The ANTI-order for cacc posets — Part I

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Abstract

This paper deals with a generalization of the following simple observation. Suppose there are distinct elements \( a, b \) of the chain complete poset \((P, \leq)\) such that \( P(\leq a) \subseteq P(\leq b) \) and \( P(\geq a) \subseteq P(\geq b) \); if \( P(\leq a) \) and \( P(\geq a) \) are both fixed point free (fpf), then \( P \) is also fpf (we say \( P \) is trivially fpf), otherwise, \( P \) has the fixed point property (fpp) if and only if \( P\setminus\{a\} \) has this property. We introduce a new quasi-order on a poset \((P, \leq)\), called the ANTI-order denoted by \( \prec \), where \( x \prec y \) holds if and only if every element strictly comparable with \( x \) is also strictly comparable with \( y \). A set \( X \subseteq P \) is an ANTI-good subset of \( P \), if \( X \) is maximal (with respect to inclusion) and its elements are \( \prec \)-maximal and pairwise \( \prec \)-incomparable. A poset \((P, \leq)\) is cacc if it is chain complete and every countably infinite antichain has a supremum (infimum) whenever the antichain is bounded above (below). The cacc property is preserved by retracts and the intersection of a decreasing chain of cacc subposets also has this property. We show that for a cacc poset \((P, \leq)\) an ANTI-good subset is a retract and it is uniquely determined up to isomorphism. Moreover, if \( P \) is not trivially fpf, then \( P \) has the fpp if and only if an ANTI-good subset has the fpp. A strictly decreasing sequence, \( P = (P_\alpha; \alpha < \lambda) \), of subsets of a cacc poset \( P \) is called an ANTI-perfect sequence of \( P \), if \( P = P_0 \) and, for each \( \xi < \lambda \), \( P_{\xi+1} \) is a \( \prec \)-good subset of \( P_\xi \), where \( \prec \) is the ANTI-order on \( P_\xi \), and \( P_\xi = \bigcap\{P_\eta; \eta < \xi\} \) when \( \xi \) is a limit ordinal, and \( P_\xi \) is a \( \prec \)-good subset of itself. We call \( P_\xi \) an ANTI-core of \( P \).

Our main result is that an ANTI-core of a cacc poset is a retract. The proof of this will be given separately in the second part of the paper [5]. In this part we establish the existence of ANTI-perfect sequences.

Keywords: Retract; Fixed point property; ANTI-order; Cacc poset

1. Basic definitions and background remarks

We shall always denote by \( P \) an arbitrary nonempty partially ordered set (poset) with a partial order \( \leq \). For any \( x, y \in P \), \( x \parallel y \) denotes that \( x \) and \( y \) are comparable.
i.e. either \( x \leq y \) or \( y \leq x \); otherwise, we say that \( x \) and \( y \) are incomparable and write \( x \perp y \). As usual, \( x < y \) means \( x \leq y \) and \( x \neq y \), and in this case we say that \( x \) and \( y \) are strictly comparable. If \( a \in P \) and \( X \subseteq P \), we define \( X(< a) = \{ x \in X : x < a \} \) and \( X(> a) = \{ x \in X : x > a \} \). When a subset of \( X \subseteq P \) is considered as a poset, its order is always the induced order, \( \leq \cap (X \times X) \). We say that a subset \( X \) of \( P \) is dominating in \( P \) if, for any \( a \in P \), there is \( x \in X \) such that \( a \leq x \). \( y \in P \) is called an upper bound (lower bound) of \( X \subseteq P \) if \( x \leq y(x \geq y) \) for all \( x \in X \). For \( X, Y \subseteq P \), an element \( y \in Y \) is called the supremum of \( X \) in \( Y \), denoted by \( \sup_Y X \), if \( y \) is an upper bound of \( X \) and \( y \leq z \) whenever \( z \in Y \) and \( z \) is an upper bound of \( X \). In this definition we do not require that \( X \) be a subset of \( Y \). The infimum of \( X \) in \( Y \) is defined dually and is denoted by \( \inf_Y X \). If \( Y = P \), these are just the usual supremum and infimum and, in this case, we write \( \sup X \) and \( \inf X \) instead of \( \sup_{\nu} X \) and \( \inf_{\nu} X \). The poset \( (P, \leq) \) is complete if every subset of \( P \) has both an infimum and a supremum. It is chain complete (cc) if every nonempty chain has an infimum and a supremum.

