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# On the Similarity between a Linear Transformation and its Adjoint

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## INTRODUCTION

Let $\mathbf{A}$ be a square matrix over a field $K$ and $\mathbf{A}'$ its transpose. It is well known that $\mathbf{A}$ and $\mathbf{A}'$ are similar, i.e., there exists a square nonsingular matrix $\mathbf{S}$ over $K$ such that $\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. Moreover, one can choose $\mathbf{S}$ to be symmetric (see [3], p. 67; [4], p. 76; [5]; [6]; [7]).

Let $\alpha \to \bar{\alpha}$ be an automorphism of $K$ such that $\bar{\bar{\alpha}} = \alpha$. We shall write $\mathbf{A}^* = \bar{\mathbf{A}}'$. $\mathbf{A}$ is said to be hermitian if $\mathbf{A}^* = \mathbf{A}$. If $\mathbf{A}$ and $\mathbf{A}^*$ are similar we shall prove that they are similar via a hermitian matrix. With some additional restrictions this was proved in [1].

## SIMILARITY OF $A$ AND $A^*$

Let $V_1$ and $V_2$ be $n$-dimensional vector spaces over $K$ and $\phi : V_1 \times V_2 \to K$ a nondegenerate sesquilinear form.

For $A \in \mathrm{End}(V_1)$ we define $A^* \in \mathrm{End}(V_2)$ by

$$\phi(xA, y) = \phi(x, yA^*).$$

THEOREM 1. *If $A$ and $A^*$ are similar then there exists a vector space isomorphism $H : V_1 \to V_2$ such that*

$$A^* = H^{-1}AH \tag{1}$$

*and*

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$$\phi(x, yH) = \overline{\phi(y, xH)} \tag{2}$$

*holds for all* $x, y \in V_1$.

COROLLARY 1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix over* $K$ *which is similar to* $\mathbf{A}^*$. *Then there exist a hermitian nonsingular matrix* $\mathbf{H}$ *such that* $\mathbf{A}^* = \mathbf{H}^{-1}\mathbf{A}\mathbf{H}$.

COROLLARY 2. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix over* $K$. *Then there exists a symmetric nonsingular matrix* $\mathbf{S}$ *such that* $\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

*Proof of Corollary* 1. We need only to fix a basis of $V_1$ and the dual basis of $V_2$ and pass to the matrices. Eq. (2) implies that the matrix of $H$ is hermitian. ■

*Proof of Corollary* 2. Take $\bar{\alpha} = \alpha$ in Corollary 1. ■

*Proof of Theorem* 1. By decomposing $V_1$ into direct sum of indecomposable cyclic subspaces we can assume without loss of generality that $V_1$ is of type 1 or type 2 described below.

*Type* 1. $V_1$ is cyclic (with respect to $A$) and its order is $f^m$ where $f$ is a monic irreducible polynomial such that $f = \bar{f}$.

*Type* 2. $V_1 = W_1 \oplus W_2$, where $W_1$ and $W_2$ are cyclic of orders $f^m$ and $\bar{f}^m$, respectively, and $f$ is a monic irreducible polynomial such that $f \neq \bar{f}$.

We shall consider these two types separately:

LEMMA 1. *Theorem* 1 *is true if* $V_1$ *is of type* 1.

*Proof.* Let $S: V_1 \to V_2$ be an isomorphism such that $SA^* = AS$. There is a unique isomorphism $S^*: V_1 \to V_2$ such that

$$\phi(x, yS) = \overline{\phi(y, xS^*)}, \qquad x, y \in V_1.$$

Since

$$\phi(x, yS^*A^*) = \phi(xA, yS^*) = \overline{\phi(y, xAS)} = \overline{\phi(y, xSA^*)}$$
$$= \overline{\phi(yA, xS)} = \phi(x, yAS^*)$$

we have $S^*A^* = AS^*$.

If $\lambda \in K$ then also

$$(\lambda S + \bar{\lambda} S^*)A^* = A(\lambda S + \bar{\lambda} S^*),$$

i.e., $HA^* = AH$ where $H = \lambda S + \bar{\lambda} S^*$. It is easy to check that this $H$ satisfies Eq. (2).

If we can choose $\lambda$ so that $H$ is nonsingular then the proof is finished. Assume that $H$ is singular for every $\lambda \in K$. Since $A$ is cyclic and $S^*S^{-1}$ and $A$ commute there exists $g \in K[X]$ such that $S^*S^{-1} = g(A)$ (see [2], p. 107). Since $H$ is singular for $\lambda = 1$ we deduce that $-1$ is an eigenvalue of $S^*S^{-1}$. Hence, there exists an eigenvalue $\alpha$ of $A$ such that $g(\alpha) = -1$. This implies that $g + 1 = fh$ for some $h \in K[X]$. It follows that all the eigenvalues of $S^*S^{-1}$ are $-1$. Since $H$ is singular for all $\lambda \in K$ we must have $\bar{\lambda} = \lambda$ for all $\lambda \in K$. Hence, $\phi$ is a symmetric form. In this case we prove that $S = S^*$ and we can take $H = S$.

If $x, y \in V_1$ then $y = xp(A)$, say, for some $p \in K[X]$. We have

$$\phi(xS^*, y) = \phi[xS^*, xp(A)] = \phi[x, xp(A)S] = \phi[x, xSp(A^*)]$$

$$= \phi[xp(A), xS] = \phi(y, xS) = \phi(xS, y)$$

for all $x, y \in V_1$. Since $\phi$ is nondegenerate we must have $S^* = S$ which completes the proof. ■

LEMMA 2. *If $V_1 = W_1 \oplus W_2$ is of type 2 then Theorem 1 is true.*

*Proof.* We have also $V_2 = W_2^{\perp} \oplus W_1^{\perp}$. The restrictions of $\phi$ to $W_1 \times W_2^{\perp}$ and $W_2 \times W_1^{\perp}$ are nondegenerate. Since $W_1$ and $W_1^{\perp}$ have the same order, namely $f^m$, there exists an isomorphism $S : W_1 \to W_1^{\perp}$ such that

$$xSA^* = xAS, \qquad x \in W_1.$$

There exists a unique vector space isomorphism $S^* : W_2 \to W_2^{\perp}$ such that

$$\phi(y, xS) = \overline{\phi(x, yS^*)}$$

for all $x \in W_1$ and $y \in W_2$. Since

$$\phi(x, yS^*A^*) = \phi(xA, yS^*) = \overline{\phi(y, xAS)} = \overline{\phi(y, xSA^*)}$$

$$= \overline{\phi(yA, xS)} = \phi(x, yAS^*)$$

we get $S^*A^* = AS^*$.

There exists a unique isomorphism $H : V_1 \to V_2$ which extends both $S$ and $S^*$. It satisfies also the condition $HA^* = AH$.

Eq. (2) follows from

$$\phi(y, xH) = \phi(y, xS) = \overline{\phi(x, yS^*)} = \overline{\phi(x, yH)},$$

$$\phi(x, yH) = \phi(x, yS^*) = \overline{\phi(y, xS)} = \overline{\phi(y, xH)},$$

where $x \in W_1$ and $y \in W_2$ are arbitrary.

This completes the proof of Lemma 2 ■ and also of Theorem 1. ■

THEOREM 2. *Let $\phi$ be a nondegenerate hermitian form on an n-dimensional vector space $V$ over $K$. If $A \in \mathrm{End}(V)$ is similar to its adjoint $A^*$ then there exists an hermitian nonsingular transformation $H$ such that $A^* = H^{-1}AH$.*

*Proof.* Take $V_1 = V_2 = V$ in Theorem 1. ■

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