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Global Theory of the Riccati Equation*

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ABSTRACT

The asymptotic theory of the matrix Riccati equation is developed using non-variational arguments. The cone of positive semi-definite matrices is shown to be invariant under the motions of the Riccati equation and questions of monotonicity and stability are resolved for motions starting in this cone.

We shall consider here the theory of the matrix Riccati equation. This theory was in part stated in Kalman and Bucy [*1*] and proved in Kalman [*2*]. The proofs in the latter paper were variational in nature and dependent on the theory of a particular control problem. In this paper we provide new and quite general proofs of this theory which are variation-free, as well as delineating some new results.

Because of the importance of asymptotic behavior of the Riccati equation for a wide variety of problems, a new derivation written for a mathematical audience seemed of interest. Most of this paper will appear later in the book by Bucy and Joseph [*3*].

NOTATION AND DEFINITIONS

We denote matrices by capital latin letters, numbers by lower-case latin letters, while boldface, lower-case latin letters denote vectors. A' denotes the transpose of A. The matrix Riccati equation is given by

$$\dot{P} = F(t)\,P + PF'(t) - PH'(t)\,R^{-1}(t)\,H(t)\,P + G(t)\,QH(t)\,G'(t),$$
$$P(t_0) = \Gamma, \tag{1}$$

where P is an $n \times n$ matrix, Γ a symmetric $n \times n$ matrix, $F(t)$ an $n \times n$ matrix, $H(t)$ an $s \times n$ matrix, $G(t)$ an $n \times r$ matrix, and $R(t)$ and $Q(t)$ a positive-definite

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$s \times s$ matrix and a positive-semidefinite $r \times r$ matrix. Further, the matrices $F(t)$, $H(t)$, $G(t)$, $Q(t)$, and $R(t)$ will be assumed continuous functions of time and often, when the context is clear, the explicit time dependence will be surpressed.

By C we denote the cone of symmetric positive-semidefinite $n \times n$ matrices and $\mathring{C}$ the interior of C, the cone of positive-definite symmetric matrices. A partial ordering $\geqslant$ of symmetric $n \times n$ matrices is induced by the cone C as $A \geqslant B (A > B)$ if, and only if, $A - B \in C(\mathring{C})$.

In the sequel, two regularity conditions will be indispensable to ensure regular asymptotic behavior; they are:

Definition A. The pair $(F(t), G(t))$ is Q completely controllable iff for every t_0 there exists a $t_1 > t_0$ such that

$$C_Q(t_1, t_0) = \int_{t_0}^{t_1} \Phi(t_1, s)\, G(s)\, Q(s)\, G'(s)\, \Phi'(s, t_1)\, ds > 0$$

or invertible [where $\Phi(t, s)$ is the fundamental matrix of $F(t)$].

Definition B. The pair $(F(t), H(t))$ is R completely observable iff for every t_1 there exists a $t_0 < t_1$ such that

$$W_R(t_1, t_0) = \int_{t_0}^{t_1} \Phi'(s, t_1)\, H'(s)\, R^{-1}(s)\, H(s)\, \Phi(s, t_1)\, ds > 0$$

or invertible.

The concepts of controllability and observability have natural interpretations in the theory of Filtering and Control; see for example Kalman and Bucy [*1*]. Further, these concepts are connected to the classical conditions of normality and disconjugacy; see Bucy [*4*].

1. Global Existence

We will first establish global existence and uniqueness by demonstrating a weak *a priori* bound on the solutions of (1) as

Lemma 1. *Let $\Pi(t, \Gamma, t_0)$ be a solution of* (1) *with $\Pi(t_0, \Gamma, t_0) = \Gamma \in C$ then for all $t > t_0$ sufficiently small*

$$0 \leqslant \Pi(t, \Gamma, t_0) \leqslant \Phi(t, t_0)\, \Gamma \Phi'(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)\, G(\tau)\, Q(\tau)\, G'(\tau)\, \phi'(t, \tau)\, d\tau$$
$$= \Phi(t, t_0)\, \Gamma \Phi'(t, t_0) + C_Q(t, t_0) \tag{2}$$

with $\Phi(t, t_0)$ the fundamental matrix of $F(t)$.