We shall introduce a variety of different orders (or quasi-orders) on subsets of the poset \( P \). To avoid ambiguity, when we use a technical term, like lower bound or antichain or chain for these other orders we always indicate which particular order is intended by prefixing the term with the appropriate order relation. For example, we write \( \leq_{\nu} \)-antichain to indicate an antichain with respect to the ANTI-order \( \leq_{\nu} \) of the subposet \( X \), and similarly for other terms. If there is no such prefix, then it is to be understood that this always refers to the usual order \( \leq \) on \( P \).

A sequence in \( P \) is a mapping from a set \( A \) of ordinals to \( P \) which we write in the form of \( (x_\eta : \eta \in A) \). The sequence is eventually bounded above (below) in \( Y \subseteq P \) if there is \( \zeta \in A \) and \( y \in Y \) so that \( x_\eta \leq y \) \((y \leq x_\eta)\) hold for all \( \eta \in A \) with \( \zeta < \eta \). The sequence \( (x_\eta : \eta \in A) \) is increasing (decreasing) if \( \eta < \zeta \Rightarrow x_\eta \leq x_\zeta \) \((x_\eta \geq x_\zeta)\) for \( \eta, \zeta \in A \).

A mapping \( f : P \rightarrow P \) is order preserving if \( x \leq y \Rightarrow f(x) \leq f(y) \) for any \( x, y \in P \). A retraction on \( P \) is an order preserving mapping \( f : P \rightarrow P \) which is idempotent, i.e. \( f(f(x)) = f(x) \) for any \( x \in P \). The image of a retraction on \( P \) is called a retract of \( P \). If \( f : P \rightarrow P \) and \( f(x) = x \), we say that \( x \) is a fixed point of \( f \). A subposet \( X \subseteq P \) has the fixed point property (fpp) if every order preserving mapping on \( X \) has a fixed point; otherwise, \( X \) is fixed point free (fpf). In particular, the empty set is fpf. The following criterion due to Abian and Brown [1] is often useful when considering fixed points in a cc poset:

1.1. An order preserving mapping \( f \) on a cc poset \( P \) has a fixed point iff there is \( x \in P \) such that \( x \parallel f(x) \).

Retracts also play an important rôle since it is well known that

1.2. If \( P \) has the fpp then every retract of \( P \) also has this property.

A procedure which is frequently effective to determine if a finite poset \( P \) has the fpp is dismantling (see [2]). This procedure leads to a retract, called the core of \( P \), which
is generally smaller and has the fpp iff $P$ has this property. This method has been
generalized to the infinite case by Li and Milner in [6,7]. We start with a new order
$\preceq$ on $P$, called the $PT$-order of $P$, which is defined by writing $x \preceq y$ iff every maximal
chain which passes through $x$ also passes through $y$, or equivalently, $z\parallel x \Rightarrow z\parallel y$
for all $z \in P$. A good subset of $P$ is a $\preceq$-dominating $\preceq$-antichain $X$. In [6] it was shown
that

1.3. If $P$ is cc, then (i) there exists a good subset; (ii) any good subset is a retract
of $P$; (iii) any two good subsets are order isomorphic; (iv) $P$ has the fpp iff a good
subset has this property.

A retract of a cc poset $P$ is also cc, and we showed in [6] that the intersection of
a decreasing sequence of cc subposets in $P$ is also cc. Therefore, we may iterate this
construction repeatedly taking good subsets to make the poset smaller and smaller and
at limit stages we take intersections. The formal definition for this procedure is the
following. A perfect sequence $II = \langle P_\xi: \xi \leq \lambda \rangle$ of a cc poset $P$ is a strictly decreasing
(with respect to containment) sequence of subsets of $P$ so that $P_{\xi+1}$ is a $\preceq_\xi$-good
subset of $P_\xi$ for $\xi < \lambda$, where $\preceq_\xi$ is the PT-order generated by the induced order on
$P_\xi$, $P_\xi = \cap \{P_\eta: \eta < \xi \}$ for a limit ordinal $\xi \leq \lambda$, and $P_\xi$ is $\preceq_\xi$-good, i.e. $P_\xi$ is a $\preceq_\xi$-
good subset of itself. $\lambda$ is called the length of the perfect sequence and $P_\lambda$ is called the
core of $P$ obtained by the perfect sequence. Since chain completeness is preserved by
retracts and the intersection of decreasing chain of cc posets, it follows that a perfect
sequence $II$ and its core are well defined.

