Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics 308 (2008) 715–725

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On bicyclic reflexive graphs

Zoran Radosavljević, Bojana Mihailović, Marija Rašajski

School of Electrical Engineering, Bulevar Kralja Aleksandra 73, Beograd, Serbia

Received 27 September 2006; accepted 11 July 2007
Available online 21 Augsust 2007

Abstract

A simple graph is said to be reflexive if the second largest eigenvalue of a (0, 1)-adjacency matrix does not exceed 2. We use graph modifications involving Smith trees to construct four classes of maximal bicyclic reflexive graphs.

MSC: 05C50

Keywords: Graph theory; Second largest eigenvalue; Reflexive graph; Bicyclic graph; Cactus

1. Introduction

For a simple graph G (a non-oriented graph without loops or multiple edges), having a matrix A as its (0, 1)-adjacency matrix, we define $P_G(\lambda) = \det(\lambda I - A)$ to be its *characteristic polynomial* and denote it simply by $P(\lambda)$ if it is clear which graph it is related to. The roots of $P_G(\lambda)$ are the *eigenvalues* of G, making up the *spectrum* of G and, since they all are real numbers, we can assume they are in non-increasing order: $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. The largest eigenvalue $\lambda_1(G)$ is also called the *index* of G. In a connected graph $\lambda_1 > \lambda_2$ holds, while in the case of a disconnected graph we can have $\lambda_1 = \lambda_2$ if these are equal indices of two components. The relation between the spectrum of a graph and the spectra of its induced subgraphs is established by the *interlacing theorem*:

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i$ $(i = 1, \ldots, m)$ hold.

Thus, e.g. if $m = n - 1$, $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \ldots$. Also $\lambda_1 > \mu_1$ if G is connected.

Graphs having $\lambda_2 \leqslant 2$ are called *reflexive graphs* (and also *hyperbolic graphs* if $\lambda_2 \leqslant 2 \leqslant \lambda_1$). They correspond to certain sets of vectors in the Lorentz space $R^{p,1}$ and have some applications to the construction and classification of reflexion groups [6]. Thus far, reflexive trees have been studied in [3,5] and bicyclic reflexive graphs with a bridge between the cycles in [11] (see also [7]). Recently, various classes of multicyclic reflexive cacti have been investigated in [4,8–10]. In this paper we continue with the investigations initiated by the article [11] and extended in the meantime by consideration of some other classes of reflexive graphs.

A *cactus*, or a *treelike graph*, is a graph in which any two cycles have at most one common vertex, i.e. are edge disjoint.

E-mail address: zorangraf@etf.bg.ac.yu (Z. Radosavljević).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.07.057

Since the property $\lambda_2 \leqslant 2$ is a *hereditary* one (any induced subgraph of a reflexive graph is reflexive itself), it is natural to present such graphs always through sets of maximal graphs. Also, since the spectrum can be extended using an arbitrary number of components, when looking for reflexive graphs we always assume them to be connected.

In order to provide the starting base and the tools for getting the results of this paper, in Section 2 we give some important general and auxiliary facts and some parts of recent results concerning maximal reflexive cacti. The remaining sections are devoted to the aim of this article, i.e. the construction of various classes of bicyclic reflexive graphs. At some stages the work has been supported by using the expert system GRAPH [2].

The terminology of the theory of graph spectra in this paper follows [1].

2. Some former, general and auxiliary results

Graphs whose index equals 2 are known as *Smith graphs*.

Lemma 1 (*Smith [13], see also Cvetković [1, p. 79]*). $\lambda_1(G) \leqslant 2$ (*resp.* $\lambda_1(G) < 2$) *if and only if each component of graph G is a subgraph* (*resp. proper subgraph*) *of one of the graphs of Fig.* 1, *all of which have index equal to* 2.

Let us emphasize the simple, but important fact, that any connected graph is either an induced subgraph or an induced supergraph of some Smith graphs.

If we form a tree T by identifying vertices u_1 and u_2 ($u_1 = u_2 = u$) of two (rooted) trees T_1 and T_2, respectively (the *coalescence* $T_1 \cdot T_2$ of T_1 and T_2), we may say that T can be *split* at its vertex u into T_1 and T_2 (Fig. 2(a)). Of course, splitting at a given vertex is not determined uniquely if its degree is greater than 2. If we split a tree T at all its vertices u in all possible ways, and in each case attach the parts at vertices of splitting u_1 and u_2 to some vertices v_1 and v_2 of a graph G (i.e. lean the parts on G by identifying u_1 with v_1 and u_2 with v_2, and vice versa), we shall say that in the obtained family of graphs the tree T *pours* between v_1 and v_2 (Fig. 2(b)). Of course, this procedure includes attachment of the intact tree T, at each vertex, to v_1 and v_2.

Pouring of Smith trees plays an important role in describing of maximal reflexive cacti [8,10].

The following formulae give useful interrelations between the characteristic polynomial of a graph and its subgraphs.

