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Non-differentiable variational principles

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Abstract

We develop a calculus of variations for functionals which are defined on a set of non-differentiable curves. We first extend the classical differential calculus in a quantum calculus, which allows us to define a complex operator, called the scale derivative, which is the non-differentiable analogue of the classical derivative. We then define the notion of extremals for our functionals and obtain a characterization in term of a generalized Euler–Lagrange equation. We finally prove that solutions of the Schrödinger equation can be obtained as extremals of a non-differentiable variational principle, leading to an extended Hamilton's principle of least action for quantum mechanics. We compare this approach with the scale relativity theory of Nottale, which assumes a fractal structure of space–time.

Résumé

(Principes variationnels non différentiable). Nous développons un calcul des variations pour des fonctionnelles définies sur un ensemble de courbes non différentiables. Pour cela, nous étendons le calcul différentiel classique, en calcul appelé *calcul quantique*, qui nous permet de définir un opérateur à valeur complexes, appelé *dérivée d'échelle*, qui est l'analogue non différentiable de la dérivée usuelle. On définit alors la notion d'extremale pour ces fonctionnelles pour lesquelles nous obtenons une caractérisation via une équation d'Euler–Lagrange généralisée. On prouve enfin que les solutions de l'équation de Schrödinger peuvent s'obtenir comme solution d'un problème variationnel non différentiable, étendant ainsi le principe de moindre action de Hamilton au cadre de la mécanique quantique. On discute enfin la connexion entre ce travail et la théorie de la relativité d'échelle développée par Nottale, et qui suppose une structure fractale de l'espace–temps.

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1. Introduction

Lagrangian mechanics describes motion of mechanical systems using differentiable manifolds. Motions of Lagrangian systems are extremals of a variational principle called "Hamilton's principle of least action" (see [1, p. 55]).

However, some important physical systems cannot be put in such a framework. For example, generic trajectories of quantum mechanics are not differentiable curves [12], such that a classical Lagrangian formalism is not possible (see however [11]).

In this article we extend the calculus of variations in order to cover sets of non-differentiable curves. We first define a quantum calculus allowing us to analyze non-differentiable functions by means of a complex operator, which generalizes the classical derivative. We then introduce functionals on Hölderian curves and study the analogue of extremals for these objects. We prove that extremals curves of our functionals are solutions of a generalized Euler–Lagrange equation, which looks like the one obtained by Nottale [17] in the context of the scale relativity theory. We then prove that the Schrödinger equation can be obtain as extremals of a non-differentiable variational problem.

The non-differentiable calculus of variations gives a rigorous basis to the scale relativity principle developed by Nottale [17] in order to recover quantum mechanics by keeping out the differentiability assumption of the space–time.

2. Quantum calculus

In this section we define the quantum calculus, which extends the classical differential calculus to non-differentiable functions. We refer to [4] and [9] for analogous ideas and the underlying physical framework leading to this extension.

2.1. Basic definitions

We denote by C^0 the set of continuous real valued functions defined on $\mathbb{R}$.

Definition 2.1. Let $f \in C^0$. For all $\epsilon > 0$, we call ϵ left and right quantum derivatives the quantities

$$\Delta_\epsilon^\sigma f(t) = \sigma \frac{f(t+\sigma\epsilon) - f(t)}{\epsilon}, \quad \sigma = \pm. \tag{1}$$

The ϵ left and right quantum derivatives of a continuous function correspond to the classical derivatives of the left and right ϵ-mean function defined by

$$f_\epsilon^\sigma(t) = \frac{\sigma}{\epsilon} \int_t^{t+\sigma\epsilon} f(s)\,ds, \quad \sigma = \pm. \tag{2}$$

Using ϵ left and right derivatives, we can define an operator which generalize the classical derivative.

Definition 2.2. Let $f \in C^0$. For all $\epsilon > 0$, the ϵ scale derivative of f at point t is the quantity denoted by $\Box_\epsilon f/\Box t$, and defined by

$$\frac{\Box_\epsilon f}{\Box t}(t) = \left(\Delta_\epsilon^+ f(t) + \Delta_\epsilon^- f(t)\right) - i\left(\Delta_\epsilon^+ f(t) - \Delta_\epsilon^- f(t)\right). \tag{3}$$

If f is differentiable, we can take the limit of the scale derivative when ϵ goes to zero. We then obtain the classical derivative of f, f'.

In the following, we will frequently denote $\Box_\epsilon x$ for $\Box_\epsilon x/\Box t$.

We also need to extend the scale derivative to complex valued functions.

Definition 2.3. Let f be a continuous complex valued function. For all $\epsilon > 0$, the ϵ scale derivative of f, denoted by $\Box_\epsilon f/\Box t$ is defined by

$$\frac{\Box_\epsilon f}{\Box t}(t) = \frac{\Box_\epsilon \operatorname{Re}(f)}{\Box t} + i\frac{\Box_\epsilon \operatorname{Im}(f)}{\Box t}, \tag{4}$$

where $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ denote the real and imaginary parts of f.

This extension of the scale derivative in order to cover complex valued functions is far from being trivial. Indeed, it mixes complex terms in a complex operator.

2.2. Basic formulas

For all $\epsilon > 0$ the scale derivative is not a derivation[1] on the set of continuous functions[2] Indeed, we have:

Theorem 2.1. *Let f and g be two functions of C^0. For all $\epsilon > 0$ we have*

$$\begin{aligned}\Box_\epsilon(fg) = {} & \Box_\epsilon f.g + f.\Box_\epsilon g \\ & + \epsilon i[\Box_\epsilon f \boxminus_\epsilon g - \boxminus_\epsilon f \Box_\epsilon g - \Box_\epsilon f \Box_\epsilon g - \boxminus_\epsilon f \boxminus_\epsilon g],\end{aligned} \tag{5}$$

where $\boxminus f$ is the complex conjugate of $\Box f$.

