Computers and Mathematics with Applications 59 (2010) 2473–2477

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analytical solutions to the pulsed Klein–Gordon equation using Modified Variational Iteration Method (MVIM) and Boubaker Polynomials Expansion Scheme (BPES)

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ARTICLE INFO

Article history:
Received 30 October 2009
Received in revised form 22 December 2009
Accepted 22 December 2009

Keywords:
Klein–Gordon equation
Stationary regime
Applied physics

ABSTRACT

In this study, we propose an analytical solution to the Klein–Gordon equation in a pulsed stationary regime. The performed protocols are based on the modified variational iteration method MVIM and Boubaker polynomials expansion scheme BPES.

The results are presented, and compared with some solutions proposed later in order to confirm the good accuracy of the protocols used.

1. Introduction

The Klein–Gordon equation is encountered in several applied physics fields such as, quantum field theory [1–6], fluid dynamics [7,8], optoelectronic devices design [9] and numerical analysis [10–21].

As it plays an important role in applied physics, the Klein–Gordon equation were paid special attention and many attempts to solve it have been published in the past decades. Wazwaz [22] developed exact travelling wave solutions, Elsayed [23] and Kaya [24] used Adomian's decomposition method in order to obtain exact solutions to the Klein–Gordon equation, and more recently, Sirendaoreji [25,26] proposed, for the same purpose, the auxiliary equation method. In the beginning of the past decade, Fu et al. [27] and Parkes et al. [28] used the Jacobi elliptic function expansion method, and proposed double periodic solutions.

The standard Klein–Gordon equation is given by the following equation:

$$\begin{cases} \dfrac{\partial^2 \xi(x,t)}{\partial t^2} + a\dfrac{\partial^2 \xi(x,t)}{\partial x^2} + g(\xi) = f(x,t) \\ x \in [0;\, 1]; \quad t \in [0;\, T] \end{cases} \tag{1}$$

where $\xi(x,t)$ is the two-variable wave unknown function, T is the system characteristic time, a is a given constant, $g(\xi)$ represents the expression of an external ξ-dependent force and $f(x,t)$ is a given function.

In this study, we propose analytical solutions to the applied physics related Klein–Gordon equation in a pulsed stationary regime.

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doi:10.1016/j.camwa.2009.12.026

2. Klein–Gordon equation solution derivation

2.1. Equation features

The discussed Klein–Gordon equation in this study is the following:

$$\begin{cases} \xi_{tt}(x,t) + a\xi_{xx}(x,t) + b\xi(x,t) = D \times \cos(Hx) \times \mathrm{e}^{-\mathrm{j}\omega t}; \quad (D,H) \in [0,+\infty[\times[0,+\infty[\\ x \in [0,1]; \quad t \in [0,T]; \end{cases} \tag{2}$$

along with the initial-boundary conditions:

$$\begin{cases} \xi(x,t)|_{x=0,t=0} = u_0 = 1 \\ \left.\dfrac{\mathrm{d}[\xi(x,t)]}{\mathrm{d}x}\right|_{x=0} = 0 \\ a = -1, \quad b = 2, \quad D = 1, \quad H = 3. \end{cases} \tag{3}$$

2.2. MVIM solution derivation

According to the modified variational iteration method MVIM principles [29–38], for a general differential equation, the correction functional is given by

$$\xi_{n+1}(x,t) = xt + \int_0^t \lambda(s)\left(\frac{\partial^2\xi_n}{\partial s^2} - \frac{\partial^2\tilde{\xi}_n}{\partial x^2} + \tilde{\xi}_n(s,t)\right)\mathrm{d}s \tag{4}$$

where subscripts n denote the nth approximation $\xi_n(x,t)$ of $\xi(x,t)$, $\tilde{\xi}_n$ is considered as a restricted variation. i.e. $\delta\tilde{\xi}_n = 0$, λ is a Lagrange multiplier which can be identified optimally via variational iteration method and s is an intermediate variable.