Lemma 2 (*Schwenk [12]*). *If* G_1 *and* G_2 *are two rooted graphs with roots* u_1 *and* u_2, *then the characteristic polynomial of their coalescence* $G_1 \cdot G_2$ *is*

$$P_{G_1 \cdot G_2}(\lambda) = P_{G_1}(\lambda) \cdot P_{G_2-u_2}(\lambda) + P_{G_1-u_1}(\lambda) \cdot P_{G_2}(\lambda) - \lambda P_{G_1-u_1}(\lambda) \cdot P_{G_2-u_2}(\lambda).$$

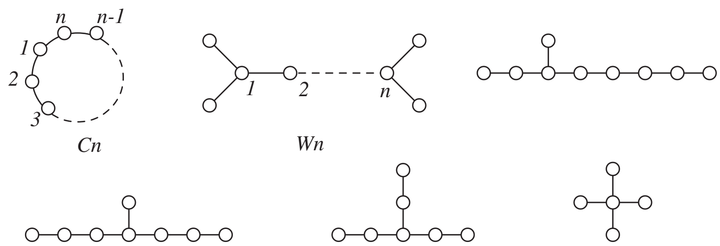

Fig. 1.

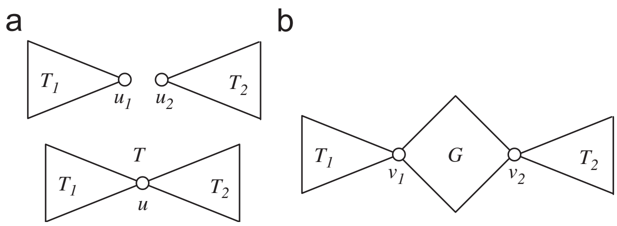

Fig. 2.

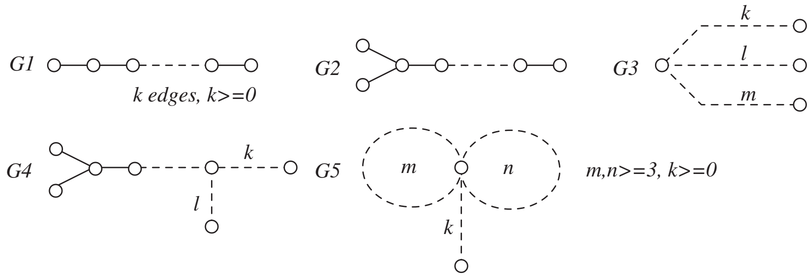

Fig. 3.

Lemma 3 (*Schwenk [12]*). *Given a graph G, let $C(v)$ $(C(uv))$ denote the set of all cycles containing a vertex v and an edge uv of it G, respectively. Then*

(i) $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u\in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2\sum_{C\in C(v)} P_{G-V(C)}(\lambda)$,
(ii) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2\sum_{C\in C(uv)} P_{G-V(C)}(\lambda)$,

where $\mathrm{Adj}(v)$ *denotes the set of neighbours of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.*

These relations have the following obvious consequences (see, e.g. [1, p. 59]).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let $G_1'(G_2')$ be the subgraph of $G_1(G_2)$ obtained by deleting the vertex $v_1(v_2)$ from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendant edge v_1v_2, v_1 being of degree* 1. *Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where $G_1(G_2)$ is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

A list of values of $P_G(2)$ for some small graphs has proved to be very useful in searching for maximal reflexive graphs.

Lemma 4 (*Radosavljević and Rašajski [9,10], Radosavljević and Simić [11]*). *Let $G_1, \ldots, G_5$ be the graphs displayed in Fig. 3. Then*

(i) $P_{G_1}(2) = k + 2$;
(ii) $P_{G_2}(2) = 4$;
(iii) $P_{G_3}(2) = -klm + k + l + m + 2$;
(iv) $P_{G_4}(2) = 4(1 - kl)$;
(v) $P_{G_5}(2) = -(3k + 2)mn$.

First induced supergraphs of Smith graphs have the following property.

Lemma 5 (*Radosavljević and Simić [11]*). *Let G be a graph obtained by extending any of Smith graphs by a vertex of arbitrary positive degree. Then $P_G(2) < 0$ (i.e. $\lambda_2(G) < 2 < \lambda_1(G)$).*

A lot of reflexive graphs can be detected by the next general theorem.

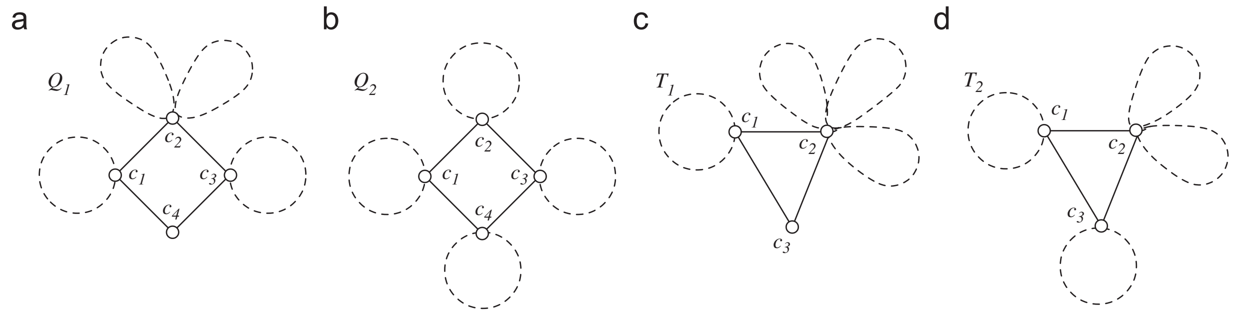

Fig. 4.