[1] We recall that a derivation on an abstract algebra A is a linear application $D : A \to A$ such that $D(xy) = D(x).y + x.D(y)$ for all $x, y \in A$.

[2] A classical result says that there exists no derivations on the set of continuous functions except the trivial one, defined by $D(f) = 0$ for all $f \in C^0$.

Of course, when we restrict our attention to differentiable functions, taking the limit of (5) when ϵ goes to zero, we obtain the classical Leibniz rule $(fg)' = f'.g + f.g'$.

Proof. Formula (5) follows from easy calculations. In particular, we use the fact that

$$\Delta_\sigma^\epsilon(fg) = \Delta_\sigma^\epsilon f.g + f.\Delta_\sigma^\epsilon g + \sigma\epsilon\Delta_\sigma^\epsilon f.\Delta_\sigma^\epsilon g, \quad \sigma = \pm, \tag{6}$$

which is a standard result of the calculus of finite differences (see [14]).

As a consequence, we have

$$\begin{aligned}\Box_\epsilon(fg) &= \Box_\epsilon f.g + f.\Box_\epsilon g\\ &\quad + \epsilon\big[\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g - \Delta_\epsilon^- f\Delta_\epsilon^- g\big) - i\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g + \Delta_\epsilon^- f\Delta_\epsilon^- g\big)\big].\end{aligned} \tag{7}$$

Moreover, we have the following formula:

$$\Box_\epsilon f\Box_\epsilon g = \frac{1}{2}\big[\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g + \Delta_\epsilon^- f\Delta_\epsilon^+ g\big) - i\big(\Delta_\epsilon^+ f\Delta_\epsilon^+ g - \Delta_\epsilon^- f\Delta_\epsilon^- g\big)\big], \tag{8}$$

and

$$\Box_\epsilon f\boxminus_\epsilon g = \frac{1}{2}\big[\big(\Delta_\epsilon^+ f\Delta_\epsilon^+ g + \Delta_\epsilon^- f\Delta_\epsilon^- g\big) - i\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g - \Delta_\epsilon^- f\Delta_\epsilon^+ g\big)\big]. \tag{9}$$

We then obtain

$$\begin{aligned}&\Delta_\epsilon^+ f\Delta_\epsilon^+ g + \Delta_\epsilon^- f\Delta_\epsilon^- g = \Box_\epsilon f\boxminus_\epsilon g + \boxminus_\epsilon f\Box_\epsilon g,\\ &-i\big(\Delta_\epsilon^+ f\Delta_\epsilon^+ g - \Delta_\epsilon^- f\Delta_\epsilon^- g\big) = \Box_\epsilon f\Box_\epsilon g - \boxminus_\epsilon f\boxminus_\epsilon g.\end{aligned} \tag{10}$$

We deduce then the following equality:

$$\begin{aligned}&\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g - \Delta_\epsilon^- f\Delta_\epsilon^- g\big) - i\big(\Delta_\epsilon^+ f\Delta_\epsilon^- g + \Delta_\epsilon^- f\Delta_\epsilon^- g\big)\\ &\quad = i(\Box_\epsilon f\Box_\epsilon g - \boxminus_\epsilon f\boxminus_\epsilon g) - i(\Box_\epsilon f\boxminus_\epsilon g + \boxminus_\epsilon f\Box_\epsilon g).\end{aligned} \tag{11}$$

This concludes the proof. □

We have the following integral formula:

$$\int_a^b \Box_\epsilon f(t)\,dt = \frac{1}{2}\big[\big(f_\epsilon^+(t) + f_\epsilon^-(t)\big) - i\big(f_\epsilon^+(t) - f_\epsilon^-(t)\big)\big]\Big|_a^b. \tag{12}$$

When ϵ goes to zero, we deduce

$$\lim_{\epsilon\to 0}\int_a^b \Box_\epsilon f(t)\,dt = f(t)|_a^b. \tag{13}$$

2.3. Hölderian functions

In the following, we consider a particular class of non-differentiable functions called *Hölderian functions* [22].

Definition 2.4. A continuous real valued function f is Hölderian of Hölder exponent α, $0 < \alpha < 1$, if for all $\epsilon > 0$, and all $t, t' \in \mathbb{R}$ such that $|t - t'| \leqslant \epsilon$, there exists a constant c such that

$$\left| f(t) - f(t') \right| \leqslant c\epsilon^{\alpha}. \tag{14}$$

In the following, we denote by H^{α} the set of continuous functions which are Hölderian of Hölder exponent α. Moreover, we say that a complex valued function $y(t)$ belongs to H^{α} if its real and imaginary parts belong to H^{α}.

We then have the following lemma.

Lemma 2.1. *If $x \in H^{\alpha}$ then $\Box_{\epsilon} x \in H^{\alpha}$ for all $\epsilon > 0$.*

This follows from the definition of $\Box_{\epsilon} x(t)$ and simple calculations.

2.4. A technical result

We derive a technical result about the scale derivative, which will be used in the last section.

