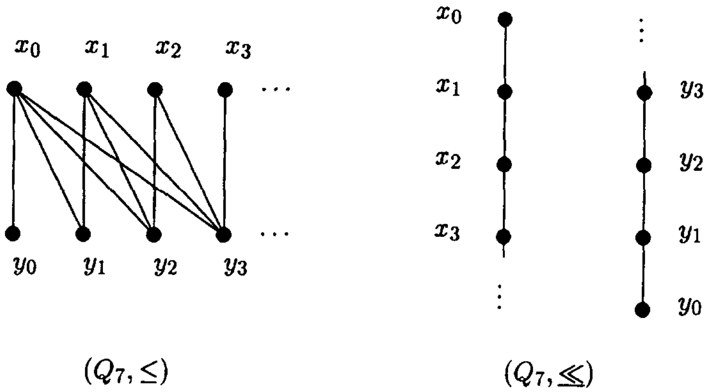

Fig. 3.

For a subset $Q \subseteq P$, We will be particularly interested in $\ll_Q$-maximal elements, i.e. elements $x \in Q$ such that $x \ll_Q y$ does not hold for any $y \in Q$. An *ANTI-good subset X* of Q (or a *$\ll_Q$-good subset* of Q) is a maximal $\ll_Q$-antichain of $\ll_Q$-maximal elements. When $Q = P$, we simply say that it is an *ANTI-good subset* or a *$\ll$-good subset*. We can understand this concept in a different way. We define an equivalence relation $\cong_Q$ on $\ll_Q$-Max, the set of all $\ll_Q$-maximal elements of Q, by writing that $x \cong_Q y$ iff both $x \ll_Q y$ and $y \ll_Q x$ hold. Then a set which intersects each $\cong_Q$-equivalence class in exactly one element will be an ANTI-good subset of Q. For example, in Fig. 3, the singleton $\{x_0\}$ is an ANTI-good subset of Q_7. Unlike in the case of the PT-order, we do not require that an ANTI-good subset of Q should be a $\ll_Q$-dominating set. For the example in Fig. 3, none of the y's are less than or equal to (w.r.t. $\ll$) x_0 , the only $\ll$-maximal element.

In general, an ANTI-good subset might behave badly, for instance it could even be empty, in which case it does not reveal much information about the original poset. To make an ANTI-good subset really 'good', we shall introduce caccc posets. P is said to be *conditionally $\aleph_0$-antichain complete* (cac) if every denumerable antichain in P has an infimun (supremum) whenever it has a lower (upper) bound. If P is cac and cc we call it a *caccc poset*. It is easy to show by transfinite induction that, in a caccc poset P, every infinite antichain with an upper (lower) bound has a supremum (infimum). Thus, it makes no difference if we redefine a caccc poset to be a cc poset in which every infinite antichain bounded above has a supremum and every infinite antichain bounded below has an infimum.

We will prove that (see Theorem 3.4), for a caccc poset P, any ANTI-good subset X is a retract of P. Also, we show that,

> Either (i) *there is some $a \in P \backslash X$ such that both $P(< a)$ and $P(> a)$ are fpf, in which case P itself is trivially fpf,* Or (ii) *P has the fpp iff X has this property.*

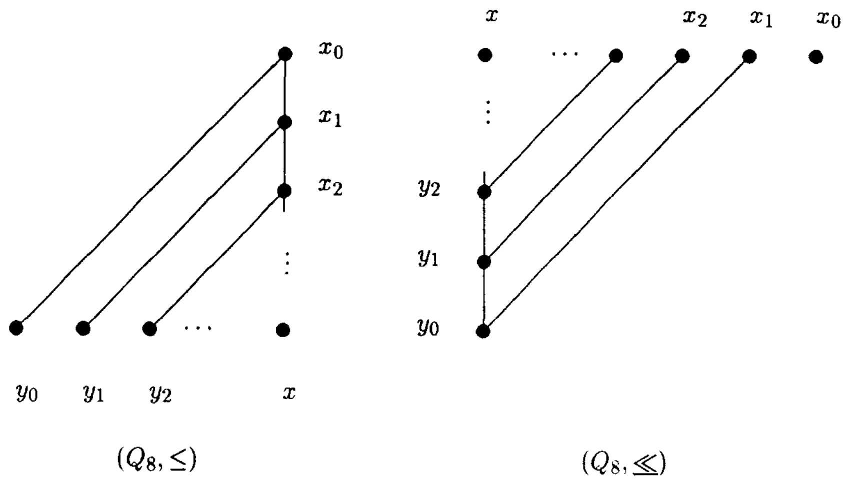

Fig. 4.