A perfect sequence is a generalization of dismantling to the case of infinite posets.
However, in general, nice properties associated with dismantling are no longer pre-
served for perfect sequences. The one-way infinite fence $F_{\omega}$ is a typical example to
show what can go wrong. $F_{\omega}$ is the poset on $\langle a_n: n < \omega \rangle$ shown in Fig. 1, in which
$a_n < a_{n+1}$ for even $n$ and $a_n > a_{n+1}$ for odd $n$ and there are no other comparabilities.
Trivially, the core of $F_{\omega}$ is the empty set, and so is not a retract. However, under
certain conditions perfect sequences do behave well as in dismantling. In [6,7] we
showed that

1.4. If $P$ is a cc poset with no infinite antichain then
(a) any two cores of $P$ are isomorphic;
(b) the core of $P$ is a finite retract of $P$;
(c) $P$ has the fpp iff the core has the fpp;
(d) any two perfect sequences of $P$ have the same length and the length is finite.

Perfect sequences also keep some of nice dismantling properties under certain weaker conditions. In [4], we proved

1.5. Let $\Pi = \langle P_\xi : \xi \leq \lambda \rangle$ be a perfect sequence of a cc poset $P$. If $P$ contains no tower (see [4] for the definition) and no one-way infinite fence then: each $P_\xi (\xi \leq \lambda)$ is a retract and $P$ has the fpp iff $P_\lambda$ has the fpp; in particular $P$ has the fpp iff the core $P_\lambda$ has the fpp.

2. Introduction

In this paper we shall use a different quasi-order on $P$, the ANTI-order, and we use this to introduce the concepts of ANTI-good subsets, ANTI-perfect sequences and ANTI-cores. These concepts look formally similar to the corresponding notions associated with the PT-order. But they are essentially different, and the ANTI-order is more difficult to handle than the PT-order. In this section we give the definitions and establish the existence of ANTI-perfect sequences for cacc posets. The rather long proof of the main result (Theorem 2.1) will be given separately in Part II [5].

The ANTI-order concept was motivated by the following consideration. Dismantling has no effect on a ramified finite poset $P$, that is a poset in which every non-maximal (non-minimal) element has at least two upper (lower) covers. The core of a ramified poset $P$ is just $P$ itself, and so is not particularly useful in deciding whether or not $P$ has the fpp. However, we may be able to use dismantling again, if we can change the poset $P$ in some way without changing its fpp status. Schröder [9] observed (it is a special case of Theorem 3.4(3)) that:

If there is a pair of distinct elements $\{x, y\}$ in a finite poset $P$ such that $P(< x) \subseteq P(< y)$ and $P(> x) \subseteq P(> y)$, then either (a) $P(< x)$ and $P(> x)$ are both ffp, in which case $P$ is also ffp, or (b) $P$ has the fpp or not according as the subposet $P\{x\}$ has this property or not.

This means that, if there is such a pair $\{x, y\}$ in $P$, whether or not $P$ has the fpp can be answered if we know how to answer the question for smaller posets. The point is that the ordinary dismantling might again be effective on these smaller posets. If there is a pair of distinct elements $\{x, y\}$ such that every element strictly comparable with $x$ is strictly comparable with $y$ and if condition (a) above is satisfied, then we say that $P$ is trivially fixed point free.

This idea works for 6 of the 10 ramified posets listed in Rutkowski [8] (see Fig. 2). In $Q_1$, we take $x = 2$, $y = 1$, since both are minimal elements and all elements strictly greater than 2 are strictly greater than 1. $Q_1(> 2)$ is dismantable since $\langle 6, 4, 8, 5, 7 \rangle$ is
a dismantling procedure. Hence $Q_1( > 2)$ has the fpp. Then, $Q_1$ has the fpp iff $Q_1 \setminus \{2\}$ has fpp. But $Q_1 - \{2\}$ does have this property since $\langle 5, 7, 8, 4, 6, 0, 3, 1 \rangle$ is a dismantling procedure. Therefore, $Q_1$ has the fpp. A similar argument works for the other 5 posets in Fig. 2 (choose $x = 9$ and $y = 1$ for $Q_2$ and $Q_3$; $x = 9$ and $y = 7$ for $Q_4$; $x = 2$ and $y = 1$ for $Q_5$ and $Q_6$).