Theorem 2.2. *Let $f(x,t)$ be a C^{n+1} function and $x(t) \in H^{1/n}$, $n \geqslant 1$. For all $\epsilon > 0$ sufficiently small, we have*

$$\frac{\Box_{\epsilon} f(x(t),t)}{\Box t} = \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{1}{j!} \frac{\partial^j f}{\partial x^j}\big(x(t),t\big)\epsilon^{j-1} a_{\epsilon,j}(t) + o(\epsilon^{1/n}), \tag{15}$$

where

$$a_{\epsilon,j}(t) = \frac{1}{2}\Big[\Big(\big(\Delta_{+}^{\epsilon} x\big)^j - (-1)^j \big(\Delta_{-}^{\epsilon} x\big)^j\Big) - i\Big(\big(\Delta_{+}^{\epsilon} x\big)^j + (-1)^j \big(\Delta_{-}^{\epsilon} x\big)^j\Big)\Big]. \tag{16}$$

The proof follows easily from the following lemma.

Lemma 2.2. *Let $f(x,t)$ be a real valued function of class C^{n+1}, $n \geqslant 1$, and $x(t) \in H^{1/n}$. For all $\epsilon > 0$ sufficiently small, the right and left quantum derivatives of $f(x(t),t)$ are given by*

$$\Delta_{\sigma}^{\epsilon} f\big(x(t),t\big) = \frac{\partial f}{\partial t}\big(x(t),t\big) + \sigma \sum_{i=1}^{n} \frac{1}{i!} \frac{\partial^i f}{\partial x^i}\big(x(t),t\big)\epsilon^{-1}\big(\sigma\epsilon\Delta_{\sigma}^{\epsilon} x(t)\big)^i + o(\epsilon^{1/n}), \tag{17}$$

for $\sigma = \pm$.

Proof. This follows from easy computations. First, we remark that, as $x(t) \in H^{1/n}$, we have $|\epsilon \Delta_{\sigma}^{\epsilon} X(t)| = o(\epsilon^{1/n})$. Moreover,

$$f\big(x(t+\epsilon), t+\epsilon\big) = f\big(x(t) + \epsilon\Delta_{+}^{\epsilon} x(t), t+\epsilon\big).$$

By the previous remark, and the fact that f is of order C^{n+1}, we can make a Taylor expansion up to order n with a controlled remainder,

$$f\big(x(t+\epsilon),t+\epsilon\big)=f\big(x(t),t\big)+\sum_{k=1}^{n}\frac{1}{k!}\sum_{i+j=k}\big(\epsilon\Delta_+^\epsilon x(t)\big)^i\epsilon^j\frac{\partial^k f}{\partial^i x\partial^j t}\big(x(t),t\big)$$
$$+o\big(\big(\epsilon\Delta_+^\epsilon x(t)\big)^{n+1}\big).$$

As a consequence, we have

$$\epsilon\Delta_+^\epsilon f\big(x(t),t\big)=\sum_{k=1}^{n}\frac{1}{k!}\sum_{i+j=k}\big(\epsilon\Delta_+^\epsilon x(t)\big)^i\epsilon^j\frac{\partial^k f}{\partial^i x\partial^j t}\big(x(t),t\big)+o\big(\big(\epsilon\Delta_+^\epsilon x(t)\big)^{n+1}\big).$$

By selecting terms of order less or equal to one in ϵ in the right of this equation, we obtain

$$\epsilon\Delta_+^\epsilon f\big(x(t),t\big)=\epsilon\left[\frac{\partial f}{\partial t}\big(x(t),t\big)+\sum_{i=1}^{n}\frac{1}{i!}\frac{\partial^i f}{\partial x^i}\big(x(t),t\big)\epsilon^{-1}\big(\epsilon\Delta_+^\epsilon x(t)\big)^i\right]$$
$$+o\big(\epsilon^2\Delta_+^\epsilon x(t)\big).$$

Dividing by ϵ, we obtain the lemma. □

3. Non-differentiable calculus of variations

3.1. Functionals

The classical calculus of variations is concerned with the extremals of functions whose domain is an infinite-dimensional space: the space of curves, which is usually the set of *differentiable* curves. We look for an analogous theory on the set of non-differentiable curves.

In all the text, α is a real number satisfying

$$0<\alpha<1,$$

and ϵ is a parameter, which is assumed to be sufficiently small, i.e.,

$$0<\epsilon\ll 1,$$

without precising its exact smallness.

We denote by $C_\epsilon^\alpha(a,b)$ the set of curves in the plane of the form

$$\gamma=\big\{\big(t,x(t)\big),\ x\in H^\alpha,\ a-\epsilon\leqslant t\leqslant b+\epsilon\big\}. \tag{18}$$

Remark 3.1. (i) In the following, we will simply write $C^\alpha(a,b)$ for $C_\epsilon^\alpha(a,b)$.

(ii) We must take $a-\epsilon\leqslant t\leqslant b+\epsilon$ in order to avoid problems with the definition of the scale derivative on the extremal points of the interval $[a,b]$.

A *functional* Φ is a map $\Phi: C^\alpha(a,b)\to\mathbb{C}$.

Remark 3.2. In classical mechanics, one usually consider real valued functionals instead of complex one.

We will restrict our attention to the following class of functionals.

Definition 3.1. Let $L:\mathbb{R}\times\mathbb{C}\times\mathbb{R}\to\mathbb{C}$ be a differentiable function of three variables (x,v,t). For all $\epsilon>0$, a functional $\Phi_\epsilon : C^\alpha(a,b)\to\mathbb{C}$ is defined by

$$\Phi_\epsilon(\gamma)=\int_a^b L\big(x(t),\Box_\epsilon x(t),t\big)\,dt, \tag{19}$$

for all $\gamma\in C^\alpha(a,b)$.

