Local Compactness in Fuzzy Convergence Spaces

Y. Boissy

Department of Mathematics, University of Central Florida, Orlando, Florida 32816-1364

P. Brock

Department of Mathematics, San Diego State University, San Diego, California 92182-7720
E-mail: pwbrock@sciences.sdsu.edu

and

G. Richardson

Department of Mathematics, University of Central Florida, Orlando, Florida 32816-1364
E-mail: garyr@pegasus.cc.ucf.edu

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Two notions of local precompactness in the realm of fuzzy convergence spaces are investigated. It is shown that the property of local precompactness possesses a “good extension.” Moreover, for each given fuzzy convergence space, there exists a coarsest locally precompact space which is finer than the original space. Continuity between fuzzy spaces is preserved between the associated locally precompact spaces. Invariance of regularity with respect to taking locally precompact modifications is discussed.

Key Words: fuzzy convergence; relative compactness; precompactness; local precompactness.

1. INTRODUCTION AND PRELIMINARIES

In the context of fuzzy convergence spaces, Jäger [2] defined and investigated the notion of a fuzzy subset being relatively compact. Our study here is a continuation of the work due to Jäger [2], where the term “relatively compact” is shortened to “precompact.” The importance of
locally compactness in the topological setting is well established, and the purpose here is to extend this concept to the realm of fuzzy convergence. Fuzzy convergence spaces have been shown by Lowen et al. [6] to possess all the desirable categorical properties needed for the study of analysis.

Given a set $X$, let $I^X$ denote the set of all functions from $X$ into $I$ (fuzzy subsets of $X$) equipped with the usual lattice structure. If $A \subseteq X$, then $1_A$ denotes the characteristic function of $A$, and $1_{\{x\}}$ is written simply as $1_x$. Suppose that $f : X \rightarrow Y$ is a function from set $X$ into set $Y$. The image of $\mu \in I^X$ under $f$ is the fuzzy subset of $Y$ defined by $f(\mu)(y) := \sup_{x \in X} 1_y \circ f(x) = \mu(x)$, where $f(\mu)(y) = 0$ when $y$ fails to be in the range of $f$. Further, if $\nu \in I^Y$, the inverse image of $\nu$ under $f$ is the fuzzy subset $f^{-1}(\nu) = \nu \circ f \in I^X$.

**Definition 1.1.** A nonempty collection $\mathcal{N}$ of elements in the lattice $I^X$ is called a *prefilter*, or fuzzy filter, on $X$ provided

1. $1_\emptyset \notin \mathcal{N}$
2. $\mu, \nu \in \mathcal{N}$ implies $\mu \land \nu \in \mathcal{N}$
3. $\nu \in \mathcal{N}$ when $\nu \geq \mu$ and $\mu \in \mathcal{N}$.

Moreover, prefilter $\mathcal{N}$ is said to be *prime* when $\mu \lor \nu \in \mathcal{N}$ implies either $\mu \in \mathcal{N}$ or $\nu \in \mathcal{N}$. The set of all (prime prefilters) prefilters on $X$ is denoted by $(P(X)) P(X)$, respectively, whereas $F(X)$ represents the set of all filters on set $X$. Given $\mathcal{N} \in P(X)$, the set of all minimal prime prefilters containing $\mathcal{N}$, denoted by $P_m(\mathcal{N})$, is fundamental to the development of fuzzy convergence spaces. The existence of members in $P_m(\mathcal{N})$ as well as their representation is given by Lowen [7].

**Definition 1.2.** A base for a prefilter on $X$ is a nonempty subset $\mathcal{B}$ of $I^X$ obeying

1. $1_\emptyset \notin \mathcal{B}$
2. $\mu, \nu \in \mathcal{B}$ implies $\mu \land \nu \geq \delta$ for some $\delta \in \mathcal{B}$.