Proof. Since (1) satisfies a local Lipshitz condition, $\Pi(t, \Gamma, t_0)$ exists and is unique for $t - t_0$ sufficiently small. Denoting ψ as the fundamental matrix of

$$F(t) - \Pi(t, \Gamma, t_0) H'(t) R^{-1}(t) H(t),$$

it follows that

$$\Pi(t, \Gamma, t_0) = \psi(t, t_0) \Gamma \psi'(t, t_0) + \int_{t_0}^{t} \psi(t, \sigma) [G(\sigma) Q(\sigma) G'(\sigma) + \Pi(\sigma, \Gamma, t_0) H'R^{-1}(\sigma) H(\sigma) \Pi(\sigma, \Gamma, t_0)] \psi'(t, \sigma) \, d\sigma,$$

so that $\Pi(t, \Gamma, t_0) \geqslant 0$ for t sufficiently small. From (1) it follows that

$$\begin{aligned}\Pi(t, \Gamma, t_0) = \Phi(t, t_0) \Gamma \Phi'(t, t_0) + \int_{t_0}^{t} \Phi(t, \sigma) [G(\sigma) Q(\sigma) G'(\sigma) \\ - \Pi(\sigma, \Gamma, t_0) H'(\sigma) R^{-1}(\sigma) \Pi(\sigma, \Gamma, t_0)] \Phi'(t, \sigma) \, d\sigma \\ \leqslant \Phi(t, t_0) \Gamma \Phi'(t, t_0) + \int_{t_0}^{t} \Phi(t, \sigma) [G(\sigma) Q(\sigma) G'(\sigma)] \Phi'(t, \sigma) \, d\sigma.\end{aligned}$$

LEMMA 2. *Equation* (1) *has a global unique solution.*

Proof. The *á priori* upper bound of Lemma 1 provides a Lipschitz constant on any finite interval no matter how large.

Since the solution of (1) exists and is unique, it induces a two-parameter semigroup of operators T_{t,t_0} defined on positive-semidefinite cone C by $T_{t,t_0}\Gamma = \Pi(t, \Gamma, t_0)$ and the content of Lemma 1 is simply that

$$T_{t,t_0} C \rightarrow C_{t,t_0}$$

where

$$C_{t,t_0} = \{A \in C \mid A \leqslant \Phi(t, t_0) \Gamma \Phi(t, t_0) + C_Q(t, t_0) \quad \text{for some} \quad \Gamma \in C.$$

We remark that this is essentially the same argument given in Kalman and Bucy [*1*] and is given here for completeness. Recently, this argument has been shown to imply global existence of an infinite dimensional generalization of the Riccati equation by Falb and Kleinman (see [*5*]).

2. Explicit Solutions and Geometry

Our aim is to obtain an explicit solution to (1), and in order to do this as well as to establish some more refined properties of T_{t,t_0}, we introduce the following linear $2n$-dimensional Hamiltonian system;

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= -F'(t)\mathbf{x} + H'(t) R^{-1}(t) H(t) \mathbf{y}, \\ \frac{d\mathbf{y}}{dt} &= G(t) Q(t) G'(t) \mathbf{x} + F(t) \mathbf{y}. \qquad (3)\end{aligned}$$

Denote by $S(t, t_0)$ the $2n \times 2n$ fundamental matrix of

$$H(t) = \begin{pmatrix} -F'(t) & H'(t) R^{-1}(t) H(t) \\ G(t) Q(t) G'(t) & F(t) \end{pmatrix},$$

with

$$S(t, t_0) = \begin{pmatrix} \theta_{11}(t, t_0) & \theta_{12}(t, t_0) \\ \theta_{21}(t, t_0) & \theta_{22}(t, t_0) \end{pmatrix},$$

where $\theta_{ij}(t, t_0)$ are $n \times n$ matrices. Since $H(t)$ is Hamiltonian, it is well known and easy to check that $H'(t) J + JH(t) = 0$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ identity matrix. $S(t, t_0)$ has the interesting and useful feature of being symplectic or canonical as:

Definition 1. A $2n \times 2n$ matrix A is symplectic iff $A'JA = J$.

Lemma 3. *For every $t > t_0$, $S(t, t_0)$ is symplectic.*

Proof. Let $\Sigma(t, t_0) = S'(t, t_0) JS(t, t_0) - J$ then $\Sigma(t, t_0) = 0$ and

$$\frac{d\Sigma(t, t_0)}{dt} = S'(t, t_0) (H'(t) J + JH(t)) S(t, t_0) = 0$$

as H is Hamiltonian and consequently $\Sigma(t, t_0) = 0$ or $S(t, t_0)$ is symplectic.