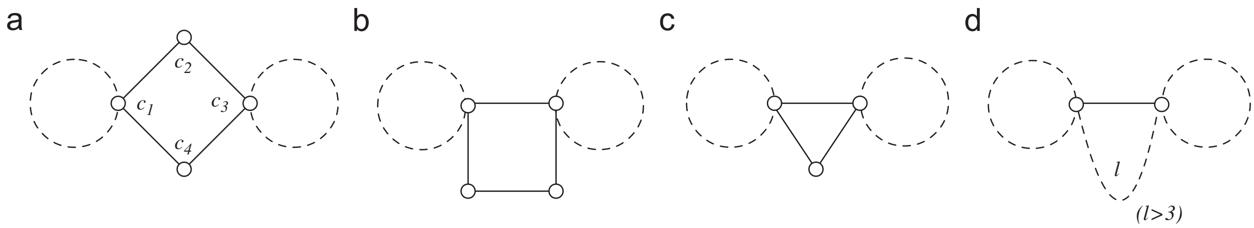

Fig. 5.

Theorem 1 (*Radosavljević and Simić [11]*). *Let G be a graph with cut-vertex u.*

(i) *If at least two components of $G-u$ are induced supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G)>2$.*
(ii) *If at least two components of $G-u$ are Smith graphs, and the rest are induced subgraphs of Smith graphs, then $\lambda_2(G)=2$.*
(iii) *If at most one component of $G-u$ is a Smith graph, and the rest are proper induced subgraphs of Smith graphs, then $\lambda_2(G)<2$.*

This theorem can be applied to a wide class of graphs with a cut-vertex: among others it covers completely all bicyclic graphs whose cycles are joined by a path of length greater than 1. But if it happens that $G-u$ has one proper induced supergraph and the rest of proper induced subgraphs of Smith graphs, Theorem 1 is not applicable and such a situation is interesting for further investigation.

If all cycles of a cactus have a unique common vertex, they are said to form a *bundle*. Since now a reflexive cactus can have an infinite number of cycles, searching for maximal reflexive cacti in this case is much harder than otherwise. That is why in former investigations bundles have been omitted.

The first class of maximal reflexive cacti to be found was that of bicyclic graphs with a bridge between its cycles. Theorem 1 does not apply here except clearly in cases when there is a cycle in which all vertices but one are of degree 2. The result includes an exceptional case of a tricyclic cactus, which appeared naturally by replacing Smith trees with cycles [11].

The next result concerns the maximum number of cycles.

Theorem 2 (*Radosavljević and Rašajski [10]*). *A treelike reflexive graph to which Theorem* 1 *cannot be applied and whose cycles do not form a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, are the four families of graphs in Fig.* 4 (*all cycles attached at the cut-vertices* (*the c-vertices*) *are of arbitrary lengths*).

Starting from these graphs, it has been possible to determine all maximal reflexive cacti with four cycles under the same two conditions—non-applicability of Theorem 1 and no bundle [10, partial results in 9]. These maximal graphs now contain cycles with only one vertex of degree $d>2$ (c-vertex), as well as those with some additional vertices with

$d > 2$. The cycles of the first kind will be called free cycles; otherwise we shall say that the corresponding vertex is loaded (by a pendant edge, a tree, etc.).

Now, using these results, one can also recognize four characteristic classes of tricyclic reflexive cacti to be subjected to further investigation [4], displayed in Fig. 5. According to Theorem 2, the first three allow the addition of more cycles, while the last one does not (i.e. always remains tricyclic). Thus far, there are some partial results for some of these classes [4,8].

3. Replacement of free cycles

Former results on maximal reflexive cacti with four and three cycles already include some cases of replacing free cycles by Smith trees (see [10, Propositions 1–3]), as well as more interesting situations in which such a substitution has been generalized into pouring of Smith trees between two vertices at which free cycles are attached [8,10].

Let us now consider the maximal reflexive cactus of Fig. 6(a), which is the coalescence of a cycle C of length n and a cactus G. Applying Lemma 2, we find that

$$P(2) = P_C(2) \cdot P_{G-v}(2) + P_{C-v}(2) \cdot P_G(2) - 2P_{C-v}(2) \cdot P_{G-v}(2).$$

If $P(2) = 0$, it follows from Lemma 4(i) that

$$n(P_G(2) - 2P_{G-v}(2)) = 0. \tag{1}$$

If C is now replaced by a Smith tree S (Fig. 6(b)), then we obtain

$$P(2) = P_{S-v}(2)(P_G(2) - 2P_{G-v}(2)) = 0. \tag{2}$$

Extensions of a maximal reflexive cactus may have $P(2) > 0$ or may have $P(2) = 0$ (e.g. if $\lambda_2 > 2$, but $\lambda_3 = 2$). Suppose that an extension of the graph G by a pendant edge gives a graph G_1 for which $P_{G_1}(2) - 2P_{G_1-v}(2) > 0$; this means that $P(2) > 0$ in (1), implying $P(2) > 0$ in (2). Also, if we extend S to S_+ by adding one new (non-isolated) vertex and assume $P_G(2), P_{G-v}(2) \neq 0$, then applying Lemma 5 we see that

$$P(2) = P_{S_+}(2) \cdot P_{G-v}(2) > 0,$$

which means that any graph in Fig. 6(b) is maximal, too.