This result provides a technique to determine whether or not a caccc poset has the fpp. We illustrate the method with an example. Suppose $Q_8 = \{x\} \bigcup \{x_n : n < \omega\} \bigcup \{y_n : n < \omega\}$ is ordered so that $\{x_n : n < \omega\}$ is a decreasing chain with x as an infimum, $\{y_n : n < \omega\}$ is an antichain, $y_m < x_n$ iff $m \geqslant n$ and there are no other comparabilities. The poset $(Q_8, \leqslant)$ and its ANTI-order $\ll$ are shown in Fig. 4.

$(Q_8, \leqslant)$ is a caccc poset and $X = \{x\} \bigcup \{x_n : n < \omega\}$ is an ANTI-good subset. An element $a \in Q_8 - X$ is some y_n, and $Q_8(> a) = \{x_0, x_1, \ldots, x_n\}$ is a finite chain and so does have the fpp. By the above result, it follows that Q_8 has the fpp iff X has this property. But X does have the fpp since, as a subposet of $(Q_8, \leqslant)$, it is a complete chain. Thus, $(Q_8, \leqslant)$ has the fpp. Of course, we can use other methods to reach the same conclusion. For instance, it is immediate from the Abian–Brown Theorem 1.1 that a cc poset with a greatest element has the fpp.

Sometimes, it is still difficult to determine whether or not an ANTI-good subset has the fpp. But we may iterate the construction and take an ANTI-good subset of the ANTI-good subset and continue. At limit stages, we take intersections. This procedure is just like the construction of a perfect sequence based on the PT-order. Formally, we define an *ANTI-perfect sequence* of a caccc poset P to be a strictly decreasing (w.r.t. containment) sequence, $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$, of subsets of P such that $P = P_0$ and, for each $\xi < \lambda$, $P_{\xi+1}$ is a $\ll_\xi$-good subset of P_ξ, where $\ll_\xi = \ll_{P_\xi}$ is the ANTI-order on the subposet P_ξ, $P_\xi = \bigcap \{P_\eta : \eta < \xi\}$ when ξ is a limit ordinal, and P_λ is *ANTI-good*, i.e. P_λ is a $\ll_\lambda$-good subset of itself. The terminal set P_λ is called an *ANTI-core* of P. Corollaries 3.6 and 3.8 ensure that the caccc property is inherited at each stage of the construction and so an ANTI-perfect sequence and an ANTI-core of a caccc poset are well defined.

Consider the following example of an ANTI-perfect sequence and the corresponding ANTI-core. Let $P = \{x_n : n < \omega\} \bigcup \{y_n : n < \omega\} \bigcup \{y\}$ be ordered so that $\{x_n : n < \omega\}$

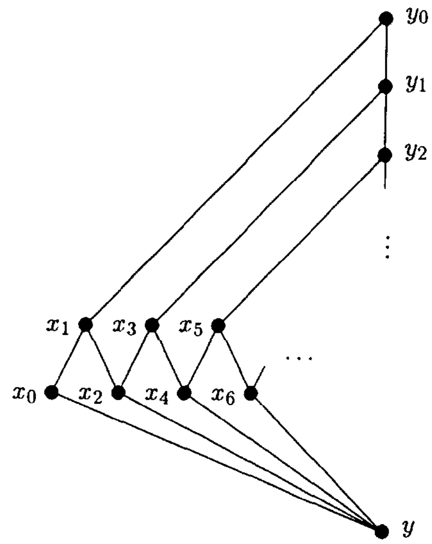

Fig. 5. $(P, \leqslant)$.