We note in passing that the symplectic matrices form a group, A symplectic implies A' and A^{-1} symplectic; see Siegel [6]. Now the following theorem provides the solution to (1) in terms of solutions of the linear system (3).

Theorem 1. *For $\Gamma \geqslant 0$ the solution of Eq. (1) is given by*

$$\Pi(t, \Gamma, t_0) = L_\Gamma(t, t_0) D_\Gamma^{-1}(t, t_0)$$

where

$$L_\Gamma(t, t_0) = \theta_{21}(t, t_0) + \theta_{22}(t, t_0) \quad \textit{and} \quad D_\Gamma(t, t_0) = \theta_{11}(t, t_0) + \theta_{12}(t, t_0) \Gamma.$$

Further, $\Gamma > 0$ implies $\Pi(t, \Gamma, t_0) > 0$ while $\Gamma \geqslant 0$ with $(F(t), G(t))$ completely Q controllable implies $\Pi(t, \Gamma, t_0) > 0$ for $t > t(t_0)$ [where $C_Q(t_0, t(t_0)) > 0$].

Proof. We shall show $D_\Gamma(t, t_0)$ is invertible for all $t \geqslant t_0$ and $\Gamma \geqslant 0$ for suppose otherwise then, there exists $t \geqslant t_0$, Γ_1 and η a nontrivial n-vector so that

$$D_{\Gamma_1}(t, t_0)\,\eta = 0;$$

but

$$\mathbf{x}'(t_1)\,\mathbf{y}(t_1) = \mathbf{x}'(t_0)\,\mathbf{y}(t_0) + \int_{t_0}^{t_1} \| G'(s)\,\mathbf{x}(s) \|_{Q(s)}^2 + \| H(s)\,\mathbf{y}(s) \|_{R^{-1}(s)}^2\, ds, \qquad (4)$$

with $\mathbf{x}(s)$, $\mathbf{y}(s)$ the solution of (3) with $x(t_0) = \eta$, $y(t_0) = \Gamma\eta$. But by assumption $\mathbf{x}(t_1) = 0$, so that

$$0 = \| \eta \|_\Gamma^2 + \int_{t_0}^{t_1} \| G'(s)\,\mathbf{x}(s) \|_{Q(s)}^2 + \| H(s)\,\mathbf{y}(s) \|_{R^{-1}(s)}^2\, ds$$

or $H(s)\,\mathbf{y}(s) = 0$, $s \in (t_0, t)$; hence

$$\frac{d\mathbf{x}(s)}{dt} = -F'(s)\,\mathbf{x}(s) + H'(s)\,R^{-1}(s)\,H(s)\,\mathbf{y}(s) = -F'(s)\,\mathbf{x}(s)$$

and $0 = \mathbf{x}(t_1) = \Phi'(t_0, t_1)\;\eta$ which implies η is trivial since Φ is a fundamental matrix. Consequently $D_\Gamma(t, t_0)$ is invertible, and $L_\Gamma(t, t_0)\;D_\Gamma^{-1}(t, t_0)$ is well defined for all $t \geqslant t_0$ and $\Gamma \geqslant 0$. An easy consequence of symplectic nature of $S^0(t, t_0)$ is that $D'_\Gamma(t, t_0)\,L_\Gamma(t, t_0) = L'_\Gamma(t, t_0)\,D_\Gamma(t, t_0)$ or $L_\Gamma(t, t_0)\;D_\Gamma^{-1}(t, t_0)$ is symmetric. Now if $L_\Gamma(t, t_0)\,D_\Gamma^{-1}(t, t_0) = K(t, t_0)$ then $K(t, t_0) = \Gamma$ and by elementary differentiation $K(t, t_0)$ satisfies (1) hence by uniquesness $\Pi(t, \Gamma, t_0) = L_\Gamma(t, t_0)\,D_\Gamma^{-1}(t, t_0)$. Suppose $\Gamma > 0$ but suppose $\| \eta^* \|_{\Pi(t,\Gamma,t_0)}^2 = 0$, then

