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Beck's Coloring of a Commutative Ring

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A commutative ring R can be considered as a simple graph whose vertices are the elements of R and two different elements x and y of R are adjacent if and only if $xy=0$. Beck conjectured that $\chi(R)=cl(R)$. We give a counterexample where R is a finite local ring with $cl(R)=5$ but $\chi(R)=6$. We show that if A is a regular Noetherian ring with maximal ideals $N_1, ..., N_s$ such that each A/N_i is finite, then for $R=A/N_1^{n_1}\cdots N_s^{n_s}$, $\chi(R)=cl(R)$. Finally, we give a complete listing of all finite rings R with $\chi(R)\leqslant 4$.

1. INTRODUCTION

In [1], I. Beck introduced the notion of coloring a commutative ring R (with identity). Here R is considered as a simple graph whose vertices are the elements of R, such that two different elements x and y of R are adjacent if and only if $xy=0$. We will use R to denote both the ring R and its associated graph. Let $\chi(R)$ denote the *chromatic number* of the graph, that is, the minimal number of colors needed to color the elements of R so that no two adjacent elements have the same color.

A subset $\{x_1, ..., x_n\}$ of R is called a *clique* if $x_ix_j=0$ for all $i\neq j$, that is, $\{x_1, ..., x_n\}$ (together with its edges) is a complete subgraph of R. (Note that we do not require a clique to be a maximal complete subgraph.) If R has a clique with n elements and no clique of greater length, we say that the *clique number* of R is n and write $cl(R)=n$. Beck showed that for a ring R the following conditions are equivalent: (1) $\chi(R)<\infty$, (2) $cl(R)<\infty$, and (3) the nilradical of R, nil(R), is finite and nil(R) is a finite intersection of prime ideals (equivalently, R has only finitely many minimal primes). Beck called a ring satisfying any of these three equivalent conditions a *coloring*. The graphs of $\mathbb{Z}_8$ and $\mathbb{Z}_2\times\mathbb{Z}_4$ are given below (Figure 1). It is easily checked that both rings R have $\chi(R)=cl(R)=3$.

It is obvious that $\chi(R)\geqslant cl(R)$; indeed this holds for any graph. We call R a *chromatic ring* if R is a coloring with $\chi(R)=cl(R)$. Beck showed that

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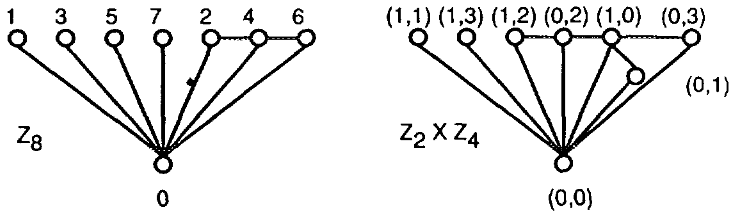

FIG. 1. The graphs of $\mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$.

if R is a finite direct product of reduced colorings and principal ideal rings, then R is a chromatic ring. He also showed that for $n = 2, 3$ or 4, $\chi(R) = n$ if and only if $cl(R) = n$ and that if $\chi(R) = 5$, then $cl(R) = 5$. Based on these positive results, Beck conjectured that $\chi(R) = cl(R)$ for each coloring R.

In Section 2 we give a strong counterexample to Beck's conjecture. We give a finite local ring (R, M) satisfying $M^3 = 0$ with $cl(R) = 5$ but $\chi(R) = 6$. Our example is a minimal counterexample in several regards.

In Section 3 we continue Beck's investigation of colorings. Among the positive results, we show (Corollary 3.3) that if A is a regular Noetherian ring (that is, A is a Noetherian ring with A_M a regular local ring for each maximal ideal M of A), $M_1, ..., M_n$ are maximal ideals of A with each A/M_i finite, and $m_1, ..., m_n$ are positive integers, then $A/M_1^{m_1} \cdots M_n^{m_n}$ is a chromatic ring and we compute its chromatic number. We also show that a Noetherian ring is a coloring if and only if it is a subring of a finite direct product of fields and a finite ring.

Beck gave a list of the finite commutative rings R with $\chi(R) \leq 3$; there were only 11 different types. We extend Beck's list by finding the finite commutative rings R with $\chi(R) = 4$. There are 31 different types of such rings.

Throughout, all rings will be commutative with identity. (R will usually denote a ring and J its nilradical.) Unreferenced results on colorings come from Beck's paper [1].

2. THE COUNTEREXAMPLE

In this section we give a strong counterexample to Beck's conjecture that $\chi(S) = cl(S)$ for each coloring S. Let $R = \mathbb{Z}_4[X, Y, Z]/(X^2 - 2,\ Y^2 - 2,\ Z^2,\ 2X,\ 2Y,\ 2Z,\ XY,\ XZ,\ YZ - 2)$. We will show that $cl(R) = 5$ but $\chi(R) = 6$.

Now (R, M) is a finite local ring with 32 elements, its maximal ideal M has 16 elements, and $R/M \approx \mathbb{Z}_2$. Here $M = \{0, 2, x, x+2,\ y, y+2,\ x+y,\ x+y+2,\ z, z+2,\ x+z,\ x+z+2,\ y+z,\ y+z+2,\ x+y+z,\ x+y+z+2\}$. The other 16 elements of R are the units $U(R)$; we have $U(R) = R - M = 1 + M = \{1 + m \mid m \in M\}$.

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A multiplication table for M is given below. We have omitted 0 and 2 since they both annihilate M. It is easily seen that $0: M = M^2 = \{0, 2\}$ and $M^3 = 0$.

To show that $cl(R) = 5$, it suffices to show that $cl(M) = 5$. By a maximal clique we mean a clique of M that can not be enlarged. (It is important to distinguish between maximal cliques and longest cliques since all maximal cliques, even if contained in the nilradical, need not have the same length. For example, in $\mathbb{Z}_{16}$, $\{\bar{0}, \bar{2}, \bar{8}\}$ and $\{\bar{0}, \bar{4}, \bar{8}, \overline{12}\}$ are both maximal cliques.) The fact that $cl(R) = 5$ follows from the following eight statements which can be verified using the mulitplication table for M.

Fact 1. Every maximal clique contains 0 and 2.

Fact 2. $\{0, 2, x, y, y+z\}$ is a maximal clique, so $cl(R) \geqslant 5$.

Fact 3. Any clique containing x or $x+2$ has at most 5 elements.