is a one-way infinite fence, $\{y_n : n < \omega\}$ is a decreasing chain with y as an infimum, $y < x_{2n}$ and $x_{2n+1} < y_n$ for all $n < \omega$ and there are no other comparabilities except for those demanded by transitivity. The poset $(P, \leqslant)$ is shown in Fig. 5. It is complete and therefore caccc. Let $P_0 = P$ and $\lll_0$ be the ANTI-order generated by $\leqslant$. Since $P(<x_0) = \{y\} = P(<x_2)$ and $P(>x_0) = \{x_1, y_0\} \subset P(>x_2) = \{x_1, x_3, y_0, y_1\}$, we have $x_0 \ll_0 x_2$. We claim that $P_1 = P \backslash \{x_0\}$ is a $\lll_0$-antichain. Assume $a, b \in P_1$ and $a \neq b$. Since $a \| b$ implies that a and b are $\lll_0$-incomparable, we may assume that $a \perp b$ and then we need only consider two possibilities: $a = x_n$ and $b = y_m$ for some $m, n < \omega$ with $1 \leqslant n < 2m$, or $a = x_n$ and $b = x_m$ for some $m, n < \omega$ with $1 \leqslant n \leqslant m-2$. In both cases, x_{n-1} is an element strictly comparable with a but incomparable with b; in the first case, y_{m+1} is an element strictly comparable with b but incomparable with a; in the second case, x_{m+1} is such an element. Therefore, a and b are $\lll_0$-incomparable. This means that P_1 is a $\lll_0$-good subset of $P = P_0$. Now, let $\lll_1$ be the ANTI-order on P_1, i.e. $\lll_1 = \lll_{P_1}$. Using the same argument, we can see that $P_2 = P \backslash \{x_0, x_1\}$ is a $\lll_1$-good subset of P_1. Continue this procedure. Let $P_n = P \backslash \{x_0, x_1, \ldots, x_{n-1}\}$ for $n < \omega$ and let $P_\omega = \bigcap \{P_n : n < \omega\} = \{y\} \bigcup \{y_n : n < \omega\}$. P_ω is ANTI-good, since it is a chain. Thus, $\Pi = \langle P_n : n \leqslant \omega \rangle$ is an ANTI-perfect sequence of length ω and P_ω is the ANTI-core generated by the ANTI-perfect sequence.

The one-way infinite fence F_ω (Fig. 1) is an awkward caccc poset; it has an empty ANTI-core. Let $P_0 = F_\omega$ and let the ANTI-order be $\lll$. It is easy to see that $a_0 \ll a_2$ and $P_1 = F_\omega \backslash \{a_0\}$ is an ANTI-good subset of P_0. Let $P_n = F_\omega \backslash \{a_0, a_1, \ldots, a_{n-1}\}$ and $P_\omega = \bigcap \{P_n : n < \omega\} = \emptyset$. Then $\Pi = \langle P_\xi : \xi \leqslant \omega \rangle$ is an ANTI-perfect sequence which generates an empty ANTI-core. Our main result says that if we exclude F_ω then an ANTI-core of a connected caccc poset must be a retract.

Theorem 2.1. *Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be an ANTI-perfect sequence of a connected caccc poset P which contains no one-way infinite fence. Then P_ξ is a retract of P for every $\xi \leqslant \lambda$, in particular, the ANTI-core P_λ is a retract of P.*

We shall give the proof of this theorem in a separate paper [5]. In this paper we shall establish the existence of ANTI-perfect sequences in caccc posets.

We believe that the ANTI-core has some connection with the fpp and we make the following conjecture (which is true for the case when λ is finite by Theorem 3.4).

Conjecture. Let $\Pi = \langle P_\xi : \xi \leqslant \lambda \rangle$ be an ANTI-perfect sequence of a connected caccc poset P which contains no one-way infinite fence. If there are $\eta < \lambda$ and $x \in P_\eta - P_{\eta+1}$ such that both $P_\eta(<x)$ and $P_\eta(>x)$ are fpf, then P is fpf; otherwise, P has the fpp iff the ANTI-core P_λ has this property.

3. ANTI-perfect sequences

In this section we show that an ANTI-perfect sequence is well defined for caccc posets. For this we need to show two things (1) that an ANTI-good subset of a caccc poset is also caccc and (2) the intersection of a decreasing sequence of caccc posets is caccc. In order to prove (1), we first show that an ANTI-good subset of a caccc poset is a retract (Theorem 3.4), and then show that the caccc condition is preserved by retracts (Lemma 3.5).

The following lemma is obvious and its proof is left to the reader.

Lemma 3.1. *If $x, y \in P$, $x \neq y$ and $x \lll y$, then x and y are incomparable, $P(<x) \subseteq P(<y)$ and $P(>x) \subseteq P(>y)$.*

We already observed that the set of all $\lll$-maximal elements in P is not necessarily $\lll$-dominating. Let

$$D(P) = \{x \in P: \text{ there is a } \lll \text{-maximal element } y \text{ such that } x \lll y\}.$$

Clearly any $\lll$-good subset X of P $\lll$-dominates the elements in $D(P)$, i.e. for any $y \in D(P)$, there is $x \in X$ such that $y \lll x$. We call $D(P)$ the *dominated part* of P.

The following lemmas show that certain elements must belong to every $\lll$-good subset.