The ANTI-order $\preceq_Q$ on a subset $Q$ of $P$ is a quasi-order generated by the induced order $\leq \cap (Q \times Q)$ in the following way: for any $x, y \in Q$, $x \preceq_Q y$ holds if and only if any element of $Q$ strictly comparable with $x$ is also strictly comparable with $y$. We write $x \ll_Q y$ when $x \preceq_Q y$ but $y \not\preceq_Q x$. If $Q = P$, we shall omit the subscript and simply write $\preceq$ or $\ll$.

We illustrate with an example, let $Q_7 = \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ and define an order $\leq$ on $Q_7$ such that $\{x_n : n < \omega\}$ and $\{y_n : n < \omega\}$ are antichains, $y_n < x_m$ iff $n \geq m$. Fig. 3 shows the order $\preceq$ and the corresponding ANTI-order $\ll$. In the order $(Q_7, \ll)$, $\{x_n : n < \omega\}$ is a $\ll$-decreasing chain and $\{y_n : n < \omega\}$ is an $\ll$-increasing chain and there are no $\ll$-comparabilities between $x$'s and $y$'s.

The main difference between the PT-order $\leq$ and the ANTI-order $\ll$ on $P$ is that, for $x, y \in P$, $x \leq y$ means that every element comparable with $x$ is comparable with $y$ while $x \ll y$ requires that every element strictly comparable with $x$ is strictly comparable with $y$. Although they look similar, they behave quite differently. For instance, when $x, y \in P$, $x \leq y$ implies $x \parallel y$, but $x \ll y$ implies $x \perp y$ if $x \neq y$.  

Fig. 2.
For a subset $Q \subseteq P$, we will be particularly interested in $\leq^Q$-maximal elements, i.e., elements $x \in Q$ such that $x \leq^Q y$ does not hold for any $y \in Q$. An ANTI-good subset $X$ of $Q$ (or a $\leq^Q$-good subset of $Q$) is a maximal $\leq^Q$-antichain of $\leq^Q$-maximal elements. When $Q = P$, we simply say that it is an ANTI-good subset or a $\leq$-good subset. We can understand this concept in a different way. We define an equivalence relation $\equiv_Q$ on $\leq^Q$-Max, the set of all $\leq^Q$-maximal elements of $Q$, by writing that $x \equiv_Q y$ iff both $x \leq^Q y$ and $y \leq^Q x$ hold. Then a set which intersects each $\equiv_Q$-equivalence class in exactly one element will be an ANTI-good subset of $Q$. For example, in Fig. 3, the singleton $\{x_0\}$ is an ANTI-good subset of $Q_7$. Unlike in the case of the PT-order, we do not require that an ANTI-good subset of $Q$ should be a $\leq$-dominating set. For the example in Fig. 3, none of the $y$'s are less than or equal to (w.r.t. $\leq$) $x_0$, the only $\leq$-maximal element.

In general, an ANTI-good subset might behave badly, for instance it could even be empty, in which case it does not reveal much information about the original poset. To make an ANTI-good subset really 'good', we shall introduce cacc posets. $P$ is said to be conditionally $\aleph_0$-antichain complete (cac) if every denumerable antichain in $P$ has an infimum (supremum) whenever it has a lower (upper) bound. If $P$ is cac and cc we call it a cacc poset. It is easy to show by transfinite induction that, in a cacc poset $P$, every infinite antichain with an upper (lower) bound has a supremum (infimum). Thus, it makes no difference if we redefine a cacc poset to be a cc poset in which every infinite antichain bounded above has a supremum and every infinite antichain bounded below has an infimum.

We will prove that (see Theorem 3.4), for a cacc poset $P$, any ANTI-good subset $X$ is a retract of $P$. Also, we show that,

Either (i) there is some $a \in P \setminus X$ such that both $P(<a)$ and $P(>a)$ are fpf, in which case $P$ itself is trivially fpf. Or (ii) $P$ has the fpp iff $X$ has this property.
This result provides a technique to determine whether or not a cacc poset has the fpp. We illustrate the method with an example. Suppose $Q_8 = \{x\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ is ordered so that $\{x_n : n < \omega\}$ is a decreasing chain with $x$ as an infimum, $\{y_n : n < \omega\}$ is an antichain, $y_m < x_n$ iff $m > n$ and there are no other comparabilities. The poset $(Q_8, \leq)$ and its ANTI-order $\ll$ are shown in Fig. 4.