Of course, when we consider differentiable curves, we can take the limit of (19) when ϵ goes to zero, and we obtain the classical functional (see [1, p. 56]):

$$\Phi(\gamma)=\int_a^b L\big(x(t),\dot{x}(t),t\big)\,dt, \tag{20}$$

where $\dot{x}=dx/dt$.

3.2. *Variations*

We first define *variations* of curves.

Definition 3.2. Let $\gamma\in C^\alpha(a,b)$. A variation γ' of γ is a curve

$$\gamma'=\big\{\big(t,x(t)+h(t)\big),\ x\in H^\alpha,\ h\in H^\beta,\ \beta\geqslant\alpha 1_{[1/2,1]}+(1-\alpha)1_{]0,1/2[},\\ h(a)=h(b)=0\big\}. \tag{21}$$

We denote this curve by $\gamma'=\gamma+h$.

As in the usual case, we look for paths of a given regularity class with prescribed end points. The condition $\beta\geqslant\alpha 1_{[1/2,1]}+(1-\alpha)1_{]0,1/2[}$ for the variation is a technical assumption, which will be used in the derivation of the non-differentiable analogue of the Euler–Lagrange equation (see Section 3.3). The minimal condition on β for which the problem of variations makes sense is $\beta\geqslant\alpha$, in order to ensure that $\gamma+h$ is again in $C^\alpha(a,b)$.

In the following, we always consider variations of a given curve γ of the form $\gamma_\mu=\gamma+\mu h$, where μ is a real parameter.

Definition 3.3. A functional Φ is called differentiable on $C^\alpha(a,b)$ if for all variations $h\in C^\beta(a,b)$, we have

$$\Phi(\gamma+h)-\Phi(\gamma)=F(\gamma,h)+R(\gamma,h), \tag{22}$$

where F depends linearly on h, i.e., $F(h_1+h_2)=F(h_1)+F(h_2)$ and $F(ch)=cF(h)$, and $R(\gamma,h)=O(h^2)$, i.e., for $|h|<\mu$ and $|\Box_\epsilon h|<\mu$, we have $|R|<C\mu^2$.

The functional F is called the differential of Φ.

In the case of functionals of the form (19), we have

Theorem 3.1. *For all $\epsilon>0$, the functional $\Phi_\epsilon(\gamma)$ defined by* (19) *is differentiable, and its derivative is given by the formula*

$$F_\epsilon^\gamma(h)=\int_a^b\left[\frac{\partial L}{\partial x}\left(x(t),\frac{\Box_\epsilon x}{\Box t},t\right)-\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial\Box_\epsilon x}\left(x(t),\frac{\Box_\epsilon x}{\Box t},t\right)\right)\right]h(t)\,dt \tag{23}$$

$$+\int_a^b\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial\Box_\epsilon x}h(t)\right)dt+iR_\epsilon^\gamma(h), \tag{24}$$

with

$$R_\epsilon^\gamma(h)=\epsilon\int_a^b\big[\Box_\epsilon f_\epsilon(t)\Box_\epsilon h(t)-\boxminus_\epsilon f_\epsilon(t)\Box_\epsilon h(t)-\Box_\epsilon f_\epsilon(t)\boxminus_\epsilon h(t)$$
$$-\boxminus_\epsilon f_\epsilon(t)\boxminus_\epsilon h(t)\big]\,dt, \tag{25}$$

where

$$f_\epsilon(t)=\frac{\partial L}{\partial\Box_\epsilon x}\big(x(t),\Box_\epsilon x(t),t\big). \tag{26}$$

Proof. We have

$$\begin{aligned}
&\Phi_\epsilon(\gamma+h)-\Phi(\gamma)\\
&\quad=\int_a^b\big[L\big(x(t)+h(t),\Box_\epsilon x(t)+\Box_\epsilon h(t),t\big)-L\big(x(t),\Box_\epsilon x(t),t\big)\big]\,dt\\
&\quad=\int_a^b\left[\frac{\partial L}{\partial x}\big(x(t),\Box_\epsilon x(t),t\big)h(t)+\frac{\partial L}{\partial\Box_\epsilon x}\big(x(t),\Box_\epsilon x(t),t\big)\Box_\epsilon h(t)\right]dt+O(h^2)\\
&\quad=F_\epsilon^\gamma(h)+R(h),
\end{aligned}$$

where

$$F_\epsilon^\gamma(h)=\int_a^b\left[\frac{\partial L}{\partial x}\big(x(t),\Box_\epsilon x(t),t\big)h(t)+\frac{\partial L}{\partial\Box_\epsilon x}\big(x(t),\Box_\epsilon x(t),t\big)\Box_\epsilon h(t)\right]dt,$$

and $R(h)=O(h^2)$.

Using (5), we deduce:

$$F_\epsilon^\gamma(h) = \int_a^b \left[\frac{\partial L}{\partial x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right) - \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial \Box_\epsilon x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right)\right)\right] h(t)\, dt$$
$$+ \int_a^b \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial \Box_\epsilon x} h(t)\right) dt$$
$$+ i\epsilon \int_a^b \left[\Box_\epsilon f_\epsilon(t) \Box_\epsilon h(t) - \boxminus_\epsilon f_\epsilon(t) \Box_\epsilon h(t) - \Box_\epsilon f_\epsilon(t) \boxminus_\epsilon h(t)\right.$$
$$\left. - \boxminus_\epsilon f_\epsilon(t) \boxminus_\epsilon h(t)\right] dt,$$

with

$$f_\epsilon(t) = \frac{\partial L}{\partial \Box_\epsilon x}\big(x(t), \Box_\epsilon x(t), t\big).$$

This concludes the proof. □

3.3. Extremal curves and Euler–Lagrange equation

The functional derivative of Φ_ϵ mix terms which are either divergent when ϵ goes to zero, or tending toward 0 with ϵ. In order to simplify our problem and to take into account only dominant terms in ϵ, we introduce the following operator.