If $P_G(2) = P_{G-v}(2) = 0$ (which simply means that the condition $\lambda_2 = 2$ has been attained before the graph has become maximal, i.e. that $\lambda_2 = 2$ is being preserved through some steps of the extension), then such cases have to be verified individually. Some of them have already been described, enabling an immediate conclusion that those graphs are also maximal [10, Propositions 1–3].

If a graph of Fig. 6(a) is a maximal reflexive cactus such that $P_G(2) - 2P_{G-v}(2) < 0$ (i.e. $\lambda_2 < 2$), then also $P(2) < 0$ in (2). Since now attaching a new pendant edge to G produces $P_{G_1}(2) - 2P_{G-v}(2) > 0$, the same holds when the cycle is replaced by a Smith tree, which means that the graphs of Fig. 6(b) cannot be extended at vertices of G. It is clear, however, that there is no guarantee that an extension of S will give $\lambda_2 > 2$. Therefore, if in a maximal reflexive cactus with $\lambda_2 < 2$ a free cycle is replaced by a Smith tree, the new cactus has $\lambda_2 < 2$ and need not necessarily be maximal.

These conclusions lead to the following theorem.

Theorem 3. *Suppose that a graph of the form shown in Fig.* 6(a) *is a maximal reflexive cactus for which* $P(2) = 0$ *and* $P_G(2) < 0$ *and for any extension* G_1 *formed by attaching to* G *a pendant edge* $P_{G_1}(2) - 2P_{G_1-v}(2) > 0$ *holds. If the free cycle* C *is replaced by an arbitrary Smith tree, then the resulting graph is again a maximal reflexive cactus.*

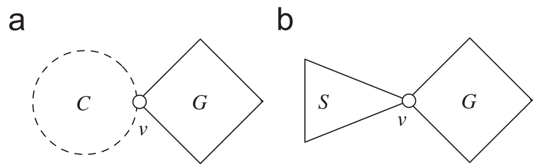

Fig. 6.

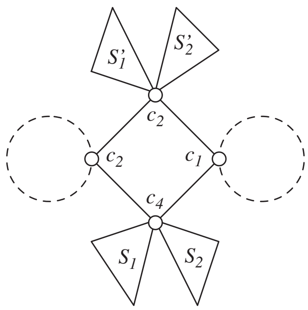

Fig. 7.

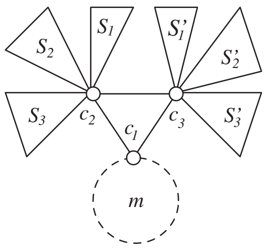

Fig. 8.

This can be the base for the construction of various classes of bicyclic reflexive cacti.

Possibilities given by the graphs Q_1 and Q_2 of Fig. 4 have already been considered in [4,8–10]. These graphs attain $\lambda_2 = 2$ already at the stage of only two cycles joined by the path of length 2. Their tricyclic induced subgraph shown in Fig. 5(a) allows pouring of a pair of Smith trees between c_2 and c_4 [4,8] (Fig. 7), and it can be verified that the replacement of any of the two free cycles (including both) by Smith trees does not change $\lambda_2 = 2$. On the other hand, if we remove, e.g. c_2 and apply Theorem 1 to c_4, we see that none of these Smith trees can be extended, and these graphs are maximal. The same holds for the graphs $H_1 - H_{48}$ defined in [9,10] whose cycles at c_2 are not free.

Unlike Q_1, and Q_2, the graphs T_1 and T_2 admit the application of Theorem 3, but since they also allow pouring of Smith trees [8,10], they will be considered in the next section.

The resulting graphs of [9] are characterized by the fact that they possess at least one cycle with attachments, but since they also have free cycles, by applying Theorem 3 they can give rise to a lot of maximal bicyclic reflexive cacti.

Theorem 3 can also be applied to graphs of Fig. 5(d) (which cannot be extended by cycles and have at least one free cycle).

A further search for maximal tricyclic reflexive cacti will at the same time yield corresponding bicyclic graphs.

4. Pouring of triples of Smith trees

The class of reflexive cacti with four cycles based on the graphs T_1 and T_2 in Fig. 4, and generated by the pouring of Smith trees between vertices c_2 and c_3, has been identified in [10]; all these graphs are maximal, except for one characteristic case (splitting of W_n into two analogous parts) which requires attachment at some vertices of the free cycle. The class of tricyclic reflexive cacti constructed by pouring of pairs of Smith trees between the same vertices has been described in [8]; most of these graphs are maximal, while two characteristic exceptions become maximal only by attachment at some vertices of the free cycle in the same way as before.

We will examine now the generalization of these cases—the pouring of triples of Smith trees.