$$\| \eta^* \|_{\Pi(t,\Gamma,t_0)}^2 = \| \eta \|_{D'_\Gamma(t,t_0)L_\Gamma(t,t_0)}^2 = \| \eta \|_\Gamma^2 + \int_{t_0}^{t} \| G'(s)\,\mathbf{x}(s) \|_{Q(s)}^2 + \| H(s)\,\mathbf{y}(s) \|_{R^{-1}(s)}^2\, ds,$$

where $\eta^* = D_\Gamma(t, t_0)\;\eta$ and $(\mathbf{x}(s), \mathbf{y}(s))$ is a solution of (3) satisfying $(\mathbf{x}(t_0), \mathbf{y}(t_0)) = (\eta, \Gamma\eta)$. But $0 < \| \eta \|_\Gamma^2 \leqslant \| \eta^* \|_{\Pi(t,\Gamma,t_0)}^2$, which implies $\eta = 0$ and, consequently, $\eta^* = 0$—a contradiction; hence $\Pi(t, \Gamma, t_0) > 0$. Now suppose $\Gamma \geqslant 0$ but

$$\int_{t_0}^{t(t_0)} \Phi(t, \sigma)\,G(\sigma)\,Q(\sigma)\,G'(\sigma)\,\Phi'(t, \sigma)\,d\sigma > 0 \qquad \text{and} \qquad \| \eta^* \|_{\Pi(t,\Gamma,t_0)}^2 = 0$$

for $t > t(t_0)$ with the same definition of $\mathbf{x}(s)$ and $\mathbf{y}(s)$ as in (4), we obtain:

$$0 = \| \eta^* \|_{\Pi(t,\Gamma,t_0)}^2 = \| \eta \|_\Gamma^2 + \int_{t_0}^{t} \| G'(s)\,\mathbf{x}(s) \|_{Q(s)}^2 + \| H(s)\,\mathbf{y}(s) \|_{R^{-1}(s)}^2\, ds.$$

But $H(s)\,\mathbf{y}(s) = 0$, so that

$$\frac{d\mathbf{x}(s)}{dt} = -F'\mathbf{x}(s),$$

$$\mathbf{x}(s) = \Phi'(t_0, s)\,\eta,$$

and hence

$$\| \Phi'(t_0, t)\,\eta \|_{C_Q(t_0, t(t_0))} = \eta' \int_{t_0}^{t(t_0)} \Phi(t_0, s)\,G(s)\,Q(s)\,G'(s)\,\Phi'(t_0, s)\,\eta\,ds = 0.$$

However, $C_Q(t_0, t(t_0)) > 0$ by complete controllability which implies $\eta = 0$ which implies $\eta^* = 0$ or by contradiction $\Pi(t, \Gamma, t_0) > 0$ for $t > t(t_0)$.

A restatement of Theorem 1 is that $T_{t,t_0} : \mathring{C} \to \mathring{C}$ and with the assumption of complete controllability $T_{t,t_0} : C \to \mathring{C}$ for $t > t_1(t_0)$. Theorem 1 is the central result here as it provides the rigorous justification for the association of a Riccati equation with a linear Hamiltonian system as well as describing the motion of the boundary points of the invariant set C under the motion induced by the Riccati equation.

3. Monotone Properties

In this section we shall restrict ourselves to *autonomous* systems so that a tutorial context may be maintained. The following theorem characterizes the directions of increase and decrease for $T_{t-t_0}\Gamma$, for the autonomous assumption replaces the two-parameter semigroup by a one-parameter semigroup.

Theorem 2. *For $\Gamma \geqslant 0$ and $t \geqslant 0$,*

$$\frac{d\Pi(t, \Gamma, 0)}{dt} = [D_\Gamma'(t)]^{-1} (F\Gamma + \Gamma F' - \Gamma H' R^{-1} H \Gamma + GQG')\, D_\Gamma^{-1}(t). \tag{5}$$