Definition 3.4. Let $a_p(\epsilon)$ be a real or complex valued function, with parameters p. We denote by $[.]_\epsilon$ the linear operator defined by

(i) $a_p(\epsilon) - [a_p(\epsilon)]_\epsilon \to_{\epsilon\to 0} 0$,
(ii) $[a_p(\epsilon)]_\epsilon = 0$ if $\lim_{\epsilon\to 0} a_p(\epsilon) = 0$.

The quantity $[a_p(\epsilon)]_\epsilon$ is called the ϵ-dominant part of $a_p(\epsilon)$.

For example, if $a(\epsilon) = \epsilon^{-1/2} + 2\epsilon + 2$, then $[a(\epsilon)]_\epsilon = \epsilon^{-1/2} + 2$.

We deduce the following properties.

Lemma 3.1. *The ϵ-dominant part is unique.*

Proof. This comes from the relation $[[.]_\epsilon]_\epsilon = [.]_\epsilon$. Indeed, by definition we have $a_p(\epsilon) = [a_p(\epsilon)]_\epsilon + r(\epsilon)$ with $\lim_{\epsilon\to 0} r(\epsilon) = 0$. Applying $[.]_\epsilon$ directly on this expression, we obtain $[a_p(\epsilon)]_\epsilon = [[a_p(\epsilon)]_\epsilon]_\epsilon$ using (ii). □

Remark 3.3. Unicity comes from condition (ii). Indeed, if we cancel this condition, we can obtain many different quantities satisfying (i). For example, if $a(\epsilon) = \alpha\epsilon^{-1/2} + \epsilon + 2$, then without (ii), we have the choice between $[a(\epsilon)]_\epsilon = \epsilon^{-1/2} + 2 + \epsilon$ and $[a(\epsilon)]_\epsilon = \epsilon^{-1/2} + 2$.

(ii) This operator can be used in the definition of left and right quantum operators by considering $\delta_\epsilon^\sigma x(t) = [\Delta_\epsilon^\sigma x(t)]_\epsilon$, $\sigma = \pm$. However, using such kind of operators lead to many difficulties from the algebraic point of view, in particular with the derivation of the analogue of the Leibniz rule.

We now introduce the non-differentiable analogue of the notion of *extremals* curves in the classical case (see [1, p. 57]).

Definition 3.5. Let $0 < \alpha \leqslant 1$. An extremal curve of the functional (19) on the space of curves of class $C^{\beta}(a,b)$, $\beta \geqslant \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}$, is a curve $\gamma \in C^{\alpha}(a,b)$ satisfying

$$\left[F_{\epsilon}^{\gamma}(h)\right]_{\epsilon} = 0, \tag{27}$$

for all $\epsilon > 0$ and all $h \in C^{\beta}(a,b)$.

The following theorem gives the analogue of the Euler–Lagrange equations for extremals of our functionals.

Theorem 3.2. *We assume that the function L defining the functional* (19) *satisfies*

$$\left\| D(\partial L/\partial v) \right\| \leqslant C, \tag{28}$$

where C is a constant, D denotes the differential, and $\|.\|$ *is the classical norm on matrices. The curve* γ: $x = x(t)$ *is an extremal curve of the functional* (19) *on the space of curves of class* $C^{\beta}(a,d)$,

$$\beta \geqslant \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}, \tag{29}$$

if and only if it satisfies the following generalized Euler–Lagrange equation:

$$\left[\frac{\partial L}{\partial x}\left(x(t), \frac{\Box_{\epsilon} x}{\Box t}, t\right) - \frac{\Box_{\epsilon}}{\Box t}\left(\frac{\partial L}{\partial \Box_{\epsilon} x}\left(x(t), \frac{\Box_{\epsilon} x}{\Box t}, t\right)\right)\right]_{\epsilon} = 0, \tag{30}$$

for $\epsilon > 0$.

Remark 3.4. Our Euler–Lagrange equation (30) looks like the one obtained by Nottale [17] in the context of the *scale relativity theory* (see Section 5.2).

Proof. The proof follow the classical derivation of Euler–Lagrange equation (see, for example, [21, pp. 432–434]). By Theorem 3.1, we have

$$\begin{aligned} F_{\epsilon}^{\gamma}(h) = {} & \int_a^b \left[\frac{\partial L}{\partial x}\left(x(t), \frac{\Box_{\epsilon} x}{\Box t}, t\right) - \frac{\Box_{\epsilon}}{\Box t}\left(\frac{\partial L}{\partial \Box_{\epsilon} x}\left(x(t), \frac{\Box_{\epsilon} x}{\Box t}, t\right)\right)\right] h(t)\, dt \\ & + \int_a^b \frac{\Box_{\epsilon}}{\Box t}\left(\frac{\partial L}{\partial \Box_{\epsilon} x} h(t)\right) dt \\ & + i\epsilon \int_a^b \left[\Box_{\epsilon} f_{\epsilon}(t) \Box_{\epsilon} h(t) - \boxminus_{\epsilon} f_{\epsilon}(t) \Box_{\epsilon} h(t) - \Box_{\epsilon} f_{\epsilon}(t) \boxminus_{\epsilon} h(t) \right. \\ & \left. - \boxminus_{\epsilon} f_{\epsilon}(t) \boxminus_{\epsilon} h(t)\right] dt, \end{aligned}$$

with

$$f_\epsilon(t) = \frac{\partial L}{\partial \Box_\epsilon x}\big(x(t), \Box_\epsilon x(t), t\big).$$

In order to conclude, we need the following lemma.