$(Q_8, \leq)$ is a cacc poset and $X = \{x\} \cup \{x_n : n < \omega\}$ is an ANTI-good subset. An element $a \in Q_8 - X$ is some $y_n$, and $Q_8(> a) = \{x_0, x_1, \ldots, x_n\}$ is a finite chain and so does have the fpp. By the above result, it follows that $Q_8$ has the fpp iff $X$ has this property. But $X$ does have the fpp since, as a subposet of $(Q_8, \leq)$, it is a complete chain. Thus, $(Q_8, \leq)$ has the fpp. Of course, we can use other methods to reach the same conclusion. For instance, it is immediate from the Abian-Brown Theorem 1.1 that a cc poset with a greatest element has the fpp.

Sometimes, it is still difficult to determine whether or not an ANTI-good subset has the fpp. But we may iterate the construction and take an ANTI-good subset of the ANTI-good subset and continue. At limit stages, we take intersections. This procedure is just like the construction of a perfect sequence based on the PT-order. Formally, we define an ANTI-perfect sequence of a cacc poset $P$ to be a strictly decreasing (w.r.t. containment) sequence, $\Pi = \{P_\xi : \xi < \lambda\}$, of subsets of $P$ such that $P = P_0$ and, for each $\xi < \lambda$, $P_{\xi+1}$ is a $\ll_{\xi}$-good subset of $P_\xi$, where $\ll_{\xi} = \ll_{P_\xi}$ is the ANTI-order on the subposet $P_\xi$, $P_\xi = \bigcap\{P_\eta : \eta < \xi\}$ when $\xi$ is a limit ordinal, and $P_\xi$ is ANTI-good, i.e. $P_\xi$ is a $\ll_{\xi}$-good subset of itself. The terminal set $P_\lambda$ is called an ANTI-core of $P$. Corollaries 3.6 and 3.8 ensure that the cacc property is inherited at each stage of the construction and so an ANTI-perfect sequence and an ANTI-core of a cacc poset are well defined.

Consider the following example of an ANTI-perfect sequence and the corresponding ANTI-core. Let $P = \{x_n : n < \omega\} \cup \{y_n : n < \omega\} \cup \{y\}$ be ordered so that $\{x_n : n < \omega\}$
is a one-way infinite fence, \( \{y_n : n < \omega \} \) is a decreasing chain with \( y \) as an infimum, \( y < x_{2n} \) and \( x_{2n+1} < y_n \) for all \( n < \omega \) and there are no other comparabilities except for those demanded by transitivity. The poset \((P, \leq)\) is shown in Fig. 5. It is complete and therefore cacc. Let \( P_0 = P \) and \( \leq_0 \) be the ANTI-order generated by \( \leq \). Since \( P(\leq x_0) = \{y\} = P(\leq x_2) \) and \( P(\geq x_0) = \{x_1, y_0\} \subseteq P(\geq x_2) = \{x_1, x_3, y_0, y_1\} \), we have \( x_0 \leq_0 x_2 \). We claim that \( P_1 = P \setminus \{x_0\} \) is a \( \leq_0 \)-antichain. Assume \( a, b \in P_1 \) and \( a \neq b \). Since \( a \parallel b \) implies that \( a \) and \( b \) are \( \leq_0 \)-incomparable, we may assume that \( a \perp b \) and then we need only consider two possibilities: \( a = x_n \) and \( b = y_m \) for some \( m, n < \omega \) with \( 1 \leq n < 2m \), or \( a = x_n \) and \( b = x_m \) for some \( m, n < \omega \) with \( 1 \leq n \leq m - 2 \). In both cases, \( x_{n-1} \) is an element strictly comparable with \( a \) but incomparable with \( b \); in the first case, \( y_{m+1} \) is an element strictly comparable with \( b \) but incomparable with \( a \); in the second case, \( x_{m+1} \) is such an element. Therefore, \( a \) and \( b \) are \( \leq_0 \)-incomparable. This means that \( P_1 \) is a \( \leq_0 \)-good subset of \( P = P_0 \). Now, let \( \leq_1 \) be the ANTI-order on \( P_1 \), i.e. \( \leq_1 = \leq_{P_1} \). Using the same argument, we can see that \( P_2 = P \setminus \{x_0, x_1\} \) is a \( \leq_1 \)-good subset of \( P_1 \). Continue this procedure. Let \( P_n = P \setminus \{x_0, x_1, \ldots, x_{n-1}\} \) for \( n < \omega \) and let \( P_\omega = \bigcap \{P_n : n < \omega\} = \{y\} \cup \{y_n : n < \omega\} \). \( P_\omega \) is ANTI-good, since it is a chain. Thus, \( \Pi = \langle P_\omega : n \leq \omega \rangle \) is an ANTI-perfect sequence of length \( \omega \) and \( P_\omega \) is the ANTI-core generated by the ANTI-perfect sequence.