A clique may not contain both x and $x+2$. We suppose that it contains x. The case where it contains $x+2$ is similar. Candidates for the clique besides 0, 2 and x include one of the pair y and $y+2$, z, $z+2$, and one of the pair $y+z$ and $y+z+2$. If we include z or $z+2$, we must exclude y, $y+2$, $y+z$ and $y+z+2$. In any case, our clique contains at most 5 elements.

Fact 4. Any clique containing y or $y+2$ has at most 5 elements.

Suppose that the clique contains y. The case where it contains $y+2$ is similar. By Fact 3 we may assume that the clique does not contain x or $x+2$. Candidates for the clique besides 0, 2 and y include one of the pair $y+z$ and $y+z+2$, $x+y+z$, and $x+y+z+2$. If we include $y+z$ or $y+z+2$, we must exclude $x+y+z$ and $z+y+z+2$. So we get at most 5 elements.

Fact 5. Any clique containing $x+y$ or $x+y+2$ has at most 5 elements.

We can assume that our clique contains $0, 2, x+y$, and $x+y+2$. We can also include at most one of the pair $x+z$ and $x+z+2$ and at most one of the pair $y+z$ and $y+z+2$. Since the product of any of these last 4 elements, one from each pair, is nonzero, we can include at most one of the elements $x+z$, $x+z+2$, $y+z$, and $y+z+2$. The result follows.

Fact 6. Any clique containing z or $z+2$ has at most 5 elements.

We may assume that our clique contains $0, 2, z$, and $z+2$, but does not contain x or $x+2$. Since the only other candidate is one of the pair $x+z$ and $x+z+2$, we get at most 5 elements.

TABLE I

Multiplication Table for M

	x	$x+2$	y	$y+2$	$x+y$	$x+y+2$	z	$z+2$	$x+z$	$x+z+2$	$y+z$	$y+z+2$	$x+y+z$	$x+y+z+2$
x	2	2	0	0	2	2	0	0	2	2	0	0	2	2
$x+2$	2	2	0	0	2	2	0	0	2	2	0	0	2	2
y	0	0	2	2	2	2	2	2	2	2	0	0	0	0
$y+2$	0	0	2	2	2	2	2	2	2	2	0	0	0	0
$x+y$	2	2	2	2	0	0	2	2	0	0	0	0	2	2
$x+y+2$	2	2	2	2	0	0	2	2	0	0	0	0	2	2
z	0	0	2	2	2	2	0	0	0	0	2	2	2	2
$z+2$	0	0	2	2	2	2	0	0	0	0	2	2	2	2
$x+z$	2	2	2	2	0	0	0	0	2	2	2	2	0	0
$x+z+2$	2	2	2	2	0	0	0	0	2	2	2	2	0	0
$y+z$	0	0	0	0	0	0	2	2	2	2	2	2	2	2
$y+z+2$	0	0	0	0	0	0	2	2	2	2	2	2	2	2
$x+y+z$	2	2	0	0	2	2	2	2	0	0	2	2	0	0
$x+y+z+2$	2	2	0	0	2	2	2	2	0	0	2	2	0	0

Fact 7. Any clique containing $x+z$, $x+z+2$, $y+z$, or $y+z+2$ has at most 5 elements.

This easily follows from the multiplication table for M and the fact that we can assume that our clique does not contain $x, x+2, y, y+2, x+y, x+y+2, z$, or $z+2$.

Fact 8. Any clique containing $x+y+z$ or $x+y+z+2$ has at most 5 elements.

This easily follows from the fact that we have already shown that any clique containing $x, x+2, y, y+2, ..., y+z$, or $y+z+2$ can have at most 5 elements.

We next show that $\chi(R)=6$. Since $\{0, 2, x, y, y+z\}$ is a clique, we require at least 5 colors, which we label as 1, 2, 3, 4, and 5. We color the elements of this clique in that order. Observe only 0 may be colored with 1 and only 2 may be colored with 2. The following three statements show that the subgraph of R with vertices $\{0, 2, x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$ cannot be colored with 5 colors, so $\chi(R) \geqslant 6$. The coloring of the subgraph $\{x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$ is shown in Figure 2 below. The subgraph $\{0, 2, x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$ is obtained by adding the vertices 0 and 2 and connecting them to each other and to all the other vertices.

Fact 9. Since $xz = x(z+2) = 0$ and $z(z+2) = 0$, we must color one of the pair z and $z+2$ with 4 and one with 5 and it doesn't matter which one is colored with 4 and which one with 5. So we color z with 4 and color $z+2$ with 5.

Fact 10. Now $(x+y)(x+y+2)=0$, so $x+y$ and $x+y+2$ must be colored different colors. Also, $(y+z)(x+y) = (y+z)(x+y+2) = 0$, so we must color $x+y$ with 3 and $x+y+2$ with 4, or vice versa, and it doesn't matter which we choose. So we color $x+y$ with 3 and $x+y+2$ with 4.

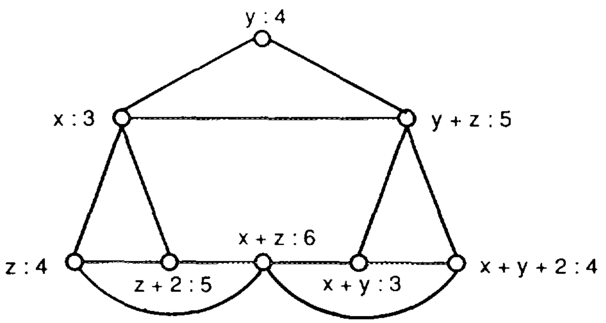

FIG. 2. Coloring of $\{x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$.

Fact 11. Now consider $x+z$. Since $(x+z)(x+y)=(x+z)z=(x+z)(z+2)=0$, $x+z$ can not be colored with 1, 2, 3, 4, or 5.

This shows that $\chi(R)\geqslant 6$. In fact, we do have $\chi(R)=6$. The following partition of R gives a coloring of R using 6 colors: $\{0\}$, $\{2\}\cup U(R)$, $\{x, x+2, x+y, x+y+z\}$, $\{y, y+2, z, x+y+2\}$, $\{y+z, y+z+2, z+2, x+y+z+2\}$, $\{x+z, x+z+2\}$. Thus we have proved the following result.

THEOREM 2.1. *The ring*

$$R=\mathbb{Z}_4[X, Y, Z]/(X^2-2, Y^2-2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ-2)$$

has $cl(R)=5$ *but* $\chi(R)=6$.