Making the correction functional stationary [36–38], the Lagrange multiplier can be identified as $\lambda(s) = (s - t)$, which leads to the iterative formula

$$\xi_{n+1}(x,t) = xt + \int_0^t (s-t)\left(\frac{\partial^2\xi_n}{\partial s^2} - \frac{\partial^2\xi_n}{\partial x^2} + \xi_n(s,t)\right)\mathrm{d}s. \tag{5a}$$

By applying the modified variational iteration method MVIM expansion [30]:

$$\xi_0 + p\xi_1 + p^2\xi_2 + \cdots = xt + p\int_0^t (s-t)\left(\left(\frac{\partial^2\xi_0}{\partial s^2} + p\frac{\partial^2\xi_1}{\partial s^2} + \cdots\right) - \left(\frac{\partial^2\xi_0}{\partial x^2} + p\frac{\partial^2\xi_1}{\partial x^2} + \cdots\right) + (\xi_0 + p\xi_1 + \cdots)\right)\mathrm{d}s \tag{5b}$$

and comparing the coefficient of like powers of p, consequently, following approximants are obtained:

$$\xi(x,t) = A + Bx - Ax^2 - \frac{1}{3}Bx^3 + \frac{1}{6}Ax^4 - \frac{44}{81}\mathrm{e}^{-\mathrm{j}\omega t}\cos^3 x + \frac{11}{27}\mathrm{e}^{-\mathrm{j}\omega t}\cos x + \frac{11}{81}\mathrm{e}^{-\mathrm{j}\omega t} - \frac{1}{9}x^2\mathrm{e}^{-\mathrm{j}\omega t}. \tag{6}$$

Or in concordance with the initial-boundary conditions (3):

$$\xi(x,t) = 1 - x^2 + \frac{1}{6}x^4 - \frac{44}{81}\mathrm{e}^{-\mathrm{j}\omega t} \times \cos^3 x + \frac{11}{27}\mathrm{e}^{-\mathrm{j}\omega t} \times \cos x + \frac{11}{81}\mathrm{e}^{-\mathrm{j}\omega t} - \frac{1}{9}x^2\mathrm{e}^{-\mathrm{j}\omega t}. \tag{7}$$

Two 3D views of this solution are presented in Fig. 1.

2.3. BPES-related solution derivation

The resolution protocol is based on the Boubaker Polynomials Expansion Scheme (BPES) [39–49], applied by setting the pulsed expression:

$$\xi(x,t) = \frac{1}{2N_0}\left(\sum_{k=1}^{N_0} \lambda_k \times B_{4k}(x \times r_k)\right)\mathrm{e}^{-\mathrm{j}\omega t} = p(x) \times \mathrm{e}^{-\mathrm{j}\omega t} \tag{8}$$

where B_{4k} are the $4k$-order Boubaker polynomials, r_k are B_{4k} minimal positive roots, N_0 is a prefixed integer, ω is the stationary regime pulsation and $\lambda_k|_{k=1\ldots N_0}$ are unknown pondering real coefficients.

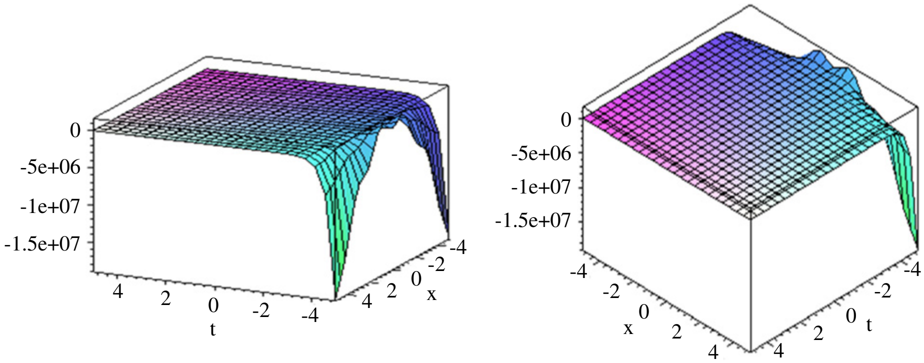

Fig. 1. Two views of the MVIM solution $\xi^{\text{sol.}}_{\text{MVIM}}(x, t)$.