The one-way infinite fence \( F_\omega \) (Fig. 1) is an awkward cacc poset; it has an empty ANTI-core. Let \( P_0 = F_\omega \) and let the ANTI-order be \( \leq \). It is easy to see that \( a_0 \ll a_2 \) and \( P_1 = F_\omega \setminus \{a_0\} \) is an ANTI-good subset of \( P_0 \). Let \( P_n = F_\omega \setminus \{a_0, a_1, \ldots, a_{n-1}\} \) and \( P_\omega = \bigcap \{P_n : n < \omega\} = \emptyset \). Then \( \Pi = \langle P_\omega : \xi \leq \omega \rangle \) is an ANTI-perfect sequence which generates an empty ANTI-core. Our main result says that if we exclude \( F_\omega \) then an ANTI-core of a connected cacc poset must be a retract.
Theorem 2.1. Let $\Pi = \{P_\xi : \xi \leq \lambda \}$ be an ANTI-perfect sequence of a connected cacc poset $P$ which contains no one-way infinite fence. Then $P_\xi$ is a retract of $P$ for every $\xi \leq \lambda$. In particular, the ANTI-core $P_\lambda$ is a retract of $P$.

We shall give the proof of this theorem in a separate paper [5]. In this paper we shall establish the existence of ANTI-perfect sequences in cacc posets.

We believe that the ANTI-core has some connection with the fpp and we make the following conjecture (which is true for the case when $\lambda$ is finite by Theorem 3.4).

Conjecture. Let $\Pi = \{P_\xi : \xi \leq \lambda \}$ be an ANTI-perfect sequence of a connected cacc poset $P$ which contains no one-way infinite fence. If there are $\eta < \xi$ and $x \in P_{\eta} - P_{\eta+1}$ such that both $P_{\eta}(x)$ and $P_{\eta}(y)$ are fpp, then $P$ is fpp; otherwise, $P$ has the fpp iff the ANTI-core $P_\lambda$ has this property.

3. ANTI-perfect sequences

In this section we show that an ANTI-perfect sequence is well defined for cacc posets. For this we need to show two things (1) that an ANTI-good subset of a cacc poset is also cacc and (2) the intersection of a decreasing sequence of cacc posets is cacc. In order to prove (1), we first show that an ANTI-good subset of a cacc poset is a retract (Theorem 3.4), and then show that the cacc condition is preserved by retracts (Lemma 3.5).

The following lemma is obvious and its proof is left to the reader.

Lemma 3.1. If $x, y \in P$, $x \neq y$ and $x \leq y$, then $x$ and $y$ are incomparable, $P( < x) \subseteq P( < y)$ and $P( > x) \subseteq P( > y)$.

We already observed that the set of all $\leq$-maximal elements in $P$ is not necessarily $\leq$-dominating. Let

$$D(P) = \{x \in P: \text{there is a } \leq \text{-maximal element } y \text{ such that } x \leq y\}.$$ 

Clearly any $\leq$-good subset $X$ of $P$ $\leq$-dominates the elements in $D(P)$, i.e. for any $y \in D(P)$, there is $x \in X$ such that $y \leq x$. We call $D(P)$ the dominated part of $P$.

The following lemmas show that certain elements must belong to every $\leq$-good subset.

Lemma 3.2. Let $Z \subseteq P$, and suppose that $z = \inf Z$ ($z = \sup Z$) exists. If $z \notin Z$, then $z$ belongs to any $\leq$-good subset.

Proof. Let $X$ be a $\leq$-good subset. If $z \notin X$, then $x \in X$ such that $z \leq x$; if $z \in P \setminus D(P)$, it is not a $\leq$-maximal element and so there is some $y \in P$ such that $z \leq y$. Thus in either case, if $z \notin X$, there is an element $a$ different from $z$ such that
Lemma 3.3. Let $P$ be a cacc poset, $X$ a $\preceq$-good subset of $P$ and $x \in P \setminus D(P)$. Then

1. If $P(< x) \neq \emptyset$ and $P(> x) \neq \emptyset$, then it has a greatest (least) element and this belongs to $X$;
2. either $P(< x) \neq \emptyset$ or $P(> x) \neq \emptyset$.