Consider the graph B of Fig. 8 and let triples of Smith trees pour between c_2 and c_3 (coalescences $S_i \cdot S_i'$, $i = 1, 2, 3$, are Smith trees). Let us introduce the labels $P_{S_i - c_2}(2) = p_i$, $P_{S_i' - c_3}(2) = p_i'$;

$$\sum_{v \in \mathrm{Adj}\, c_2 \cap S_i} P_{S_i - c_2 - v}(2) = \Sigma_i, \qquad \sum_{v \in \mathrm{Adj}\, c_3 \cap S_i'} P_{S_i' - c_3 - v}(2) = \Sigma_i' \quad (i = 1, 2, 3),$$

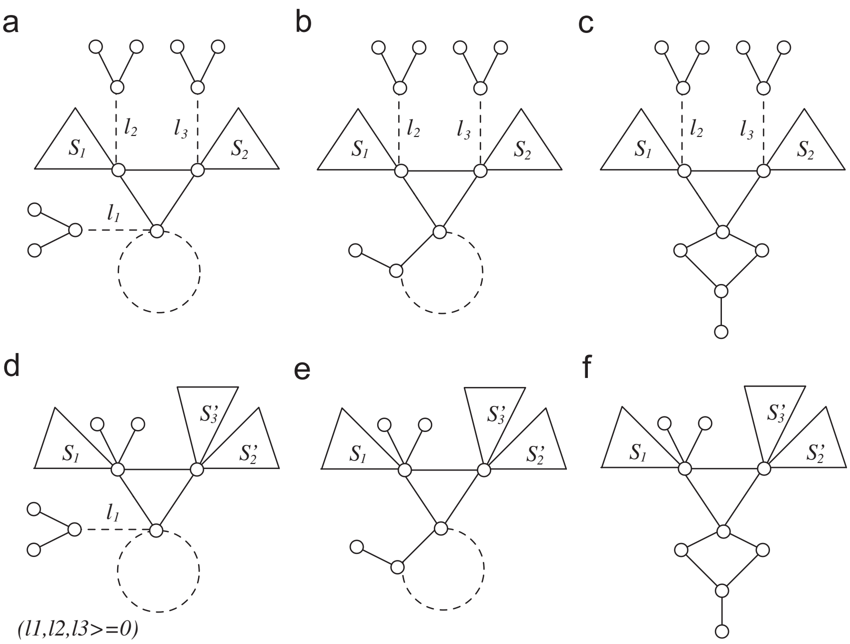

Fig. 9.

where Adj c_i denotes the set of vertices adjacent to c_i $(i=2,3)$. Applying Lemma 3(i) to c_1 we find that

$$\begin{aligned} P_B(2) = &-m[p_1p_2p_3(2p_1'p_2'p_3' - \Sigma_1'p_2'p_3' - p_1'\Sigma_2'p_3' - p_1'p_2'\Sigma_3') \\ &+ p_1'p_2'p_3'(2p_1p_2p_3 - \Sigma_1p_2p_3 - p_1\Sigma_2p_3 - p_1p_2\Sigma_3) + 2p_1p_2p_3p_1'p_2'p_3'] \\ = &-m[p_2p_3p_2'p_3'(2p_1p_1' - \Sigma_1p_1' - \Sigma_1'p_1) + p_1p_3p_1'p_3,(2p_2p_2' - \Sigma_2p_2' - \Sigma_2'p_2) \\ &+ p_1p_2p_1'p_2'(2p_3p_3' - \Sigma_3p_3' - \Sigma_3'p_3)]. \end{aligned} \tag{3}$$

On the other hand, applying the same Lemma to the splitting vertex of a Smith tree, we get

$$2p_ip_i' - \Sigma_ip_i' - \Sigma_i'p_i = 0 \quad (i=1,2,3), \tag{4}$$

implying $P_B(2)=0$ in (3). If some of the three coalescences $S_i \cdot S_i'$ are proper induced subgraphs of a Smith tree, the corresponding expression in (4) is positive, implying $P_B(2)<0$, while in the case of a proper induced supergraph, because of Lemma 5, $P_B(2)>0$. Thus, we have reflexive graphs, which are maximal in the sense that none of them can be extended at any vertex of S_i or S_i' $(i=1,2,3)$.

Now, a question arises whether it is possible to add something at c_1 or some of other vertices of the cycle. If we attach a pendant edge to c_1, by applying Corollary 2 we see that the property $P(2)=0$ will be preserved if and only if $P_E(2)=0$, where E is the component of $B-c_1$ different from the path of length $m-2$.

If at least two complete Smith trees are attached at c_2, for example, Theorem 1(iii) gives $P_E(2)<0$. If two intact Smith trees S_1 and S_2 are attached at the opposite vertices of the bridge c_2c_3, while the third one pours, the situation is analogous to that considered in [8]. Using Corollary 1 we see that $P_E(2)=p_1p_2(\Sigma_3\Sigma_3' - p_3p_3')$ and since for all Smith graphs $\Sigma_3\Sigma_3' - p_3p_3' < 0$ except in the case $p_3 = p_3' = \Sigma_3 = \Sigma_3' = 4$ (splitting of W_n into two analogous parts), we come to the case which can be extended as far as the graph of Fig. 9(a). In an analogous way one can make sure that the same holds for the remaining two exceptions of Fig. 9(b, c).