Lemma 3.2. *Let* $0 < \epsilon$, $a, b \in \mathbb{R}$, $h \in H^\beta$, $\beta \geqslant \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}$, *such that* $h(a) = h(b) = 0$, *and* $f_\epsilon : \mathbb{R} \to \mathbb{C}$ *such that*

$$\sup_{s\in\{t,t+\sigma\epsilon\}} \big|f_\epsilon(s)\big| \leqslant C\epsilon^{\alpha-1}, \tag{31}$$

for all $t \in [a,b]$. *Then, we have*

$$\int_a^b \frac{\Box_\epsilon}{\Box t}\big(f_\epsilon(t)h(t)\big)\,dt = O(\epsilon^{\alpha+\beta-1}) \tag{32}$$

and

$$\epsilon \int_a^b Op_\epsilon(f_\epsilon)Op'_\epsilon(h)\,dt = O(\epsilon^{\alpha+\beta}),$$

where Op_ϵ *and* Op'_ϵ *are either* $\Box_\epsilon$ *or* $\boxminus_\epsilon$.

The proof is given in the next section.

Using condition (28), we obtain

$$\sup_{s\in\{t,t+\sigma\epsilon\}} |\partial L/\partial \Box_\epsilon x| \leqslant C'\epsilon^{\alpha-1},$$

as $\sup_{s\in\{t,t+\sigma\epsilon\}}[\max(|x(s)|, |\Box_\epsilon x(s)|, |s|)] \leqslant C''\epsilon^{\alpha-1}$.

Using Lemma 3.2 with $f_\epsilon(s) = (\partial L/\partial \Box_\epsilon t)(x(s), \Box_\epsilon(s), s)$, and condition (29), we deduce that

$$\lim_{\epsilon\to 0}\int_a^b \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial \Box_\epsilon x}h(t)\right)dt = 0$$

and

$$\lim_{\epsilon\to 0}\epsilon\int_a^b Op_\epsilon\left(\frac{\partial L}{\partial \Box_\epsilon x}\right)Op'_\epsilon\big(h(t)\big)\,dt = 0,$$

for Op_ϵ and Op'_ϵ which are either $\Box_\epsilon$ or $\boxminus_\epsilon$.

Hence, applying the operator $[.]_\epsilon$, we obtain

$$\big[F_\epsilon^\gamma(h)\big]_\epsilon = \left[\int_a^b \left[\frac{\partial L}{\partial x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right) - \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial \Box_\epsilon x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right)\right)\right]h(t)\,dt\right]_\epsilon$$

$$= \int_a^b \left[\frac{\partial L}{\partial x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right) - \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial L}{\partial \Box_\epsilon x}\left(x(t), \frac{\Box_\epsilon x}{\Box t}, t\right)\right)\right]_\epsilon h(t)\, dt.$$

The rest of the proof follows as in the classical case (see [1, pp. 57–58]). □

Remark 3.5. The special form of condition (29) comes from the two following constraints: one must have $\beta \geqslant \alpha$ in order to preserve the regularity of perturbed curves $\gamma + h$, and $\beta \geqslant 1 - \alpha$ in order to ensure that the first quantity of Eq. (32) goes to zero when ϵ goes to zero. Note that $\alpha = 1/2$ plays a special role for these sets of conditions, as this is the only one for which the regularity of curves and variations are equal.

4. Proof of Lemma 3.2

This comes essentially from the integral formula (12). Indeed,

$$\int_a^b \frac{\Box_\epsilon}{\Box t}\big(f_\epsilon(t)h(t)\big)\, dt$$

is a combination of the following quantities:

$$\frac{1}{2\epsilon}\int_t^{t+\sigma\epsilon} f_\epsilon(s)h(s)\, ds, \quad \sigma = \pm,$$

for $t = a$ or $t = b$.

As $h(a) = h(b) = 0$ and $h \in C^\beta(a, b)$, we have for $t = a$ or $t = b$,

$$\sup_{s\in\{t,t+\sigma\epsilon\}} \big|h(s)\big| = \sup_{s\in\{t,t+\sigma\epsilon\}} \big|h(s) - h(t)\big| \leqslant C\epsilon^\beta,$$

for some constant C. Moreover, using condition (31), we easily obtain

$$\sup_{s\in\{t,t+\sigma\epsilon\}} \big|f_\epsilon(s)h(s)\big| C'\epsilon^{\alpha+\beta-1},$$

where C' is a constant. Using this inequality, we deduce

$$\left|\int_a^b \frac{\Box_\epsilon}{\Box t}\big(f_\epsilon(s)h(s)\big)\, ds\right| = O(\epsilon^{\alpha+\beta-1}).$$

We only prove the second inequality of Eq. (32) for $Op_\epsilon = \Box_\epsilon$ and $Op'_\epsilon = \Box_\epsilon$. The remaining cases are proved in the same way.