Eq. (1) is hence written:

$$\left[\frac{-\omega^2}{2N_0}\left(\sum_{k=1}^{N_0}\lambda_k \times B_{4k}(x\times r_k)\right) + a\frac{-\omega^2}{2N_0}\left(\sum_{k=1}^{N_0}\lambda_k \times r_k^2\frac{\mathrm{d}^2B_{4k}(x\times r_k)}{\mathrm{d}x^2}\right)\right]\mathrm{e}^{-\mathrm{j}\omega t} + b\xi(x,t) = f(x,t). \tag{9}$$

By decoupling the t-dependent component, and for the particular case:

$$\begin{cases} f(x,t) = D\times\cos(Hx)\times \mathrm{e}^{-\mathrm{j}\omega t}, \quad (D,H)\in[0,+\infty[\times[0,+\infty[\\ g(\xi) = b\times\xi, \quad b\in]-\infty,+\infty[\end{cases} \tag{10}$$

the main equation becomes:

$$\frac{-\omega^2+b}{2N_0}\left(\sum_{k=1}^{N_0}\lambda_k \times B_{4k}(x\times r_k)\right) + a\frac{-\omega^2}{2N_0}\left(\sum_{k=1}^{N_0}\lambda_k \times r_k^2\frac{\mathrm{d}^2B_{4k}(x\times r_k)}{\mathrm{d}x^2}\right) = D\cos(Hx). \tag{11}$$

Or, using a standardized form:

$$\begin{cases} \displaystyle\sum_{k=1}^{N_0}\lambda_k\times\Theta_k(x) = D\cos(Hx) \\ \Theta_k(x) = \dfrac{1}{2N_0}\left((-\omega^2+b)B_{4k}(x\times r_k) - a\omega^2 r_k^2\dfrac{\mathrm{d}^2B_{4k}(x\times r_k)}{\mathrm{d}x^2}\right). \end{cases} \tag{12}$$

The following step consists of evaluating the coefficients $\lambda'_k|_{k=1,\dots,N_0}$ that verify:

$$D\cos(Hx) = \sum_{k=1}^{N_0}\lambda'_k\times B_{4k}(x\times r_k). \tag{13}$$

This operation leads to the weak solution defined by the system (14):

$$\begin{cases} \displaystyle\sum_{k=1}^{N_0}\lambda_k\times I_k = \sum_{k=1}^{N_0}\lambda'_k\times J_k \\ I_k = \displaystyle\int_0^1\Theta_k(x)\mathrm{d}x; \qquad J_k = \int_0^1 B_{4k}(x\times r_k)\mathrm{d}x. \end{cases} \tag{14}$$

The values of $I_k|_{k=1\dots N_0}$ and $J_k|_{k=1\dots N_0}$ are calculated using Eq. (14) along with the arithmetical properties of the Boubaker polynomials [40–47].

Finally, the coefficients $\lambda_k^{\text{sol.}}|_{k=1\dots N_0}$ are deduced by identification:

$$\begin{cases} \lambda_k^{\text{sol.}} = \lambda'_k\times\dfrac{J_k}{I_k} \\ k=1,2,\dots,N_0. \end{cases} \tag{15}$$

The solution $\xi^{\text{sol.}}_{\text{BPES}}(x,t)$ is hence:

$$\xi^{\text{sol.}}_{\text{BPES}}(x,t) = \frac{1}{2N_0}\left(\sum_{k=1}^{N_0}\lambda_k^{\text{sol.}}\times B_{4k}(x\times r_k)\right)\mathrm{e}^{-\mathrm{j}\omega t}. \tag{16}$$