The prefilter generated by $\mathcal{B}$ is denoted by $[\mathcal{B}] := \{\nu \in I^X : \nu \geq \mu, \mu \in \mathcal{B}\}$; however, $[(\alpha 1_\emptyset)]$ is written simply as $\alpha 1_x$, where $0 < \alpha \leq 1$. Given $\mathcal{F} \in F(X)$, a corresponding prefilter on $X$ is given by $\omega \mathcal{F} := \{1_F : F \in \mathcal{F}\}$. Conversely, if $\mathcal{F} \in P(X)$, then $\mathcal{N} := \{1_\mu : \mu \in \mathcal{N}\}$ is a filter on $X$, where $1_\mu = \{x \in X : \mu(x) > 0\}$. The characteristic value of $\mathcal{N} \in P(X)$ is defined by $c(\mathcal{N}) := \inf_{\mu \in \mathcal{N}} \sup_{x \in X} \mu(x)$.

**Theorem 1.3.** The following relations are valid:

1. $\omega \mathcal{F} = \mathcal{F}$, $\mathcal{F} \in F(X)$
2. $\omega \mathcal{N} \subseteq \mathcal{N}$, $\mathcal{N} \in P(X)$
(c) \( \widehat{\mathcal{H}} \in P_p(x) \) if and only if \( \nu \widehat{\mathcal{H}} \) is an ultrafilter on \( X \).

(d) \( P_n(\widehat{\mathcal{H}}) = (\omega \mathcal{U} \vee \widehat{\mathcal{H}} : \mathcal{U} \supseteq \nu \widehat{\mathcal{H}}, \text{ for some ultrafilter } \mathcal{U} \text{ on } X), \widehat{\mathcal{H}} \in P(X) \).

(e) if for each \( \emptyset \in \mathcal{F}_n \) there exists \( \mu_{\emptyset} \in \emptyset \), then for any finite subcollection, \( \sup_{1 \leq i \leq n} \mu_{\emptyset_i} \in \widehat{\mathcal{H}}, \) where \( \widehat{\mathcal{H}} \in P(X) \).

Proofs of (d) and (e) are more involved and can be found in [7, Theorem 3.2; 8, Lemma 3.2].

**Definition 1.4** [6]. Given a set \( X \), the pair \((X, \lim)\) is called a fuzzy convergence space, where \( \lim : P(X) \to I^X \), provided:

- (PST) \( \lim \widehat{\mathcal{H}} = \inf_{\emptyset \in \mathcal{F}_n(\widehat{\mathcal{H}})} \lim \emptyset, \widehat{\mathcal{H}} \in P(X) \)
- (F1p) \( \lim \widehat{\mathcal{H}} \leq c(\widehat{\mathcal{H}}), \widehat{\mathcal{H}} \in P_p(X) \)
- (F2p) \( \lim \emptyset \leq \lim \widehat{\mathcal{H}} \) when \( \widehat{\mathcal{H}} \subseteq \emptyset \), \( \widehat{\mathcal{H}}, \emptyset \in P_p(X) \)
- (C1) \( \lim \alpha \mathbf{1}_x \geq \alpha \mathbf{1}_x \), for \( 0 < \alpha \leq 1 \) and \( x \in X \).

A mapping \( f : (X, \lim_X) \to (Y, \lim_Y) \) between two fuzzy convergence spaces is called continuous when \( \lim_X \widehat{\mathcal{H}}(x) \leq \lim_Y f(\widehat{\mathcal{H}}(x)) \) for each \( x \in X \) and \( \widehat{\mathcal{H}} \in P(X) \) (equivalently, \( \widehat{\mathcal{H}} \in P_p(X) \)), where \( f \widehat{\mathcal{H}} \) denotes the prefilter \([f(\mu) : \mu \in \widehat{\mathcal{H}}]\)). Given fuzzy convergence spaces \((X, \lim_1)\) and \((X, \lim_2)\), \((X, \lim_1)\) is defined to be finer than \((X, \lim_2)\) or \((X, \lim_2)\) is coarser than \((X, \lim_1)\), provided the identity map \( 1_X : (X, \lim_1) \to (X, \lim_2) \) is continuous; equivalently, \( \lim_1 \emptyset \leq \lim_2 \widehat{\mathcal{H}} \) for each \( \widehat{\mathcal{H}} \in P_p(X) \). Moreover, \((X, \lim)\) is called \( T_2 \) if for each \( \widehat{\mathcal{H}} \in P_p(X) \), \( \lim \widehat{\mathcal{H}}(x) > 0 \) for at most one \( x \in X \), and it is regular \((\omega\text{-regular})\) provided \( \lim \widehat{\mathcal{H}} \geq \lim \widehat{\mathcal{H}} \) \((\lim \omega \widehat{\mathcal{H}} \geq \lim \omega \widehat{\mathcal{H}})\) for each \( \widehat{\mathcal{H}} \in P_p(X) \), respectively, where \( \widehat{\mathcal{H}} = [\mu : \mu \in \widehat{\mathcal{H}}] \) and \( \mu(x) = \sup(\lim \emptyset(x) : \emptyset \in \emptyset) \). X \). The above two notions of regularity in FCS were introduced by Minkler et al., where “weakly regular” is shortened to “\( \omega \text{-regular} \)” here. The category whose objects consist of all the fuzzy convergence spaces and whose morphisms are the continuous maps between objects is denoted by FCS. The class of all objects is denoted by [FCS]. It is shown [6, Theorem 3.11] that the category FTS of fuzzy topological spaces is a bireflective subcategory of FCS.