As $h \in C^\beta(a, b)$, we have

$$\sup_{t\in[a,b]} \big|\Box_\epsilon h(t)\big| \leqslant C\epsilon^\beta,$$

for some constant C (see Lemma 2.1). Moreover, using (31), we obtain

$$\sup_{s\in[a,b]} \big|\Delta_\epsilon^\sigma(f_\epsilon)(s)\big| \leqslant C^\sigma\epsilon^{\alpha-1},$$

for some constant C^σ, $\sigma = \pm$. We deduce

$$\sup_{s\in[a,b]} \left|\Box_\epsilon f_\epsilon(s)\right| \leqslant C'\epsilon^{\alpha-1}.$$

As a consequence, we obtain the inequality

$$\left|\epsilon \int_a^b \frac{\Box_\epsilon f_\epsilon}{\Box t}\frac{\Box_\epsilon h}{\Box t}\,dt\right| \leqslant C''\epsilon^{\alpha+\beta},$$

for some constant C''. This concludes the proof of Lemma 3.2.

5. Application: least action principle and non-linear Schrödinger equations

5.1. Least action principle and the Schrödinger equation

In this section we gives a variational principle whose extremals are solutions of the *Schrödinger equation*.

We consider the following non-linear Schrödinger's equation (obtained in [4,9]):

$$\left[2i\gamma m\left[-\frac{1}{\psi}\left(\frac{\partial\psi}{\partial x}\right)^2\left(i\gamma+\frac{a_\epsilon(t)}{2}\right)+\frac{\partial\psi}{\partial t}+\frac{a_\epsilon(t)}{2}\frac{\partial^2\psi}{\partial x^2}\right]\right.$$
$$\left. = \big(U(x)+\alpha(x)\big)\psi\right]_\epsilon, \tag{33}$$

where $m>0$, $\gamma\in\mathbb{R}$, $U:\mathbb{R}\to\mathbb{R}$, $a_\epsilon:\mathbb{R}\to\mathbb{C}$, $\alpha(x)$ is an arbitrary continuous function.

The main result of this section is an analogue of the *Hamilton's principle of least action* (see [1, p. 59]) for (33).

Theorem 5.1. *Solutions of the non-linear Schrödinger equation* (33) *coincide with extremals of the functional associated to*

$$L\big(x(t),\Box_\epsilon x(t),t\big) = (1/2)m\big(\Box_\epsilon x(t)\big)^2 + U(x), \tag{34}$$

on the space of $C^{1/2}$ curves, where $x(t)$ and $\psi_\epsilon(x,t)$ are related by

$$\frac{\Box_\epsilon x}{\Box t} = -i2\gamma\frac{\partial \ln(\psi(x,t))}{\partial x}, \tag{35}$$

and if $a_\epsilon(t)$ is such that

$$a_\epsilon(t) = \frac{1}{2}\big[\big(\Delta_\epsilon^+ x(t)\big)^2 - \big(\Delta_\epsilon^- x(t)\big)^2\big] - i\frac{1}{2}\big[\big(\Delta_\epsilon^+ x(t)\big)^2 + \big(\Delta_\epsilon^- x(t)\big)^2\big]. \tag{36}$$

Remark 5.1. (i) The non-linear Schrödinger equation (33) was derived in [4] using an analogue of the Euler–Lagrange equation (30) proposed by Nottale [17] in the context of the Scale relativity theory. This derivation was done in the framework of the *local fractional calculus* developed in [3] and under an assumption concerning the existence of solutions to a particular fractional differential equation. However, as proved in ([9, part I, Section 4.3], [5]) such assumptions cannot be satisfied.

(ii) In [9], Eq. (33) was derived using a "scale quantization procedure," which gives a way to pass from classical mechanics to quantum mechanics, avoiding the problems of [4]. However, the Euler–Lagrange equation used in [9] comes from scale quantization, which is an abstract and formal way to derive the analogue of (30) from the classical Euler–Lagrange equation (see Section 5.2).

Proof. As $\partial L/\partial \Box_\epsilon x = m\Box_\epsilon x$, its differential is given by

$$D(\partial L/\partial \Box_\epsilon x) = (0, m, 0),$$

so that condition (28) is satisfied. By Theorem 3.2, extremals of our functional satisfy the Euler–Lagrange equation

$$\left[m\frac{\Box_\epsilon \Box_\epsilon x(t)}{\Box t} = \frac{dU}{dx}(x)\right]_\epsilon. \tag{37}$$

We denote

$$f(x,t) = \frac{\partial \ln(\psi(x,t))}{\partial x}(x,t).$$

We apply Theorem 2.2 with $n = 2$, in order to compute $\Box_\epsilon f(x(t),t)/\Box t$. We have

$$\begin{aligned}\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial \ln(\psi)}{\partial x}\big(x(t),t\big)\right) &= \frac{\Box_\epsilon x}{\Box t}\frac{\partial}{\partial x}\left(\frac{\partial \ln(\psi(x,t))}{\partial x}\right)\big(x(t),t\big) \\ &\quad + \frac{\partial}{\partial t}\left(\frac{\partial \ln(\psi(x,t))}{\partial x}\right)\big(x(t),t\big) \\ &\quad + \frac{1}{2}a_\epsilon(t)\frac{\partial^2}{\partial x^2}\left(\frac{\partial \ln(\psi(x,t))}{\partial x}\right)\big(x(t),t\big) + o(\epsilon^{1/2}). \end{aligned} \tag{38}$$

Elementary calculus gives

$$\frac{\partial \ln(\psi(x,t))}{\partial x} = \frac{1}{\psi}\frac{\partial \psi}{\partial x}$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial x}\right) = \frac{1}{\psi}\frac{\partial^2 \psi}{\partial^2 x} - \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2.$$