If an intact Smith tree S_1 is attached at c_2 and the remaining two pour, the corresponding expression becomes

$$P_E(2) = -p_1(2\Sigma_2\Sigma_3p_2'p_3' - \Sigma_2p_3\Sigma_2'p_3' - p_2\Sigma_3p_2'\Sigma_3' + p_2p_3p_2'p_3'). \tag{5}$$

If the two pouring Smith trees are split in such a way that S_2 and S_3 are K_2 (pendant edges), we have $p_i=2$, $\Sigma_i=1$, $p_i' = \frac{2}{3}\Sigma_i'$, $(i=2,3)$ and $P_E(2)=0$ in (5) (the exceptional case of Fig. 9(d)). Otherwise $P_E(2)<0$ (for all other ways of

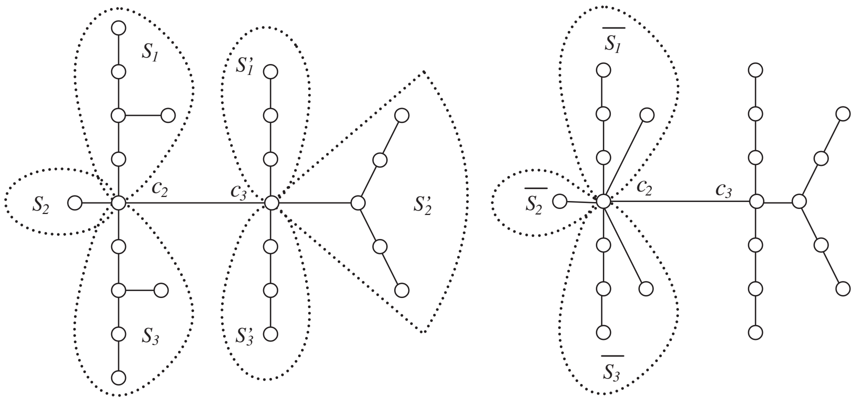

Fig. 10.

splitting $p_i' \geqslant \frac{3}{4}\Sigma_i'$ and the proof is identical to that in [8]). The remaining two exceptions (Fig. 9(e, f)) can be obtained in an analogous way.

Suppose now that none of the six parts of the three Smith trees is empty. The application of Corollary 1 gives

$$P_E(2) = (2p_1p_2p_3 - \Sigma_1p_2p_3 - p_1\Sigma_2p_3 - p_1p_2\Sigma_3)(2p_1'p_2'p_3' - \Sigma_1'p_2'p_3' - p_1'\Sigma_2'p_3' - p_1'p_2'\Sigma_3') - p_1p_2p_3p_1'p_2'p_3', \quad (6)$$

and now we can determine whether $P_E(2) < 0$ or $P_E(2) = 0$.

A simple inspection of the possibilities for splitting Smith trees and corresponding values of p_i, p_i', Σ_i, Σ_i', supported by the application of Lemma 4, shows that all their splittings can be classified into six classes. If W_n is split into two analogous parts, $p_1 = p_1' = \Sigma_1 = \Sigma_1' = 4$. All other splittings always give one simple path, say S_i. It turns out that $\Sigma_i = \alpha p_i$, $\Sigma_i' = (2-\alpha)p_i'$, where $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ depending on whether the mentioned path is of length 1, 2, 3, 4, 5, respectively. Numerical examination of (6) has shown that all the exceptional cases ($P_E(2) = 0$), can be described by referring to those illustrated in Fig. 9. Thus, we can now formulate the conclusion.

Theorem 4. *Let a bicyclic graph G consist of a cycle of arbitrary length and a triangle, let them have the common vertex c_1 and let triples of Smith trees pour between the remaining two vertices c_2 and c_3 (Fig. 8). Then G is maximal reflexive graph, with the following exceptions*:

(1) *A complete Smith tree is attached at c_2 and another at c_3, while the third (pouring) tree is W_n, split as shown in Fig. 9*(a–c), *in which case these three (families of) graphs are maximal reflexive graphs.*
(2) *A complete Smith tree S_1 is attached at c_2, and each of the remaining two Smith trees is split into K_2 and S_i' ($i = 1, 2$), as shown in Fig. 9*(d–f), *in which case these three (families of) graphs are maximal reflexive graphs.*
(3) *For one of the two coalescences of three parts of three pouring Smith trees, say S_1, S_2, S_3, there exist corresponding parts $\bar{S}_1, \bar{S}_2, \bar{S}_3$ such that S_i and $\bar{S}_i$ ($i = 1, 2, 3$) have the same values p_i and Σ_i (i.e. belong to the same one of the formerly described six classes) which, of course, includes the possibility $S_i = \bar{S}_i$ for some i, and such that their analogous coalescence consists of a complete Smith tree and two additional pendant edges at c_1 (as in Fig. 9*(d–f)), *in which case the three exceptional (families of) graphs are formed in the same way as in former cases.*

The graphs of Fig. 10 illustrate the description of case (3) of this theorem.