Lemma 3.2. *Let $Z \subseteq P$, and suppose that $z = \inf Z$ ($z = \sup Z$) exists. If $z \notin Z$, then z belongs to any $\lll$-good subset.*

Proof. Let X be a $\lll$-good subset. If $z \in D(P) \backslash X$, there is $x \in X$ such that $z \lll x$; if $z \in P \backslash D(P)$, it is not a $\lll$-maximal element and so there is some $y \in P$ such that $z \lll y$. Thus in either case, if $z \notin X$, there is an element a different from z such that

$z \underline{\ll} a$. Then a is also a lower bound of Z. By Lemma 3.1, z and a are incomparable, contradicting the assumption that z is the infimum of Z. $\square$

Lemma 3.3. *Let P be a cac poset, X a $\underline{\ll}$-good subset of P and $x \in P \backslash D(P)$. Then*

(1) *If $P(<x) \neq \emptyset$ ($P(>x) \neq \emptyset$), then it has a greatest (least) element and this belongs to X;*

(2) *either $P(<x) \neq \emptyset$ or $P(>x) \neq \emptyset$.*

Proof. Since $x \notin D(P)$, there is a strictly $\underline{\ll}$-increasing sequence $\langle x_n : n < \omega \rangle$, where $x_0 = x$. By Lemma 3.1, $\{x_n : n < \omega\}$ is a denumerable antichain.

(1) If $c \in P(<x)$, then $c < x_n$ for $1 \leqslant n < \omega$, since $x_0 \ll x_n$, and so c is a lower bound of $\{x_n : n < \omega\}$. Since P is cac, $a = \inf\{x_n : n < \omega\}$ exists and $c \leqslant a$. Since c is an arbitrary element of $P(<x)$, it follows that a is the greatest element of $P(<x)$, and $a \in X$ by Lemma 3.2.

(2) Since $x \ll x_1$, there is $y \in P$ strictly comparable with x_1 but incomparable with x and so either $P(>x_1) \neq \emptyset$ or $P(<x_1) \neq \emptyset$. By (1) applied to x_1, it follows that $X \neq \emptyset$. This fact implies that either $P(<x) \neq \emptyset$ or $P(>x) \neq \emptyset$; otherwise, $x \underline{\ll} y$ for any $y \in X$ and $x \in D(P)$, contrary to the hypothesis. $\square$

Let X be a $\underline{\ll}$-good subset of a caccc poset P. We define a mapping $g : P \to X$ as follows. For $x \in X$, $g(x) = x$; for $x \in D(P) \backslash X$, $g(x)$ is an element in X such that $x \underline{\ll} g(x)$. If $x \in P \backslash D(P)$ then, by Lemma 3.3, either the greatest element b of $P(<x)$ belongs to X or the smallest element a of $P(>x)$ belongs to X. If the first possibility occurs we define $g(x) = b$, otherwise we define $g(x) = a$. Of course, the definition of g depends not only upon X, but also on the choices of the $g(x)$ for $x \in D(P) \backslash X$. We call a map of this type an *ANTI-good map onto X*. The main result of this section is the following theorem which shows that every such map is a retraction.

Theorem 3.4. *Let X be a $\underline{\ll}$-good subset of a caccc poset P and let g be an ANTI-good map onto X. Then,*

(1) *if $x, y \in P$ and $x < y$, then $g(x) \leqslant y$ and $x \leqslant g(y)$;*

(2) *g is a retraction;*

(3) *if there is $a \in P \backslash X$ such that both $P(<a)$ and $P(>a)$ are fpf, then P is also fpf; otherwise P has the fpp iff X has the fpp;*

(4) *any two $\underline{\ll}$-good subsets are isomorphic.*

Proof. (1) We prove that $g(x) \leqslant y$, the proof that $x \leqslant g(y)$ is similar. If $x \in X$, the result is obvious since $g(x) = x$. If $x \in D(P) \backslash X$, $x \underline{\ll} g(x)$ and so, by Lemma 3.1, $g(x) < y$. If $x \in P \backslash D(P)$, x and $g(x)$ are strictly comparable. If $g(x) < x$, then obviously $g(x) < y$. If $g(x) > x$, then $g(x)$ is the least element of $P(>x)$ and so $g(x) \leqslant y$.

(2) By definition, g is idempotent. We need only to show that it is order preserving. Let $x, y \in P$ and $x < y$. Then, by (1), $x \leqslant g(y)$. If $x = g(y)$, then $x \in X$ and so $g(x) = x = g(y)$. If $x < g(y)$, then applying (1) again to the pair x, $g(y)$, we conclude that $g(x) \leqslant g(y)$.