Hence, we obtain

$$\begin{aligned}\frac{\Box_\epsilon x}{\Box t}\frac{\partial}{\partial x}\left(\frac{\partial \ln(\psi(x,t))}{\partial x}\right)\big(x(t),t\big) &= -i2\gamma\frac{\partial \ln(\psi)}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \ln(\psi)}{\partial x}\right)\big(x(t),t\big) \\ &= -i\gamma\frac{\partial}{\partial x}\left[\left(\frac{\partial \ln(\psi)}{\partial x}\right)^2\right]\big(x(t),t\big) \\ &= -i\gamma\frac{\partial}{\partial x}\left[\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2\right]\big(x(t),t\big).\end{aligned}$$

We then have

$$\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial \ln(\psi(x,t))}{\partial x}\big(x(t),t\big)\right)$$
$$= \frac{\partial}{\partial x}\left[-i\gamma\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{\partial \ln(\psi)}{\partial t} + \frac{1}{2}a_\epsilon(t)\left[\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x}\right)^2\right]\right] + o(\epsilon^{1/2})$$
$$= \frac{\partial}{\partial x}\left[-\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial\psi}{\partial t} + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}\right] + o(\epsilon^{1/2}).$$

As a consequence, Eq. (38) is equivalent to

$$\frac{\partial}{\partial x}\left[i2\gamma m\left[-\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial\psi}{\partial t}\right] + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}\right] = \frac{\partial U}{\partial x}.$$

By integrating with respect to x, we obtain

$$i2\gamma m\left[-\frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial x}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial\psi}{\partial t}\right] + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}$$
$$= U(x) + \alpha(x) + o(\epsilon^{1/2}),$$

where $\alpha(x)$ is an arbitrary function. This concludes the proof. □

A great deal of efforts have been made in order to generalize the classical linear Schrödinger equation (see, for example, de Broglie [6,7] and Lochak [13]). However, these generalizations are in general ad hoc one, choosing some particular non-linear terms in order to solve some specific problems of quantum mechanics (see, for example, [2,19,20]). On the contrary, the non-differentiable least action principle impose a fixed non-linear term.

In order to recover the classical linear Schrödinger equation, we must specialize the functional space on which we work. Precisely, we have

Theorem 5.2. *Solutions of the Schrödinger equation*

$$\left[i\bar{h}\frac{\partial\psi}{\partial t} + \frac{\bar{h}^2}{2m}\frac{\partial^2\psi}{\partial x^2} = U(x)\psi\right]_\epsilon, \tag{39}$$

where $\bar{h} = h/2\pi$, coincide with extremals of the functional associated to

$$L\big(x(t), \Box_\epsilon x(t), t\big) = (1/2)m\big(\Box_\epsilon x(t)\big)^2 + U(x), \tag{40}$$

on the space of $C^{1/2}$ curves γ: $x = x(t)$ satisfying

$$\frac{1}{2}\Big[\big(\Delta_\epsilon^+ x(t)\big)^2 - \big(\Delta_\epsilon^- x(t)\big)^2\Big] - i\frac{1}{2}\Big[\big(\Delta_\epsilon^+ x(t)\big)^2 + \big(\Delta_\epsilon^- x(t)\big)^2\Big] = -i\bar{h}/m, \tag{41}$$

where $x(t)$ and $\psi_\epsilon(x,t)$ are related by

$$\frac{\Box_\epsilon x}{\Box t} = -i\frac{\bar{h}}{m}\frac{\partial \ln(\psi(x,t))}{\partial x}. \tag{42}$$

Proof. This follows easily from the calculations made in the proof of Theorem 5.1.

For different derivations of the Schrödinger equation, we refer to the work of Nelson on *stochastic mechanics* [15,16] and Feynman [11], where he developed a principle of least action, different from the one presented here.

5.2. About the scale relativity theory

This final section is informal and discuss the connexion between our non-differentiable variational principle and the scale relativity theory. In the following, we do not give a precise definition to the word *fractal*. The only property which is assumed is that fractals are scale dependent objects. We refer to [10] for more details.

The *scale relativity theory* developed by Nottale [17], gives up the assumption of the differentiability of space–time by considering what he calls a *fractal space–time*, and extending the Einstein's principle of relativity to scales.

One of the consequences of such a theory is that there exists an infinity of geodesics[3] and that geodesics are fractal curves. On such curves, one must develop a new differential calculus taking into account the non-differentiable character of the curve [8]. The scale derivative introduced by Nottale is the analogue of the scale derivative introduced in this paper.

The *scale relativity principle* can be state as follows: *The equations of physics keep the same form under scale transformations* (see [17]).

As a consequence, the scale relativity principle allows us to pass from classical mechanics to quantum mechanic via a simple procedure: *one must change the classical derivative in Newton's fundamental equation of dynamics by the scale derivative* (see [18]).

As Newton's equation is written via an Euler–Lagrange equation of the form

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}\right] = \frac{\partial L}{\partial x}, \tag{43}$$

this procedure, called *scale quantization* in [9], gives a quantum analogue of the form

$$\frac{\Box_\epsilon}{\Box t}\left[\frac{\partial L}{\partial v}\right] = \frac{\partial L}{\partial x}, \tag{44}$$

where v is of course a complex quantity defined by

$$v = \frac{\Box_\epsilon x}{\Box t}. \tag{45}$$

As a consequence, scale quantization gives an Euler–Lagrange equation similar to the one obtained via the non-differentiable variational principle introduced in this paper. The non-differentiable variational principle can be considered as an attempt to develop the mathematical foundations of the scale relativity principle.

[3] This notion is not well defined, and we refer to [17] for more details.

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