5. A case of two free cycles

The classes of bicyclic reflexive graphs discussed above do not admit two free cycles. Therefore, let us now consider a pair of free cycles (of arbitrary lengths m and n) with a common vertex c to which we attach trees. Theorem 1 says that we can attach to c infinitely many trees without violating the property $\lambda_2 \leqslant 2$ and in view of the multitude of possible

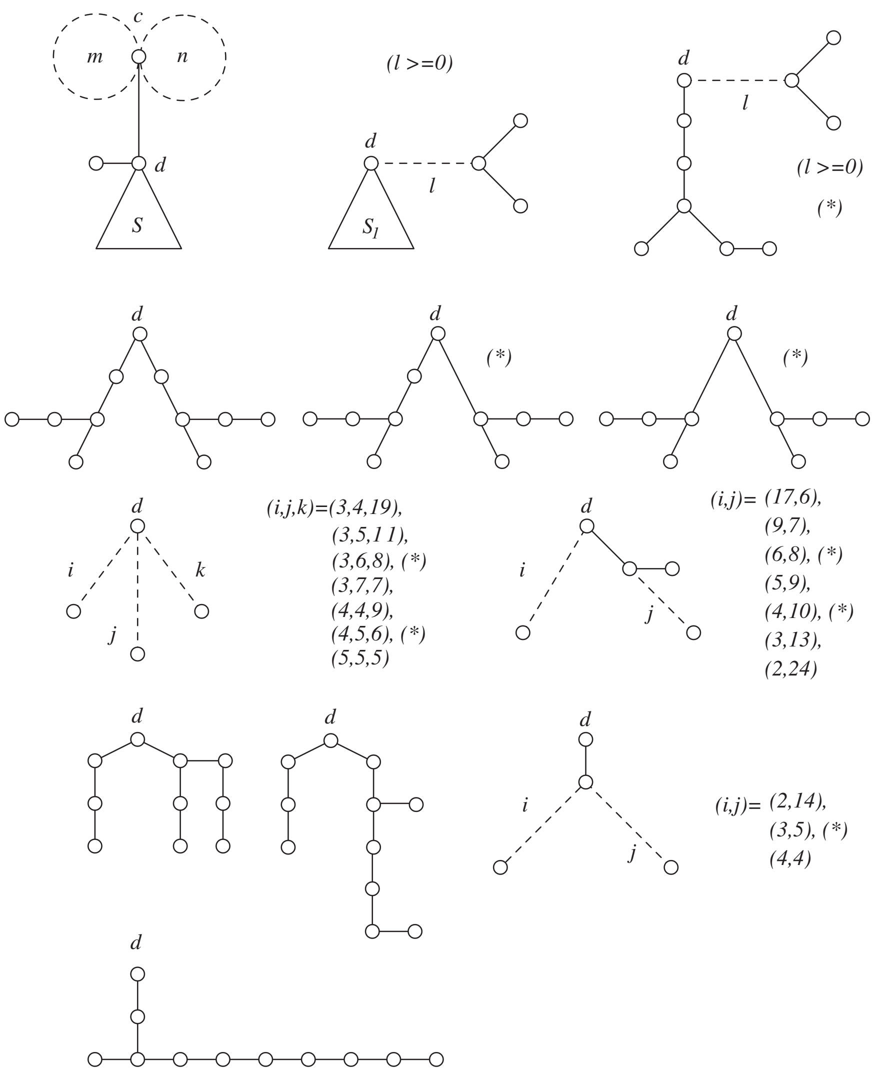

Fig. 11.

cases we limit ourselves to a narrower class. Thus, suppose that c is of degree exactly 5 and that we are seeking only those maximal reflexive graphs which cannot be detected by Theorem 1.

Let us denote by d the vertex adjacent to c not belonging to the cycles. If we attach to d a Smith tree S and an additional pendant edge, the application of Corollaries 1 and 2 gives $\lambda_2 = 2$. Since all trees are comparable with Smith trees, this family covers all cases with a pendant edge at d. Next we examine various shapes of trees attached to d, starting with Corollary 1 (applied to the edge cd) and applying Lemma 4. The resulting set of maximal reflexive graphs is displayed in Fig. 11.

The graphs S_1 are induced subgraphs of Smith trees obtained by removing a pendant edge, which means that the edge dc augments them to Smith trees. The cases for which maximal graphs have $\lambda_2 < 2$ are marked by asterisk.

Theorem 5. *A bicyclic graph having two free cycles with a common vertex of degree* 5, *to which Theorem* 1 *cannot be applied*, *is reflexive if and only if it is an induced subgraph of some graph in Fig*. 11.

6. θ-graphs

If two cycles of a bicyclic graph have a common path, we shall say that they form a θ-graph and the same name will be used for any bicyclic graph with such cycles.

If we apply Lemma 3 to the θ-graph of Fig. 12 and make use of Lemma 4, we get

$$P_\theta(2) = klm - 2(kl + km + lm). \tag{7}$$

This expression gives bounds for a reflexive θ-graph. Assuming $k \leqslant l \leqslant m$ we find that for $k = 1, 2$ the parameters l and m are not limited, while for $k = 3$, $l = 3, 4, 5, 6$ and $k = l = 4$ the parameter m can be arbitrary. Otherwise k, l, m are bounded and $k \leqslant 6$. Any of these particular cases can be investigated by attaching trees to its vertices in order to find the corresponding set of maximal reflexive graphs. Also, some initial extensions of the starting graph can cause unlimited lengths to become bounded. For example, if $k = l = 4$ and if a pendant edge is attached to a vertex on the third path, then we have $m \leqslant 10$.