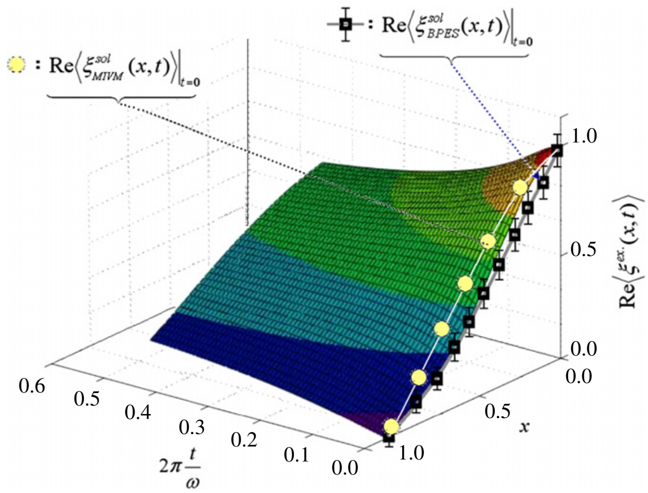

Fig. 2. Exact solution $\xi^{\text{ex.}}(x,t)$ and $\xi^{\text{sol.}}_{\text{BPES}}(x,t)|_{t=0}$ $\text{Re}\langle\xi^{\text{sol.}}_{\text{MIVM}}(x,t)\rangle|_{t=0}$ real parts.

Table 1
Parameter values.

BPES solution		Exact solution	
Parameter	Value	Parameter	Value
N_0	25	ω	$\sqrt{10}$
D	1	φ	0
a	-1	A	1
H	3	b	2

2.4. Exact solution derivation

The exact solution can be derived by combining Eqs. (1) and (2):

$$-\omega^2 p(x) + a\frac{\mathrm{d}^2 p(x)}{\mathrm{d}x^2} + bp(x) = D\cos(Hx). \tag{17}$$

Or simply:

$$\frac{\mathrm{d}^2 p(x)}{\mathrm{d}x^2} - \frac{\omega^2 - b}{a}p(x) = \frac{D}{a}\cos(Hx). \tag{18}$$

An exact solution to Eq. (17) is:

$$p_{\text{ex.}}(x) = A\cos\left(\pm\sqrt{\frac{b-\omega^2-D}{a}}x + \varphi\right) \tag{19}$$

with A and φ arbitrary constants. The complete exact solution $\xi^{\text{ex.}}(x,t)$ is finally:

$$\xi^{\text{ex.}}(x,t) = A\cos\left(\pm\sqrt{\frac{b-\omega^2-D}{a}}x + \varphi\right)\mathrm{e}^{-\mathrm{j}\omega t}. \tag{20}$$

3. Results and discussions

The BPES-related solutions $\text{Re}\langle\xi^{\text{sol.}}_{\text{BPES}}(x,t)\rangle|_{t=0}$ and $\text{Re}\langle\xi^{\text{sol.}}_{\text{MIVM}}(x,t)\rangle|_{t=0}$ have been plotted along with the exact solution $\text{Re}\langle\xi^{\text{ex.}}(x,t)\rangle$ in Fig. 2.

The parameter values for both BPES-related and exact solutions are presented in Table 1.

The results show a good agreement between the exact and the proposed solution along the $t = 0$-plane (Fig. 2). The concordance for higher values of t is simply predictable since the t-dependent terms, although eliminated at the beginning of the resolution process (Section 2.3), remain the same for both exact and proposed solutions. This feature means that it is possible to extract analytical solutions when exact ones are difficult or impossible to obtain.

4. Conclusion

This work proposes an analytical solution to well-known applied-physics-related Klein–Gordon equation. The analytical approaches of the first method: The Modified Variational Iteration Method MVIM has been performed as an enhanced form of the protocols developed earlier by Wang et al. [50] involving the Exp-function method, Adomian [51] and Gerogiev et al. [52] who extended the solution to n-dimensional cases, or Wang [53] using the generalized solitary wave approaches. A given example gives good fundaments to the performed protocols MVIM and BPES, particularly when exact solution expressions are difficult to establish.

Acknowledgement

The authors would like to express their gratitude to Pr. K. Boubaker from Tunisia for the stimulating support and help.

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