As previously mentioned, (R, M) is a local ring with 32 elements, $M^2\neq 0$, but $M^3=0$. Our counterexample is minimal in a number of senses. First, it has the smallest possible clique or chromatic number. For a coloring S with either $cl(S)\leqslant 4$ or $\chi(S)\leqslant 5$ has $cl(S)=\chi(S)$ [1, Theorem 7.3]. Second, it is minimal in the sense that a coloring S with $\text{nil}(S)^2=0$ has $\chi(S)=cl(S)=|\text{nil}(S)|+1$ (Theorem 3.1). Finally, it has the smallest number of elements possible. First, suppose that (S, N) is a finite local ring with $cl(S)<\chi(S)$. Then $N^2\neq 0$ and N can not be principal. So $|S|=|S/N|\,|N/N^2|\,|N^2|\geqslant |S/N|^4$. Thus $|S|\geqslant 32$ unless $S/N\approx\mathbb{Z}_2$, $\dim_{S/N} N/N^2=2$, N^2 is principal, and $N^3=0$. In this case, $|S|=16$. However, it can be shown that such a ring has $cl(S)=4$ and hence $cl(S)=\chi(S)$ or $cl(S)=\chi(S)=5$. Using the fact that a finite commutative ring is a direct product of local rings, it can be shown that a finite commutative ring S with $|S|\leqslant 31$ has $cl(S)=\chi(S)$.

3. SOME POSITIVE RESULTS

In the previous section we showed that we may have $cl(R)<\chi(R)$. In this section we give some positive results.

Let R be a coloring. Then the set of associated primes $\text{Ass}(R)=\{P \text{ a prime} \mid P=0:x \text{ for some } x\in R\}$ is finite. Every minimal prime of R is an associated prime of R and if $P\in\text{Ass}(R)$, then either R_P is a field (and hence P is minimal) or R/P is finite (and hence P is maximal). Let $\varepsilon(R)$ be the number of primes with R_P a field. Then $cl(R)=cl(\text{nil}(R))+\varepsilon(R)$ and $\chi(R)=\chi(\text{nil}(R))+\varepsilon(R)$. Hence $cl(R)=\chi(R)$ if and only if $cl(\text{nil}(R))=\chi(\text{nil}(R))$. These facts about colorings were all given by Beck [1]. Our counterexample had $\text{nil}(R)^3=0$, but $\text{nil}(R)^2\neq 0$. If $\text{nil}(R)^2=0$, R is easily seen to be a chromatic ring.

THEOREM 3.1. *Let R be a coloring with nilradical J.*

(1) *If $J^2=0$, then R is a chromatic ring with $\chi(R)=cl(R)=|J|+\varepsilon(R)$.*

(2) *If $J^2\neq 0$, then $\chi(R)\geqslant cl(R)\geqslant |J\cap(0:J)|+\varepsilon(R)+1$. If $cl(R)=|J\cap(0:J)|+\varepsilon(R)+1$, then $\chi(R)=cl(R)$.*

Proof. (1) We always have $cl(R)=cl(J)+\varepsilon(R)$ and $\chi(R)=\chi(J)+\varepsilon(R)$. If $J^2=0$, then certainly $cl(J)=|J|=\chi(R)$ and (1) follows.

(2) Suppose that $J^2\neq 0$. Then $J\supsetneq J\cap(0:J)$. For $j\in J-J\cap(0:J)$, $J\cap(0:J)\cup\{j\}$ is a clique. So $cl(J)\geqslant|J\cap(0:J)|+1$. Suppose that $cl(J)=|J\cap(0:J)|+1$. Then for $x, y\in J-(0:J)$ with $x\neq y$, $xy\neq 0$. Thus the elements of $J-(0:J)$ can all be colored with the same color. Thus $\chi(J)=|J\cap(0:J)|+1$, too.

Beck showed [1, Proposition 2.3] that if $p_1, ..., p_k, q_1, ..., q_r$ are different prime numbers and if $N=p_1^{2n_1}\cdots p_k^{2n_k}q_1^{2m_1+1}q_r^{2m_r+1}$, then $\chi(\mathbb{Z}_N)=cl(\mathbb{Z}_N)=p_1^{n_1}\cdots p_k^{n_k}q_1^{m_1}\cdots q_r^{m_r}+r$. Our next theorem will be used to give a broad generalization of this result. We show in Corollary 3.3 that if A is any regular Noetherian ring and $M_1, ..., M_s$ is any set of maximal ideals of A with each residue field A/M_i finite, then $R=A/M_1^{n_1}\cdots M_s^{n_s}$ is a coloring with $\chi(R)=cl(R)$ and we explicitly compute $\chi(R)$. The idea used to give the coloring is essentially that used by Beck, but from a different point of view. For a nilpotent ideal I, the index of nilpotency of I is the smallest positive integer e with $I^e=0$.

THEOREM 3.2. *Let $R_1, ..., R_k, R_{k+1}, ..., R_{k+r}$ be colorings. Put $R=R_1\times\cdots\times R_k\times R_{k+1}\times\cdots\times R_{k+r}$. Let J_i be the nilradical of R_i (so J_i is finite). Suppose that the index of nilpotency for J_i is $2n_i$ for $i=1, ..., k$ and $2m_i-1$ for $i=k+1, ..., k+r$ $(n_i, m_i\geqslant 1)$. Then*

$$|J_1^{n_1}|\cdots|J_k^{n_k}|\,|J_{k+1}^{m_{k+1}}|\cdots|J_{k+r}^{m_{k+r}}|+r\leqslant cl(R)\leqslant\chi(R).$$

Furthermore, suppose for each i, the following condition holds:
For $1\leqslant i\leqslant k$, $x, y\in R_i$ with $xy=0$ implies $x\in J_i^{n_i}$ or $y\in J_i^{n_i}$ and if $x\notin J_i^{n_i+1}$, then $y\in J_i^{n_i}$.

(∗) *For $k+1\leqslant i\leqslant k+r$, $x, y\in R_i$ with $xy=0$ implies $x\in J_i^{m_i}$ or $y\in J_i^{m_i}$.*

Then $\chi(R)=cl(R)=|J_1^{n_1}|\cdots|J_k^{n_k}|\,|J_{k+1}^{m_{k+1}}|\cdots|J_{k+r}^{m_{k+r}}|+r$.