Proof. Let

$$S = \begin{pmatrix} GQG' & F \\ F' & -H'R^{-1}H \end{pmatrix} = JH$$

and consider

$$R(t) = e^{H't} S e^{Ht} - S$$

so that

$$R(0) = 0 \qquad \text{and} \qquad \frac{dR(t)}{dt} = e^{H't}(H'S + SH)\,e^{Ht} = 0$$

since $H'S + SH = (H'J + JH)H = 0$ as H is Hamiltonian. Since $R(t) = 0$; all t we have

$$e^{H't}Se^{Ht} = S \qquad \text{all} \qquad t$$

or in component form

$$\begin{aligned} GQG' &= \theta_{11}'(t)\, GQG'\theta_{11}(t) + \theta_{11}'(t) F\theta_{21}(t) + \theta_{21}'(t) F'\theta_{11}(t) - \theta_{21}'(t)\, H'R^{-1}H\theta_{21}(t), \\ F' &= \theta_{11}'(t)\, GQG'\theta_{12}(t) + \theta_{21}'(t) F'\theta_{12}(t) + \theta_{11}'(t) F\theta_{22}(t) - \theta_{21}'(t)\, H'R^{-1}H\theta_{22}(t), \\ H'R^{-1}H &= \theta_{12}'(t)\, GQG'\theta_{12}(t) + \theta_{12}'(t) F\theta_{22}(t) + \theta_{22}'(t) F'\theta_{12}(t) - \theta_{22}'(t)\, H'R^{-1}H\theta_{22}(t). \end{aligned} \tag{6}$$

Now (6) can be used to obtain

$$GQG' + F\Gamma + \Gamma F' - \Gamma H'R^{-1}H\Gamma$$
$$= D_\Gamma'(t)\,\{GQG' + F\Pi(t, \Gamma, 0) + \Pi(t, \Gamma, 0)F' - \Pi(t, \Gamma, 0)\, H'R^{-1}H\Pi(t, \Gamma, 0)\}\, D_\Gamma(t)$$

which of course implies (5).

Corollary 2. *$\Pi(t, 0, o)$ is monotone-nondecreasing in t.* [I.e., $t > s > 0$ $\Pi(t, 0, o) \geqslant \Pi(s, 0, o)$].

Proof. For every $\mathbf{x}$,

$$\| \mathbf{x} \|_{\Pi(t,0,o)}^2 = \| \mathbf{x} \|_{\Pi(s,0,o)}^2 + \int_s^t \| D_\Gamma^{-1}(v)\, \mathbf{x} \|_{G'QG}^2\, dv$$

by Theorem 2, so that

$$\| \mathbf{x} \|_{\Pi(t,0,o)}^2 \geqslant \| \mathbf{x} \|_{\Pi(s,0,o)}^2$$

and the result follows.

Corollary 3. *If* $\Gamma_0 \in S = \{A \mid FA + AF' - AH'R^{-1}HA + GQG \leqslant 0\}$, *then* $\Pi(t, \Gamma_0, t_0)$ *is monotone-nonincreasing.*

Corollary 4. *The matrix*

$$D_\Gamma'(t)\, GQG'D_\Gamma(t) + D_\Gamma'(t) FL_\Gamma(t) + L_\Gamma'(t) F'D_\Gamma(t) - L_\Gamma(t)\, H'R^{-1}HL_\Gamma(t)$$

is constant.

4. Uniform *á priori* Bounds

In this section, we return the general non-autonomous Riccati equation and construct majorants and minorants in the ordering of C for the solutions of the Riccati equation under assumptions of uniform controllability and observability. Not only are these bounds interesting in that they give a geometric picture of the Riccati equation motion, but also they will prove useful in the stability considerations of the next section. The following definitions is necessary:

Definition 2. $(F, H)\{(F, G)\}$ is uniformly $R\{Q\}$ observable {controllable} iff there exist positive real numbers α, β and $\sigma\{\tau\}$ such that

$$\beta I > \int_{t_0-\sigma}^{t_0} \Phi'(s, t_0^{-\sigma})\, H'(s)\, R^{-1}(s)\, H(s)\, \Phi(s, t_0^{-\sigma})\, ds > \alpha I,$$

$$\left\{\beta I > \int_{t_0}^{t_0+\tau} \Phi(t_0 + \tau, s)\, G(s)\, Q(s)\, G'(s)\, \Phi'(t_0 + \tau, s)\, ds > \alpha I\right\},$$

for all t_0

($\sigma\{\tau\}$ is often referred to as the interval of observability controllability}).