Proof. Since $x \not\in D(P)$, there is a strictly $\preceq$-increasing sequence $(x_n : n < \omega)$, where $x_0 = x$. By Lemma 3.1, \{x_n : n < \omega\} is a denumerable antichain.

1. If $c \in P(< x)$, then $c < x_n$ for $1 \leq n < \omega$, since $x_0 \preceq x_n$, and so $c$ is a lower bound of \{x_n : n < \omega\}. Since $P$ is cacc, $a = \inf\{x_n : n < \omega\}$ exists and $c \preceq a$. Since $c$ is an arbitrary element of $P(< x)$, it follows that $a$ is the greatest element of $P(< x)$, and $a \in X$ by Lemma 3.2.

2. Since $x \preceq x_1$, there is $y \in P$ strictly comparable with $x_1$ but incomparable with $x$ and so either $P(> x_1) \neq \emptyset$ or $P(< x_1) \neq \emptyset$. By (1) applied to $x_1$, it follows that $X \neq \emptyset$. This fact implies that either $P(< x) \neq \emptyset$ or $P(> x) \neq \emptyset$; otherwise, $x \preceq y$ for any $y \in X$ and $x \in D(P)$, contrary to the hypothesis.

Let $X$ be a $\preceq$-good subset of a cacc poset $P$. We define a mapping $g : P \to X$ as follows. For $x \in X$, $g(x) = x$; for $x \in D(P) \setminus X$, $g(x)$ is an element in $X$ such that $x \preceq g(x)$. If $x \in P \setminus D(P)$ then, by Lemma 3.3, either the greatest element $b$ of $P(< x)$ belongs to $X$ or the smallest element $a$ of $P(> x)$ belongs to $X$. If the first possibility occurs we define $g(x) = b$, otherwise we define $g(x) = a$. Of course, the definition of $g$ depends not only upon $X$, but also on the choices of the $g(x)$ for $x \in D(P) \setminus X$. We call a map of this type an ANTI-good map onto $X$. The main result of this section is the following theorem which shows that every such map is a retraction.

Theorem 3.4. Let $X$ be a $\preceq$-good subset of a cacc poset $P$ and let $g$ be an ANTI-good map onto $X$. Then,

1. if $x, y \in P$ and $x < y$, then $g(x) \preceq y$ and $x \preceq g(y)$;
2. $g$ is a retraction;
3. if there is $a \in P \setminus X$ such that both $P(< a)$ and $P(> a)$ are fpf, then $P$ is also fpf; otherwise $P$ has the fpp iff $X$ has the fpp;
4. any two $\preceq$-good subsets are isomorphic.

Proof. (1) We prove that $g(x) \preceq y$, the proof that $x \preceq g(y)$ is similar. If $x \in X$, the result is obvious since $g(x) = x$. If $x \in D(P) \setminus X$, $x \preceq g(x)$ and so, by Lemma 3.1, $g(x) < y$. If $x \in P \setminus D(P)$, $x$ and $g(x)$ are strictly comparable. If $g(x) < x$, then obviously $g(x) < y$. If $g(x) > x$, then $g(x)$ is the least element of $P(> x)$ and so $g(x) \preceq y$.

(2) By definition, $g$ is idempotent. We need only to show that it is order preserving. Let $x, y \in P$ and $x < y$. Then, by (1), $x \preceq g(y)$. If $x = g(y)$, then $x \in X$ and so $g(x) = x = g(y)$. If $x < g(y)$, then applying (1) again to the pair $x, g(y)$, we conclude that $g(x) \preceq g(y)$. 
(3) Assume that there is $a \in P \setminus X$ such that both $P(\leq a)$ and $P(\geq a)$ are fpf. Let $t : P(\geq a) \rightarrow P(\geq a)$ and $s : P(\leq a) \rightarrow P(\leq a)$ be fpf order preserving mappings. If $a \in D(P) \setminus X$, $a \leq g(a)$ and if $a \in P \setminus D(P)$, $a$ is not $\leq$-maximal and so there is $c \in P$ such that $a \ll c$. Thus, in either case, there is $b \in P$ such that $a \neq b$ and $a \ll b$. Then it is easy to see that the mapping $h : P \rightarrow P$ is fpf and order preserving, where $h$ is defined as follows: $h(a) = b$; $h(x) = t(x)$ if $x \in P(\geq a)$; $h(x) = s(x)$ if $x \in P(\leq a)$; $h(x) = a$ if $x \perp a$.