Proof. Let $S=J_1^{n_1}\times\cdots\times J_k^{n_k}\times J_{k+1}^{m_{k+1}}\times\cdots\times J_{k+r}^{m_{k+r}}$. For each i with $k+1\leqslant i\leqslant k+r$, choose $\bar{y}_i\in J_i^{m_i-1}$ and put $y_i=(0, ..., 0, \bar{y}_i, 0, ..., 0)$ where every coordinate is 0 except the ith and its entry is $\bar{y}_i$. Note that for $i\neq j$,

$y_i y_j = 0$, for each i, $y_i S = 0$ and $S^2 = 0$. Thus $C = S \cup \{y_{k+1}, ..., y_{k+r}\}$ is a clique with $|S| + r$ elements. The first result follows.

Suppose that (*) holds. Choose a distinct color for each element of C. Let $f(y)$ denote the color of an element $y \in R$ and color the remaining elements as follows. For each i with $1 \leqslant i \leqslant k$, choose $\bar{x}_i \in J_i^{n_i} - J_i^{n_i+1}$. Put $x_i = (0, ..., 0, \bar{x}_i, 0, ..., 0)$ where every coordinate is 0 except the ith coordinate and its entry is $\bar{x}_i$. So $\{x_i\} \subseteq C$ and hence has already been colored. Let $z = (z_1, ..., z_{k+r}) \in R - C$. First suppose that for each i with $1 \leqslant i \leqslant k$, $z_i \in J_i^{n_i}$. Since $z \notin S$, there is a coordinate l, $k+1 \leqslant l \leqslant k+r$ with $z_l \notin J_l^{m_l}$. We then define $f(z) = f(y_j)$ where $j = \min\{l \mid k+1 \leqslant l \leqslant k+r \text{ with } z_l \notin J_l^{m_l}\}$. If for some i with $1 \leqslant i \leqslant k$, $z_i \notin J_i^{n_i}$, we set $f(z) = f(x_j)$ where $j = \min\{i \mid 1 \leqslant i \leqslant k \text{ with } z_i \notin J_i^{n_i}\}$. It remains to verify that this is indeed a coloring.

Suppose that two distinct elements of R, $a = (a_1, ..., a_{k+r})$ and $b = (b_1, ..., b_{k+r})$, have the same color. We need that $ab \neq 0$, that is, $a_i b_i \neq 0$ for some i. First, suppose that $f(a) = f(b) = f(y_j)$. Then $a_j \notin J_j^{m_j}$ and $b_j \notin J_j^{m_j}$. (Note that this also holds if a or b is y_j.) Thus by (*), $a_j b_j \neq 0$. Next, suppose that $f(a) = f(b) = f(x_j)$. If $a \neq x_j$ and $b \neq x_j$, then $a_j \notin J_j^{n_j}$ and $b_j \notin J_j^{n_j}$, so $a_j b_j \neq 0$. If, say, $a = x_j$, then $a_j = \bar{x}_j \in J_j^{n_j} - J_j^{n_j+1}$ and $b_j \notin J_j^{n_j}$ since $b \neq x_j$. Again by (*), $a_j b_j \neq 0$.

Let R be a coloring with nilradical J and e the index of nilpotency. First, note that if (*) holds, then J is prime. For if $xy \in J$, then $x^e y^e = (xy)^e = 0$. If $x \notin J$, then $x^e \notin J$; so by (*), $y^e \in J^{[(e+1)/2]} \subseteq J$. For $e = 1$, that is, for R reduced, (*) holds if and only if J is prime, that is, R is an integral domain. For $e = 2$, (*) holds if and only if J is prime and $0 = J^2$ is primary. Thus (*) holds for a finite local ring (R, M) with $M^2 = 0$.

Suppose that R satisfies (**): if $0 = xy$ where $x \in J^n - J^{n+1}$ and $y \in J^m - J^{m+1}$ $(n, m \geqslant 0)$, then $n + m \geqslant e$. Then (**) $\Rightarrow$ (*) and conversely if $e \leqslant 2$. Suppose that (D, N) is a local domain which satisfies $x \in N^n - N^{n+1}$ and $y \in N^m - N^{m+1}$ implies $xy \in N^{n+m} - N^{n+m+1}$ (equivalently, the associated graded ring $D/N \oplus N/N^2 \oplus \cdots$ is an integral domain). If D/N is finite, then D/N^e is a coloring satisfying (**). Thus if (D, N) is a regular local ring with finite residue field, then D/N^e satisfies (**) and hence by Theorem 3.2 is a chromatic ring. Actually, a more general result is true, which we state as Corollary 3.3.

COROLLARY 3.3. *Let A be a regular Noetherian ring (that is, A_N is a regular local ring for each maximal ideal N of A). Let $N_1, ..., N_k$, $N_{k+1}, ..., N_{k+r}$ be maximal ideals of A with each residue field A/N_i finite. Consider the ring $R = A/N_1^{2n_1} \cdots N_k^{2n_k} N_{k+1}^{2m_{k+1}-1} \cdots N_{k+r}^{2m_{k+r}-1}$. Then R is a chromatic ring with*

$$\chi(R) = cl(R) = |A/N_1|^{\alpha_1} \cdots |A/N_k|^{\alpha_k} \, |A/N_{k+1}|^{\beta_{k+1}} \cdots |A/N_{k+r}|^{\beta_{k+r}} + r$$

where

$$\alpha_i = \sum_{l=n_i}^{2n_i-1} \binom{l-1+\operatorname{ht} N_i}{\operatorname{ht} N_i - 1} \quad \text{and} \quad \beta_i = \sum_{l=m_i+1}^{2m_i-2} \binom{l-1+\operatorname{ht} N_i}{\operatorname{ht} N_i - 1}.$$