Lemma 4. *Suppose (F, H) is uniformly R observable, then for every $\Delta < (t - t_0)$, where Δ is the interval of observability, and $\Gamma \geqslant 0$,*

$$\Pi(t, \Gamma, t_0) \leqslant W_R^{-1}(t, t - \Delta) + C_Q(t, t - \Delta),$$

where

$$W_R(t, t - \Delta) = \int_{t-\Delta}^{t} \Phi'(s, t)\, H'R^{-1}H\Phi(s, t)\, ds.$$

Proof. Let K be the solution of

$$\dot{K} = FK + KF' - KH'R^{-1}HK,$$
$$K(t_0) = \Gamma_0 \geqslant 0,$$

and $\Delta(t) = \Pi(t, \Gamma_0, t_0) - K(t)$; then

$$\dot{\Delta} = \bar{F}(t)\Delta + \Delta\bar{F}'(t) - \Delta H'R^{-1}H\Delta + GQG',$$

where $\bar{F}(t) = F - K(t)\, H'R^{-1}H$. Now $\Pi \geqslant K$ as

$$\Delta(t) = \int_{t_0}^{t} \psi(t, \sigma)\, [GQG' + \Delta H'R^{-1}H\Delta]\, \psi'(t, \sigma)\, d\sigma \geqslant 0, \qquad (6')$$

with $\psi(t, \sigma)$ the fundamental matrix of $\bar{F} - \Delta(t) H'R^{-1}H$. However, (1) implies

$$\Pi(t, \Gamma_0, t-\Delta) = \Phi(t, t-\Delta)\,\Gamma_0\Phi'(t, t-\Delta) + C_Q(t, t-\Delta)$$
$$+ \int_{t-\Delta}^{t} \Phi(t,\sigma)\,[-\Pi(\sigma, \Gamma_0, t-\Delta)\,H'R^{-1}H\Pi(\sigma, \Gamma_0, t-\Delta)]\,\Phi'(t,\sigma)$$
$$\leqslant \Phi(t, t-\Delta)\,\Gamma_0\Phi'(t, t-\Delta) + C_Q(t, t-\Delta)$$
$$+ \int_{t-\Delta}^{t} \Phi(t,\sigma)\,[-K(\sigma)\,H'R^{-1}HK(\sigma)]\,\Phi'(t,\sigma)\,d\sigma \tag{7}$$

since

$$0 \leqslant K(\sigma) \leqslant \Pi(\sigma, \Gamma_0, t-\Delta),$$
$$-K(\sigma)\,H'R^{-1}HK(\sigma) \geqslant -\Pi(\sigma, \Gamma_0, t-\Delta)\,H'R^{-1}H\Pi(\sigma, \Gamma_0, t-\Delta).$$

Now (7) implies that

$$\Pi(t, \Gamma_0, t-\Delta) \leqslant C_Q(t, t-\Delta) + K(t, \Gamma_0, t-\Delta) \tag{8}$$

since

$$K(t, \Gamma_0, t-\Delta) = \Phi(t, t-\Delta)\,\Gamma_0\Phi'(t, t-\Delta)$$
$$- \int_{t-\Delta}^{t} \Phi(t,\sigma)\,K(\sigma, \Gamma_0, t-\Delta)\,H'R^{-1}HK(\sigma, \Gamma_0, t-\Delta)\,\Phi'(t,\sigma)\,d\sigma$$

where $\Gamma_0 \geqslant 0$ it follows that $K(s, \Gamma_0, t-\Delta) = A(I + A'W_RA)^{-1}A'$ where $A = \Phi(t, t_0)\,\Gamma^{1/2}$.

We wish to show

$$A(I + A'W_RA)^{-1}A' \leqslant W_R^{-1}.$$

Let $S^{-1} = I + A'W_{R^{-1}}A$; then by Shur's relations it follows that

$$S = I - A'(AA' + W_R^{-1})^{-1}A,$$
$$ASA' = AA' - AA'(AA' + W_R^{-1})^{-1}AA',$$

or

$$ASA' = AA'(AA' + W_R^{-1})^{-1}W_R^{-1}.$$

Therefore, it follows that

$$ASA' - W_R^{-1} = (AA'(AA + W_R^{-1})^{-1} - I)\,W_R^{-1} = -W_R^{-1}(AA' + W_R^{-1})^{-1}W_R^{-1} \leqslant 0.$$

Now (8) becomes, with $\Gamma' = \Pi(t - \Delta, \Gamma, t_0)$,

$$\Pi(t, \Gamma, t_0) \leqslant C_Q(t, t - \Delta) + W_R^{-1}(t, t - \Delta).$$

W^{-1} exists by uniform observability.