Let us examine here only one of the boundary cases $k = l = m = 6$, giving $P_\theta(2) = 0$ in (7) (Fig. 13(a)). If we remove the three c-vertices, the remaining pair of Smith trees shows that an extension of the starting graph is possible at none of their vertices. Thus, it remains to test the extension at c-vertices.

Theorem 6. *An extension of a θ-graph with $k = l = m = 6$ is reflexive if and only if it is an induced subgraph of some of the four graphs of Fig.* 13(b).

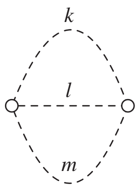

Fig. 12.

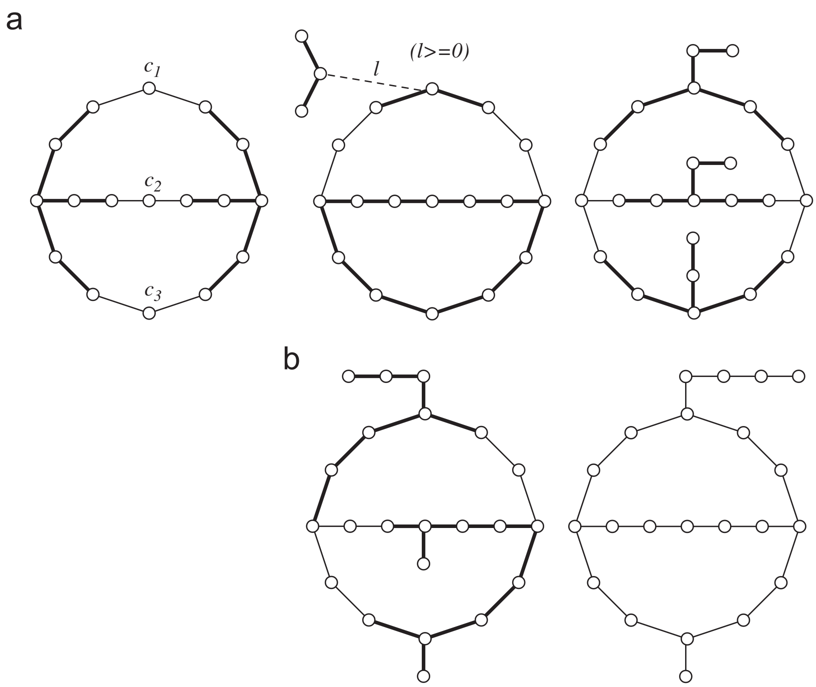

Fig. 13.

In some of these resulting graphs one can recognize again pairs of Smith trees (as indicated in the drawings) to make sure that they are maximal.

Acknowledgements

The work on this article, including the related former results of these authors, has been facilitated by the programming package GRAPH [2].

The authors are thankful to the Serbian Ministry of Science and Environment Protection for the financial support.

References

[1] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs—Theory and Application, Deutscher Verlag der Wissenschaften, Academic Press, Berlin, New York, 1980 (second edition 1982; third edition, Johann Ambrosius Barth Verlag, Heidelberg, Leipzig, 1995).
[2] D. Cvetković, L. Kraus, S. Simić, Discussing graph theory with a computer, Implementation of algorithms. Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 716 (734) (1981) 100–104.
[3] G. Maxwell, Hyperbolic trees, J. Algebra 54 (1978) 46–49.
[4] B. Mihailović, Z. Radosavljević, On a class of tricyclic reflexive cactuses, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. 16 (2005) 55–63.
[5] A. Neumaier, The second largest eigenvalue of a tree, Linear Algebra Appl. 46 (1982) 9–25.
[6] A. Neumaier, J.J. Seidel, Discrete hyperbolic geometry, Combinatorica 3 (1983) 219–237.
[7] M. Petrović, Z. Radosavljević, Spectrally constrained graphs, Faculty of Science, Kragujevac, Serbia, 2001.
[8] Z. Radosavljević, B. Mihailović, M. Rašajski, Decomposition of Smith graphs in maximal reflexive cacti, Discrete Math. (2007), doi: 10.1016/j.disc.2006.11.049.
[9] Z. Radosavljević, M. Rašajski, A class of reflexive cactuses with four cycles, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. 14 (2003) 64–85.
[10] Z. Radosavljević, M. Rašajski, Multicyclic treelike reflexive graphs, Discrete Math. 296 (1) (2005) 43–57.
[11] Z. Radosavljević, S. Simić, Which bicyclic graphs are reflexive?, Univ. Beograd Publ. Elektroteh. Fak. Ser. Mat. 7 (1996) 90–104.
[12] A.J. Schwenk, Computing the characteristic polynomial of a graph, in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Mathematics, vol. 406, Springer, Berlin, Heidelberg, New York, 1974, pp. 153–172.
[13] J.H. Smith, Some properties of the spectrum of a graph, in: R. Guy, H. Hanani, N. Sauer, J. Schonheim (Eds.), Combinatorial Structures and Their Applications, Gordon and Breach, Science, New York, London, Paris, 1970, pp. 403–406.