Proof. For $i = 1, ..., k+r$, put $R_i = A_i/N_i^{e_i}$ where $e_i = 2n_i$ or $2m_i - 1$, depending on i. By the Chinese Remainder Theorem $R \approx R_1 \times \cdots \times R_{k+r}$. Now $R_i \approx A_{N_i}/N_{i_{N_i}}^{e_i}$ is a local coloring with maximal ideal $M_i = N_i/N_i^{e_i} \approx N_{i_{N_i}}/N_{i_{N_i}}^{e_i}$ and residue field $R_i/M_i \approx A/N_i$. Now A_{N_i} is a regular local ring, so R_i satisfies (**) and hence (*). By Theorem 3.2, $\chi(R) = cl(R) = |M_1^{n_1}| \cdots |M_k^{n_k}|\, |M_{k+1}^{m_{k+1}+1}| \cdots |M_{k+r}^{m_{k+r}+1}| + r$. First suppose that i satisfies $1 \leqslant i \leqslant k$. Then $|M_i^{n_i}| = |M_i^{n_i}/M_i^{n_i+1}|\, |M_i^{n_i+1}/M_i^{n_i+2}| \cdots |M_i^{2n_i-2}/M_i^{2n_i-1}|\, |M_i^{2n_i-1}|$. Now if $\operatorname{ht} N_i = s$ and $N_{i_{N_i}}$ has $z_1, ..., z_s$ as a minimal basis, then $N_{i_{N_i}}^j$ has as a minimal basis all power products $z_1^{\gamma_1} \cdots z_s^{\gamma_s}$ where $\gamma_1 + \cdots + \gamma_s = j$. Thus $\dim_{A/N_i} N_i^j/N_i^{j+1} = \binom{j+s-1}{s-1}$, so $|M_i^j/M_i^{j+1}| = |N_i^j/N_i^{j+1}| = |A/N_i|^{\binom{j+s-1}{s-1}}$. A similar statement holds for each i with $k+1 \leqslant i \leqslant k+r$. The result now easily follows.

Beck [1, Theorem 6.13] showed that if a coloring R is finite product of reduced rings and principal ideal rings, then $\chi(R) = cl(R)$. To recapture this result using Theorem 3.2 or Corollary 3.3, we need the following lemma.

LEMMA 3.4. *Let R be a ring and A an overring of R. Then $cl(R) = cl(A)$ and $\chi(R) = \chi(A)$. In particular, R is a coloring if and only if $T(R)$, the total quotient ring of R, is a coloring, and in this case $cl(R) = cl(T(R))$ and $\chi(R) = \chi(T(R))$.*

Proof. Since $R \subseteq A \subseteq T(R)$, we certainly have $cl(R) \leqslant cl(A) \leqslant cl(T(R))$ and $\chi(R) \leqslant \chi(A) \leqslant \chi(T(R))$. But since $T(R)$ is a localization of R, we have $cl(T(R)) \leqslant cl(R)$ and $\chi(T(R)) \leqslant \chi(R)$ by [1, Theorem 5.8].

COROLLARY 3.5. *Let R be a coloring that is a finite direct product of reduced rings, principal ideal rings, and finite rings with index of nilpotency 2. Then $\chi(R) = cl(R)$.*

Proof. By Lemma 3.4 we may assume that R is a total quotient ring. First, note that a reduced coloring that is a total quotient ring is a direct product of fields. Second, a principal ideal ring is a direct product of principal ideal domains and special principal ideal rings [2, Section 39]. Thirdly, a finite ring with index of nilpotency 2 is a direct product of fields and local rings with index of nilpotency 2. But each type of these rings clearly satisfies hypothesis (*) of Theorem 3.2. Hence by Theorem 3.2 their product satisfies $\chi(R) = cl(R)$.

If R is a subring of a finite direct product of integral domains and finite rings, certainly R is a coloring. For Noetherian colorings the converse is true.

THEOREM 3.6. *Let R be a Noetherian ring. Then R is a coloring if and only if R is a subring of a finite direct product of fields and a finite ring.*

Proof. Let $0 = Q_1 \cap \cdots \cap Q_n$ be a normal decomposition for 0 where Q_i is P_i-primary. Now $R \subseteq R/Q_1 \times \cdots \times R/Q_n$. If P_i is not maximal, then $Q_i = P_i$ and R/Q_i is an integral domain and hence is a subring of its quotient field. Suppose that P_i is maximal. (We may suppose that $Q_i \neq P_i$, for if $Q_i = P_i$, then R/Q_i is a field.) Then R/Q_i is a zero-dimensional local ring with finite residue field R/P_i and hence R/Q_i is finite.

Whether every coloring is a subring of a finite direct product of fields and a finite ring remains open. Such a coloring must have the property that 0 has a primary decomposition.

However, for many purposes it is sufficient to consider only Noetherian colorings. For let R be a coloring with nilradical J and let $R_0 = \mathbb{Z} \cdot 1 + J$. Note that R_0 is Noetherian and that J is also the nilradical of R_0. So $cl(R_0) = \varepsilon(R_0) + cl(J)$, $cl(R) = \varepsilon(R) + cl(J)$, $\chi(R_0) = \varepsilon(R_0) + \chi(J)$, and $\chi(R) = \varepsilon(R) + \chi(J)$. Hence $\chi(R_0) = cl(R_0)$ if and only if $\chi(R) = cl(R)$. Note that if char $R > 0$, then R_0 is a finite ring. If char $R = 0$, then $J \cap \mathbb{Z} \cdot 1 = 0$ and $R_0/J \approx \mathbb{Z}$, so R_0 is an indecomposable coloring.

We end this section with two methods for constructing colorings.

EXAMPLE 3.7. Let R be a ring with ideals $I_1, \ldots, I_n$ such that each R/I_i is a coloring. Then $R/I_1 \cap \cdots \cap I_n$ being a subring of $R/I_1 \times \cdots \times R/I_n$, is a coloring. For example, let (R, M) be a local ring with finite residue field. Let Q be M-primary and let $P_1, \ldots, P_n$ be prime ideals of R. Then $R/Q \cap P_1 \cap \cdots \cap P_n$ is a coloring.

EXAMPLE 3.8. Let R be a ring and M and R-module. Then the idealization (or trivial extension) $R_0 = R \oplus M$ of R and M is a coloring if and only if R is a coloring and M is finite. This easily follows from the facts that $(0 \oplus M)^2 = 0$ and $R_0/(0 \oplus M) \approx R$.

4. FINITE COMMUTATIVE RINGS WITH $\chi(R) \leqslant 4$

In this section we characterize the finite commutative rings R with $\chi(R) \leqslant 4$ (or equivalently, $cl(R) \leqslant 4$, since for a ring with $cl(R) \leqslant 4$ or $\chi(R) \leqslant 4$, $cl(R) = \chi(R)$).