If (F, H) is completely observable, we shall show that there exists a $\bar{P} \in C$ with the fixed point property that is $\Pi(t, \bar{P}, t_0) = \bar{P}$ as

THEOREM 3. *Suppose in the autonomous case* (F, H) *is completely observable, then*

$$\lim_{t_0 \to -\infty} \Pi(t, 0, t_0)$$

exists and equals $\bar{P}$ *a constant in* C. *Further,* $\bar{P}$ *is a solution of* (1), *and is a positive-semi-definite solution of*

$$F\bar{P} + \bar{P}F' - \bar{P}H'R^{-1}H\bar{P} + GQG' = 0.$$

Proof. Corollary 3 shows $\Pi(t, 0, t_0)$ is monotone nondecreasing in t or in $-t_0$ in the ordering of C as in the autonomous case complete observability implies uniform observability. Lemma 4 shows under the assumption of complete observability $\Pi(t, 0, t_0)$ is uniformly bounded in t_0 as $t_0 \downarrow -\infty$ and consequently there exists $\bar{P} \in C$ so that $\Pi(t, 0, t_0) \to \bar{P}$ as $t_0 \downarrow -\infty$.

Since for $\Delta > 0$

$$\Pi(t, 0, t_0 - \Delta) = \Pi(t, \Pi(t_0, 0, t_0 - \Delta), t_0) \tag{9}$$

and by continuity of solutions with respect to initial conditions (9) implies as, $\Delta \to +\infty$,

$$\bar{P} = \Pi(t, \bar{P}, t_0);$$

hence $\bar{P}$ is a fixed-point solution of (1).

From Theorem 1, if (F, G) is uniformly Q controllable, then $\Pi(t, \Gamma, t_0)$ is invertible for $t > t_0 + \Delta$ (with Δ the interval of controllability), for $\Gamma \geqslant 0$, and $\Pi^{-1}(t, \Gamma, t_0)$ satisfies the Riccati equation

$$\frac{d\Pi^{-1}}{dt} = -F'\Pi^{-1} - \Pi^{-1}F - \Pi^{-1}GQG'\Pi^{-1} + H'R^{-1}H,$$

$$\Pi^{-1}(t_0 + \Delta) = \Pi^{-1}(t_0 + \Delta, \Gamma, t_0) > 0,$$

and Lemma 4 immediately implies $\Pi^{-1}(t, \Gamma, t_0) \leqslant C_Q^{-1}(t, t - \Delta) + W_R(t, t - \Delta)$ or; since $A \leqslant B \Rightarrow A^{-1} \geqslant B^{-1}$ for $A, B \in C$ (see Beckenbach and Bellman [7]), it follows that:

LEMMA 5. *If (F, G) is uniformly Q controllable, then*

$$(C_Q^{-1}(t, t - \Delta) + W_R(t, t - \Delta))^{-1} \leqslant \Pi(t, \Gamma, t_0)$$

for all $\Gamma \geqslant 0$ and $\Delta < t - t_0$, with Δ the interval of controllability.

5. EXPODENTIAL STABILITY

Our major result of this paper is Theorem 4, which will be proven under somewhat stronger hypothesis than is necessary in order to obtain a quick and tutorial proof. Basically the theorem can be extended to situations which are uniformly controllable and observable where the system matrices have uniform behavior. This theorem illustrates the fundamental importance of the concepts of controllability and observability for regular behavior of the Riccati equation.

THEOREM 4. *Suppose $(F(t), G(t))$ and $(F(t), H(t))$ are uniformly Q controllable and uniformly R observable, respectively, and that*

(1) $G(t) Q(t) G'(t) > \alpha_0 I$,

(2) $\| \Phi(t_1, t_2) \| \leqslant \gamma(t_1 - t_2)$,

(3) $| H'(t) R^{-1}(t) H(t) | \leqslant \delta$,

with α_0 and δ positive real numbers and γ a continuous function. Then the unforced equation

$$\frac{d\hat{\mathbf{x}}}{dt} = [F(t) - P(t) H'(t) R^{-1} H(t)] \hat{\mathbf{x}} \tag{10}$$

is uniformly asymptotically stable and the solution of the Riccati equation $\Pi(t, \Gamma, \sigma)$ satisfies, for $t > t_0$,

$$\| \Pi(t, \Gamma_1, t_0) - \Pi(t, \Gamma_2, t_0) \| \leqslant A e^{-\beta(t-t_0)},$$

with $A > 0$ and $\beta > 0$ depending only on Γ_1 and Γ_2.