We assume next that at least one of $P(\geq a)$ and $P(\leq a)$ has the fpp for every element $a \in P \setminus X$. Since the fpp is preserved by retracts, we need only show that $P$ has the fpp if $X$ has this property. Let $f : P \rightarrow P$ be an order preserving mapping. Then, $(g \circ f) \mid X$ is order preserving on $X$ and so it has a fixed point, say $z = g(f(z)) \in X$. Let $x = f(z)$ so that $g(x) = z$. If $x \in X$, then $f(z) = x = g(x) = z$, i.e. $z$ is a fixed point of $f$. If $x \in D(P) \setminus X$, it follows from the definition of $g$ that $x \ll z$ and $x \neq z$. For any $y \in P(\geq a)$, we have $y > z$ and then $f(y) \geq f(z) = x$. If $f(y) = x$, $y$ and $f(y)$ are comparable and so, by the Abian–Brown Theorem 1.1, $f$ has a fixed point since $P$ is cc. So, we may assume that $f(y) > x$, i.e. $f(y) \in P(\geq a)$, for any $y \in P(\geq a)$. Thus, $f \mid P(\geq a)$ is an order preserving mapping on $P(\geq a)$. By a similar argument, we may also assume that $f \mid P(\leq a)$ is an order preserving mapping on $P(\leq a)$. At least one of these two mappings has a fixed point which, of course, is a fixed point of $f$. Finally, suppose $x \in P \setminus D(P)$. In this case, $z = g(x)$ is either the greatest element of $P(\leq x)$ or the smallest element of $P(\geq x)$. In other words, $z \ll x = f(z)$, and so again by the Abian–Brown Theorem $f$ has a fixed point.

(4) Let $X$ and $Y$ be $\leq$-good subsets of $P$. Since $X \subseteq D(P)$, for each $x \in X$, there is a $y \in Y$ such that $x \ll y$, and since $x$ and $y$ are both $\ll$-maximal elements, $y \cong x$. Define $y = f(x)$ to be the unique element of $Y$ such that $x \cong y$. By symmetry, $f$ is a bijective mapping from $X$ to $Y$. Suppose that $x, x' \in X$ and $x < x'$. Since $x' \ll f(x')$, we have $x < f(x')$, and since $x \ll f(x)$ it follows that $f(x) < f(x')$. Therefore, $f$ is an isomorphism of $X$ onto $Y$. □

**Lemma 3.5.** The cac and caccc properties of a poset are preserved by retracts.

**Proof.** Suppose that $X$ is a retract of a cac poset $P$ and that $f : P \rightarrow X$ is the corresponding retraction onto $X$. Let $A$ be a denumerable antichain in $X$ with a lower bound in $X$. Then $A$ has an infimum, say $b$, in $P$. Then $f(b)$ is an infimum of $A$ in $X$, and so $X$ is also cac. Since chain completeness is also preserved by retracts, a retract of a caccc poset is also caccc. □

**Corollary 3.6.** An ANTI-good subset of a caccc poset is also caccc.

That our definition of an ANTI-perfect sequence is meaningful for caccc posets depends upon the fact (2) that the intersection of a decreasing sequence of caccc posets is also caccc. This is an immediate corollary of the following Lemma of [6].
Lemma 3.7. Let $\xi$ be a limit ordinal and let $\Pi = (P_\eta : \eta \leq \xi)$ be a decreasing sequence of subsets of a poset $P$ such that $P_\xi = \bigcap \{P_\eta : \eta < \xi\}$, and suppose that each $P_\eta$ ($\eta < \xi$) is cc. If $X \subseteq P$ and $\operatorname{sup} P_\eta X$ (\operatorname{inf} P_\eta X) exist for all $\eta < \xi$, then $\operatorname{sup} P_\xi X$ (\operatorname{inf} P_\xi X) also exists.

Corollary 3.8. Let $\xi$ be a limit ordinal and let $\Pi = (P_\eta : \eta \leq \xi)$ be a decreasing sequence of subsets of a poset $P$ such that $P_\xi = \bigcap \{P_\eta : \eta < \xi\}$. If $P_\eta$ are caccc for all $\eta < \xi$, then $P_\xi$ is also caccc.

It follows from the above results that ANTI-perfect sequences for caccc posets are well defined.

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