Beck [1] has already characterized the finite commutative rings R with $\chi(R) \leqslant 3$. They are

$\chi(R)=2$: (i) R a finite field

(ii) $R \approx \mathbb{Z}_4$

(iii) $R \approx \mathbb{Z}_2[X]/(X^2)$

$\chi(R)=3$: (i) $R \approx$ a product of two finite fields

(ii) $R \approx k \times \mathbb{Z}_4$, k *a* finite field

(iii) $R \approx k \times \mathbb{Z}_2[X]/(X^2)$, k a finite field

(iv) $R \approx \mathbb{Z}_8$

(v) $R \approx \mathbb{Z}_9$

(vi) $R \approx \mathbb{Z}_3[X]/(X^2)$

(vii) $R \approx \mathbb{Z}_2[X]/(X^3)$

(viii) $R \approx \mathbb{Z}_4[X]/(2X, X^2-2)$.

It is interesting to note that each of these rings is a principal ideal ring.

Let R be a finite commutative ring with $\chi(R)=4$. First, suppose that R is *not* indecomposable. Then $R=R_1 \times R_2$, so $4=cl(R_1 \times R_2) \geqslant cl(R_1)+cl(R_2)-1$, and thus $cl(R_1)+cl(R_2) \leqslant 5$. Therefore we can assume that either $cl(R_1)=cl(R_2)=2$ or $cl(R_1)=2$ and $cl(R_2)=3$.

Assume that $cl(R_1)=cl(R_2)=2$. If R_1 or R_2 is a finite field, then $cl(R_1 \times R_2)=3$. However, $R=\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$, and $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$ each have $\chi(R)=cl(R)=4$.

So suppose that $cl(R_1)=2$ and $cl(R_2)=3$. If $R_1=k_1$ is a finite field, then the eight rings $k_1 \times k_2 \times k_3$, $k_1 \times k_2 \times \mathbb{Z}_4$, $k_1 \times k_2 \times \mathbb{Z}_2[X]/(X^2)$, $k_1 \times \mathbb{Z}_8$, $k_1 \times \mathbb{Z}_9$, $k_1 \times \mathbb{Z}_3[X]/(X^2)$, $k_1 \times \mathbb{Z}_2[X]/(X^3)$, and $k_1 \times \mathbb{Z}_4[X]/(2X, X^2-2)$ (where k_2 and k_3 are finite fields) each have $\chi(R)=cl(R)=4$. If $R_1=\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and $cl(R_2)=3$, then it is easily seen that $cl(R_1 \times R_2)=5$ unless $R_2=k_1 \times k_2$(where k_1 and k_2 are finite fields) and we have already counted this type of ring.

This gives a total of eleven types of decomposable finite commutative rings R with $\chi(R)=4$:

(i) $R \approx \mathbb{Z}_4 \times \mathbb{Z}_4$

(ii) $R \approx \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$

(iii) $R \approx \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$

(iv) $R \approx k_1 \times k_2 \times k_3$, each k_i a finite field

(v) $R \approx k_1 \times k_2 \times \mathbb{Z}_4$, each k_i *a* finite field

(vi) $R \approx k_1 \times k_2 \times \mathbb{Z}_2[X]/(X^2)$, each k_i a finite field

(vii) $R \approx k \times \mathbb{Z}_8$, k a finite field

(viii) $R \approx k \times \mathbb{Z}_9$, k a finite field

(ix) $R \approx k \times \mathbb{Z}_3[X]/(X^2)$, k a finite field

(x) $R \approx k \times \mathbb{Z}_2[X]/(X^3)$, k a finite field

(xi) $R \approx k \times \mathbb{Z}_4[X]/(2X, X^2-2)$, k a finite field.

Each of these rings is a principal ideal ring.

Thus we are reduced to characterizing finite indecomposable commutative rings R with $\chi(R)=4$. Such a ring is necessarily local. Let (R, M) be a finite local ring with $\chi(R)=4$. Suppose that $M^n=0$, but $M^{n-1}\neq 0$. If $n\geqslant 5$, $(M^{n-2})^2=0$, so M^{n-2} is a clique. Also $M^2M^{n-2}=0$, so there is a $x\in M^2-M^{n-2}$ with $xM^{n-2}=0$. Thus $M^{n-2}\cup\{x\}$ is a clique. So $cl(R)\geqslant |M^{n-2}|+1=|M^{n-2}/M^{n-1}|\,|M^{n-1}|+1\geqslant |R/M|^2+1\geqslant 5$. We may therefore suppose that $M^4=0$.

First, suppose that $0=M^4\subsetneq M^3$. Then M^2 is a clique, so $|M^2|=4$, $|M^3|=2$, and $R/M\approx\mathbb{Z}_2$. Since $M^2/M^3\approx R/M$, M^2 is principal, say $M^2=Rb$. Certainly, $0:b\supseteq Rb$ and if $x\in(0:b)-Rb$, $Rb\cup\{x\}$ is a clique with 5 elements. Hence $(0:b)=Rb$. So $Rb\approx R/(0:b)$ gives that $|R|=16$. Note that $|M|=8$ and since $|M/M^2|=2$, M is principal, say $M=Ra$. If char $R=16$, then $\mathbb{Z}_{16}\subseteq R$ so $R\approx\mathbb{Z}_{16}$. If char $R=2$, then an easy application of the structure theory for complete local rings gives that $R\approx\mathbb{Z}_2[X]/(X^4)$. The case char $R=8$ can not occur. For $2^2\neq 0$ implies $2\in M-M^2$ and hence $M=2R$ since M is principal. But then $0=2^3R=M^3$, a contradiction. So we are left with the case char $R=4$. Now $\mathbb{Z}_4[a]\subseteq R$ and since $0, 1, 2, 3, a, 1+a, 2+a, 3+a$, and one of the pair $2a, a^2$ are distinct elements, $8<|\mathbb{Z}_4[a]|\leqslant|R|=16$. Hence $R=\mathbb{Z}_4[a]$. If $2\in M^3$, then $2=a^3$ and $2a=0$. Now the epimorphism $\varphi:\mathbb{Z}_4[X]\to R$ given by $\varphi(X)=a$ has $(2X, X^3-2, X^4)=(2X, X^3-2)\subseteq\ker\varphi$. But $\mathbb{Z}_4[X]/(2X, X^3-2)$ has at most 16 elements, so $R\approx\mathbb{Z}_4[X]/(2X, X^3-2)$. Next suppose that $2\in M^2-M^3$, so $M^2=2R$. If $2=a^2$, then it is easily seen using an argument similar to the case where $2\in M^3$ that $R\approx\mathbb{Z}_4[X]/(X^2-2)$. If $2\neq a^2$, then $2R=M^2=\{0, 2, a^2, 2+a^2\}$, so $2(1+a)=a^2$. In this case it is easily seen that $R\approx\mathbb{Z}_4[X]/(X^2+2X+2)$. Note that $\mathbb{Z}_4[X]/(X^2-2)$ and $\mathbb{Z}_4[X]/(X^2+2X+2)$ are not isomorphic since 2 is not a square in the second ring. Since each of the five rings (R, M) is a local principal ideal ring with 16 elements, it follows that $|M^2|=4$ and M^2 is a longest clique, so $cl(R)=4$. Thus we have proved the following result.