Proof. In view of our assumptions, the map $x_{t_0} : R^n \to R^n$ defined by

$$x_{t_0}(\mathbf{c}) = \phi(t_0 + \sigma, \mathbf{c}, t_0)$$

is a continuous function of $\mathbf{c}$, uniformly in t_0, where ϕ is the solution of (10), and σ is larger than the maximum of the length interval of observability and controllability.

In view of this uniform continuity, it suffices to demonstrate the uniform asymptotic stability of the system

$$\dot{\mathbf{x}} = \bar{F}(t)\,\mathbf{x},$$

where

$$\bar{F}(t) = F(t) - \Pi(t, \Gamma, t_0 - \sigma)\, H'(t)\, R^{-1}(t)\, H(t)$$

on intervals of form $(t_0, +\infty)$, we prove the uniform asymptotic stability by exhibiting a Liapunov function as

$$V(\mathbf{x}, t) = \|\mathbf{x}\|^2_{\Pi^{-1}(t, \Gamma, t_0 - \sigma)}$$

Now, by the choice of σ it follows that

$$\alpha_0 \|\mathbf{x}\|^2 \leqslant V(\mathbf{x}, t) \leqslant \alpha_1 \|\mathbf{x}\|^2$$

for positive constants α_0 and α_1, in view of Lemmas 4 and 5. Further, an easy calculations reveals

$$\left.\frac{dV}{dt}\right|_{\text{motion}(10)} = -\|\mathbf{x}\|^2_{H'RH + \Pi^{-1}GQG\Pi^{-1}} \leqslant -\beta \|\mathbf{x}\|^2 < 0,$$

for β positive using the *à priori* bounds for Π^{-1} and Assumption 1. Consequently, the Liapunov stability theorem implies that (10) is uniformly asymptotically stable and, in consequence, exponentially stable by the Perron-Liapunov-Malkin theorem (Hahn [*8*]). Hence, there exist real positive numbers γ and A, so that

$$\|\psi_\Gamma(t, \sigma)\| \leqslant A e^{-\gamma(t_2 - \sigma)},$$

where ψ_Γ is the fundamental matrix of $F(t) - \Pi(t, \Gamma, t_0)\, H(t)\, R^{-1}(t)\, H(t)$. Now let

$$\Delta = \Pi(t, \Gamma_1, t_0) - \Pi(t, \Gamma_2, t_0);$$

then

$$\dot{\Delta} = (F - \Pi(t, \Gamma_1, t_0)\, H'R^{-1}H)\,\Delta + \Delta(F' - H - R^{-1}H\Pi(t, \Gamma_2, t_0)),$$

$$\Delta(t_0) = \Gamma_1 - \Gamma_2 .$$

Hence

$$\|\Delta\| = \|\psi_{\Gamma_1}(t, t_0)\,(\Gamma_1 - \Gamma_2)\,\psi'_{\Gamma_2}(t, t_0)\| \leqslant \|\Gamma_1 - \Gamma_2\|\, e^{-2\gamma(t - t_0)},$$

and the desired result follows.

In the autonomous case, the following corollary characterizes the asymtotic behavior as:

COROLLARY 4. *Suppose F, G, H, Q and R, are autonomous but the system is completely controllable and completely observable. Then, for every* $\Gamma \in C$,

$$\Pi(t, \Gamma, t_0) \xrightarrow{t_0 \to -\infty} \bar{P} = \lim_{t_0 \to -\infty} \Pi(t, 0, t_0).$$

Corollary 4 clearly also holds in the non-autonomous case, under uniform conditions, once one shows that $\lim_{t_0 \to -\infty} \Pi(t, 0, t_0)$ exists which is quite easy using a variational argument, but seemingly difficult by a nonvariational argument.

We mention that the monotone properties of Section 3 seem difficult to generalize to the nonautonomous case.

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