(3) Assume that there is $a \in P \backslash X$ such that both $P(<a)$ and $P(>a)$ are fpf. Let $t : P(>a) \to P(>a)$ and $s : P(<a) \to P(<a)$ be fpf order preserving mappings. If $a \in D(P) \backslash X$, $a \lll g(a)$ and if $a \in P \backslash D(P)$, a is not $\lll$-maximal and so there is $c \in P$ such that $a \ll c$. Thus, in either case, there is $b \in P$ such that $a \neq b$ and $a \lll b$. Then it is easy to see that the mapping $h : P \to P$ is fpf and order preserving, where h is defined as follows: $h(a) = b$; $h(x) = t(x)$ if $x \in P(>a)$; $h(x) = s(x)$ if $x \in P(<a)$; $h(x) = a$ if $x \perp a$.

We assume next that at least one of $P(>a)$ and $P(<a)$ has the fpp for every element $a \in P \backslash X$. Since the fpp is preserved by retracts, we need only show that P has the fpp if X has this property. Let $f : P \to P$ be an order preserving mapping. Then, $(g \circ f) \mid X$ is order preserving on X and so it has a fixed point, say $z = g(f(z)) \in X$. Let $x = f(z)$ so that $g(x) = z$. If $x \in X$, then $f(z) = x = g(x) = z$, i.e. z is a fixed point of f. If $x \in D(P) \backslash X$, it follows from the definition of g that $x \lll z$ and $x \neq z$. For any $y \in P(>x)$, we have $y > z$ and then $f(y) \geqslant f(z) = x$. If $f(y) = x$, y and $f(y)$ are comparable and so, by the Abian–Brown Theorem 1.1, f has a fixed point since P is cc. So, we may assume that $f(y) > x$, i.e. $f(y) \in P(>x)$, for any $y \in P(>x)$. Thus, $f \mid P(>x)$ is an order preserving mapping on $P(>x)$. By a similar argument, we may also assume that $f \mid P(<x)$ is an order preserving mapping on $P(<x)$. At least one of these two mappings has a fixed point which, of course, is a fixed point of f. Finally, suppose $x \in P \backslash D(P)$. In this case, $z = g(x)$ is either the greatest element of $P(<x)$ or the smallest element of $P(>x)$. In other words, $z \| x = f(z)$, and so again by the Abian–Brown Theorem f has a fixed point.

(4) Let X and Y be $\lll$-good subsets of P. Since $X \subseteq D(P)$, for each $x \in X$, there is a $y \in Y$ such that $x \lll y$, and since x and y are both $\lll$-maximal elements, $y \cong x$. Define $y = f(x)$ to be the unique element of Y such that $x \cong y$. By symmetry, f is a bijective mapping from X to Y. Suppose that $x, x' \in X$ and $x < x'$. Since $x' \lll f(x')$, we have $x < f(x')$, and since $x \lll f(x)$ it follows that $f(x) < f(x')$. Therefore, f is an isomorphism of X onto Y. □

Lemma 3.5. *The cac and caccc properties of a poset are preserved by retracts.*

Proof. Suppose that X is a retract of a cac poset P and that $f : P \to X$ is the corresponding retraction onto X. Let A be a denumerable antichain in X with a lower bound in X. Then A has an infimum, say b, in P. Then $f(b)$ is an infimum of A in X, and so X is also cac. Since chain completeness is also preserved by retracts, a retract of a caccc poset is also caccc. □

Corollary 3.6. *An ANTI-good subset of a caccc poset is also caccc.*

That our definition of an ANTI-perfect sequence is meaningful for caccc posets depends upon the fact (2) that the intersection of a decreasing sequence of caccc posets is also caccc. This is an immediate corollary of the following Lemma of [6].

Lemma 3.7. *Let ξ be a limit ordinal and let $\Pi = \langle P_\eta : \eta \leqslant \xi \rangle$ be a decreasing sequence of subsets of a poset P such that $P_\xi = \bigcap\{P_\eta : \eta < \xi\}$, and suppose that each P_η $(\eta < \xi)$ is cc. If $X \subseteq P$ and $\sup_{P_\eta} X$ $(\inf_{P_\eta} X)$ exist for all $\eta < \xi$, then $\sup_{P_\xi} X$ $(\inf_{P_\xi} X)$ also exists.*

Corollary 3.8. *Let ξ be a limit ordinal and let $\Pi = \langle P_\eta : \eta \leqslant \xi \rangle$ be a decreasing sequence of subsets of a poset P such that $P_\xi = \bigcap\{P_\eta : \eta < \xi\}$. If P_η are caccc for all $\eta < \xi$, then P_ξ is also caccc.*

It follows from the above results that ANTI-perfect sequences for caccc posets are well defined.

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