LEMMA 4.1. *Let (R, M) be a finite local ring with $\chi(R)=4$ and $0=M^4\subsetneq M^3$. Then R is a principal ideal ring with 16 elements and R is isomorphic to one of the following five rings*: $\mathbb{Z}_{16}$, $\mathbb{Z}_2[X]/(X^4)$, $\mathbb{Z}_4[X]/(2X, X^3-2)$, $\mathbb{Z}_4[X]/(X^2-2)$, *or* $\mathbb{Z}_4[X]/(X^2+2X+2)$.

We next handle the case where (R, M) is a finite local ring with $\chi(R)=4$ and $M^2=0$. The following well-known and easily proved proposition will be used.

PROPOSITION 4.2. *Let (R, M) be a finite local ring (but not a field) with $M^2=0$. Suppose that* char $R/M = p$ *and* $s=\dim_{R/M} M$.

1. *If* char $R=p$, *then* $R \approx (R/M)[X_1, \ldots, X_s]/(X_1, \ldots, X_s)^2$.
2. *If* char $R=p^2$, *then* $R \approx \mathbb{Z}_{p^2}[X_1, \ldots, X_s]/((p, X_2, \ldots, X_s)^2 + (f(X_1)))$ *where* $f(X_1) \in \mathbb{Z}_{p^2}[X_1]$ *is irreducible* mod p *with* $\mathbb{Z}_p[X_1]/(\bar{f}(X_1)) \approx R/M$.

Proof. If char $R=p$, then R contains a coefficient field F (i.e., $F \approx R/M$). Map $F[X_1, \ldots, X_s]$ onto R by sending $X_i \to x_i$ where $x_1, \ldots, x_s$ is a minimal basis for M. It is obvious that this map has kernel $(X_1, \ldots, X_s)^2$.

Suppose that char $R=p^2$. Then p is part of a minimal basis for M; so extend to a basis $p, x_2, \ldots, x_s$ for M. Now $R/M=\mathbb{Z}_p[\bar{r}]$ for some $r \in R$. Let $f(X_1) \in \mathbb{Z}_{p^2}[X_1]$ so that $f(X_1)$ is the minimal polynomial for $\bar{r}$ mod p. Then $R/M \approx \mathbb{Z}_p[X_1]/(\bar{f}(X_1))$. It is easily seen that $R=\mathbb{Z}_{p^2}[r, x_2, \ldots, x_s]$ (for $\mathbb{Z}_{p^2}[r, r_2, \ldots, x_s]$ is a local subring of R with the same residue field as R, the same number of elements in a minimal basis for its maximal ideal as for M and hence has the same number of elements as R). The map $\mathbb{Z}_{p^2}[X_1, \ldots, X_s] \to R$ given by $X_1 \to r$ and $X_i \to x_i$ for $2 \leqslant i \leqslant s$ is an epimorphism with kernel $(p, X_2, \ldots, X_s)^2 + (f(X_1))$.

LEMMA 4.3. *Let (R, M) be a finite local ring with $\chi(R)=4$ and $M^2=0$. Then R is isomorphic to one of the following four rings*: $GF(2^2)[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2+X+1)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$ *or* $\mathbb{Z}_4[X]/(2, X)^2$.

Proof. Here $4=\chi(R)=cl(R)=|M|=|R/M|^s$ where $s=\dim_{R/M} M$. So either $R/M \approx GF(2^2)$ and $s=1$ or $R/M \approx \mathbb{Z}_2$ and $s=2$. The result follows from Proposition 4.2.

Finally, we do the case where (R, M) is a finite local ring with $cl(R)=4$ and $M^3=0$, but $M^2 \neq 0$. Let $x \in M-0:M$; then $0:M \cup \{x\}$ is a clique. So $|0:M|+1 \leqslant 4$ and hence $|0:M| \leqslant 3$. Thus $R/M \approx \mathbb{Z}_2$ or $\mathbb{Z}_3$ and $M^2=0:M$ is principal.

Suppose that $R/M \approx \mathbb{Z}_3$; so $|M^2|=3$. Let $x \in M-M^2$. Then $M^2 \cup \{x\}$ is a clique of length 4, so $0:x \subseteq M^2 \cup \{x\}$. But 3 divides $|0:x|$, so $|0:x|=3$. Now $Rx \approx R/(0:x)$, so $|Rx|=|R|/3$. Therefore $|R|=3\,|Rx| \leqslant |R/M|\,|M|=|R|$. Thus $|Rx|=|M|$ and hence $M=Rx$; so R is a local principal ideal ring. Hence $|R|=27$. If char $R=27$, then $R \approx \mathbb{Z}_{27}$. If char $R=3$, then $R \approx \mathbb{Z}_3[X]/(X^3)$. Suppose that char $R=9$. So $3 \in M^2$ and hence $M^2=\{0, 3, 6\}$. Let $M=Ra$. Then $a^2 \in M^2$, so $a^2=3$ or $a^2=6$. Now $\mathbb{Z}_9 \subsetneq \mathbb{Z}_9[a] \subseteq R$, so $\mathbb{Z}_9[a]=R$ since $\mathbb{Z}_9$ has 9 elements and R has 27 elements. First suppose that $a^2=3$. Then $\mathbb{Z}_9[X]/(X^2-3, 3X)$ maps onto R

and has at most 27 elements, so $R \approx \mathbb{Z}_9[X]/(X^2-3, 3X)$. If $a^2=6$, then by similar reasoning, $R \approx \mathbb{Z}_9[X]/(X^2-6, 3X)$. Note that these two rings are not isomorphic since 3 is a square in the first, but not in the second. Thus $R/M \approx \mathbb{Z}_3$ implies that $R \approx \mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2-3, 3X)$, or $\mathbb{Z}_9[X]/(X^2-6, 3X)$. Each of these rings is a local principal ring with 27 elements and hence has $cl(R)=4$.

So suppose that $R/M \approx \mathbb{Z}_2$. Then M can not be principal, for then we would have $cl(R)=3$, a contradiction. Let $x, y \in M-M^2$ with $x-y \notin M^2$. Then $(x-y)^2 = x^2-2xy+y^2 = x^2+y^2$. If $x^2=y^2$, then $x^2-y^2=2x^2=0$, so $(x-y)^2=0$. Thus either $x^2=0$, $y^2=0$ or $(x-y)^2=0$. So we can assume that there is an element $x \in M-M^2$ with $x^2=0$. Then Rx is a clique, so $|Rx| \leqslant 4$. If $|Rx|=2$, then $MRx=0$, so $x \in 0:M=M^2$, a contradiction. Thus $|Rx|=4$. Now $Rx \subseteq 0:x$ and if $r \in 0:x-Rx$, then $Rx \cup \{r\}$ is a clique with 5 elements, a contradiction. So $Rx=0:x$. From $Rx \approx R/(0:x)$ and $|Rx|=|0:x|=4$, it follows that $|R|=16$. From $|R/M|=|M^2|=2$, it follows that $|M/M^2|=4$, so M has a minimal basis consisting of 2 elements and we can assume that one of the basis elements is 2-nilpotent.

So suppose that $M=(a,b)$ where $a^2=0$. Then $b^3=0$, but $ab \neq 0$ for this would imply that $Ra \cup \{b\}$ is a clique with 5 elements. Therefore $M^2=\{0, ab\}$. Hence $b^2=ab$ or $b^2=0$.

Suppose that char $R=2$. Consider the homomorphism $\varphi: \mathbb{Z}_2[X, Y] \to R$ given by $\varphi(X)=a$ and $\varphi(Y)=b$. Certainly φ is onto. If $b^2=ab$, then $(X^2, Y^2-XY) \subseteq \ker \varphi$. Since $\mathbb{Z}_2[X, Y]/(X^2, Y^2-XY)$ has 16 elements, $R \approx \mathbb{Z}_2[X, Y]/(X^2, Y^2-XY)$. If $b^2=0$, then $(X^2, Y^2) \subseteq \ker \varphi$. Since $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$ has 16 elements, $R \approx \mathbb{Z}_2[X, Y]/(X^2, Y^2)$. It easily checked that both rings R do indeed have $cl(R)=4$.

Next suppose that char $R=8$, so 2 can be part of a minimal basis for M and $M^2=\{0,4\}$. Note that $a, 2$ is a basis for M for if $a-2 \in M^2$, then $0=(a-2)^2=a^2-4a+4=4$, a contradiction. The homomorphism $\varphi: \mathbb{Z}_8[X] \to R$ with $\varphi(X)=a$ is surjective since $\mathbb{Z}_8[a]=R$. We must have $2a=4$ since $2a=0$ would give a clique with 5 elements. So $(2X-4, X^2) \subseteq \ker \varphi$. Since $\mathbb{Z}_8[X]/(2X-4, X^2)$ has 16 elements, $R \approx \mathbb{Z}_8[X]/(2X-4, X^2)$. It is easily seen that this ring does have $cl(R)=4$.

It remains to do the case char $R=4$. First, suppose that $2 \notin M^2$, so 2 is part of a minimal basis for M and $2^2=0$. Let b be the other basis element. Since $2R$ is a clique of length 4, we must have $2b \neq 0$. So $M^2=\{0, 2b\}$. Thus $b^2=0$ or $b^2=2b$. Note that $R=\mathbb{Z}_4[b]$ since $2b \neq 0$. So the homomorphism $\varphi: \mathbb{Z}_4[X] \to R$ given by $\varphi(X)=b$ is surjective. If $b^2=0$, $X^2 \in \ker \varphi$. Since $\mathbb{Z}_4[X]/(X^2)$ has 16 elements, $R \approx \mathbb{Z}_4[X]/(X^2)$. If $b^2=2b$, $X^2-2X \in \ker \varphi$. Since $\mathbb{Z}_4[X]/(X^2-2X)$ has 16 elements, $R \approx \mathbb{Z}_4[X]/(X^2-2X)$. It is easily checked that both rings R have $cl(R)=4$ and that they are not isomorphic. Thus we are reduced to the case $2 \in M^2$.

So $M^2 = \{0, 2\}$. Again, let a, b be a basis for M where $a^2 = 0$. Then $ab \neq 0$ gives $ab = 2$. Now $b^2 = 0$ or $b^2 = ab = 2$. It is easily verified that $\mathbb{Z}_4[a, b] = R$. Thus we have an epimorphism $\varphi : \mathbb{Z}_4[X, Y] \to R$ given by $\varphi(X) = a$ and $\varphi(Y) = b$. Now $2X, 2Y, X^2, XY - 2 \in \ker \varphi$. If $b^2 = 0$, then $Y^2 \in \ker \varphi$. Since $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$ has 16 elements, $R \approx \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$. If $b^2 = ab$, then $Y^2 - XY \in \ker \varphi$. Since $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$ has 16 elements, $R \approx \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$. It is easily checked that each of these two rings have clique number 4 and that they are not isomorphic since 2 is not a square in the first, but is in the second. This finishes the classification of the finite commutative rings with $\chi(R) = 4$.

THEOREM 4.4. *Let R be a finite commutative ring with $\chi(R) = 4$. Then R is isomorphic to one of the following* 31 *types of rings* (*where k_i is a finite field*):

$$\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2),$$

$$k_1 \times k_2 \times k_3, k_1 \times k_2 \times \mathbb{Z}_4, k_1 \times k_2 \times \mathbb{Z}_2[X]/(X^2), k_1 \times \mathbb{Z}_8, k_1 \times \mathbb{Z}_9,$$

$$k_1 \times \mathbb{Z}_3[X]/(X^2), k_1 \times \mathbb{Z}_2[X]/(X^3), k_1 \times \mathbb{Z}_4[X]/(2X, X^2 - 2),$$

$$\mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4), \mathbb{Z}_4[X]/(2X, X^3 - 2), \mathbb{Z}_4[X]/(X^2 - 2),$$

$$\mathbb{Z}_4[X]/(X^2 + 2X + 2), GF(2^2)[X]/(X^2), \mathbb{Z}_4[X]/(X^2 + X + 1),$$

$$\mathbb{Z}_2[X, Y]/(X, Y)^2, \mathbb{Z}_4[X]/(2, X)^2, \mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3),$$

$$\mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X), \mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY),$$

$$\mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_8[X]/(2X - 4, X^2), \mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X),$$

$$\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y) \text{ or}$$

$$\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2Y, 2Y).$$

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