**Definition 1.5.** A convergence space is a pair \((X, q)\), where \( X \) is a set and \( q : F(X) \to 2^X \) satisfies:

- (CS1) \( x \in q(\mathcal{F}) \) for each \( x \in X \), where \( \mathcal{F} \) is the ultrafilter containing \( x \)
- (CS2) \( q(\mathcal{F}) \subseteq q(\mathcal{G}) \) when \( \mathcal{F} \subseteq \mathcal{G} \)
- (CS3) \( q(\mathcal{F}) = \bigcap\{q(\mathcal{G}) : \mathcal{F} \subseteq \mathcal{G} \} \) is an ultrafilter containing \( \mathcal{F} \).
Axiom (CS3) is stronger than normally required of a convergence structure but is suited for our purposes. The above definition is sometimes referred to as a Choquet, or pseudotopological, convergence space. Quite often \( x \in q(F) \) is denoted by \( F \xrightarrow{q} x \), and is interpreted as “\( F \)-converges to \( x \).” A map \( f : (X, q) \to (Y, p) \) between two convergence spaces is said to be continuous if \( F \xrightarrow{q} x \) implies \( fF \xrightarrow{p} f(x) \). The category whose objects are all the convergence spaces and whose morphisms are the continuous maps between convergence spaces is denoted by \( \text{CONV} \).

Given \((X, q), (X, p) \in |\text{CONV}|\), define \( q \leq p \) when \( F \xrightarrow{q} x \) implies that \( F \xrightarrow{p} x \). Then the order \( \leq \) determines a complete lattice on the set of all convergence structures for \( X \); indeed, \( F \xrightarrow{q} x \) if and only if \( \sup_{\alpha \in \Lambda} q_\alpha(x) \) for each \( \alpha \in \Lambda \). A subset \( A \) of \((X, q)\) is called precompact when each ultrafilter containing \( A \) \( q \)-converges to some \( x \in X \). Moreover, \((X, q)\) is said to be locally precompact provided each convergent filter contains a precompact member. This condition is also equivalent to the requirement that each convergent ultrafilter contains a precompact element.

A convergence space \((X, q)\) is called regular when \( F \xrightarrow{q} x \) implies \( \sup_{\alpha \in \Lambda} q_\alpha(x) = 1 \), where \( F \) denotes the filter with base \( \{F : F \in \mathcal{F}\} \), and \( F = \{x \in X : \mathcal{F} \xrightarrow{q} x \} \) and \( F \in \mathcal{F} \) for some \( F \in \mathcal{F}(X) \). The following result is needed later.

**Theorem 1.6** [5]. An object \((X, q) \in |\text{CONV}|\) is regular provided \( F \xrightarrow{q} x \) when \( F \xrightarrow{q} x \), for each ultrafilter \( F \) on \( X \).

### 2. LOCAL PRECOMPACTNESS

Basic properties of local precompactness are developed in this section. Jäger [2] uses the term “relatively compact” rather than “precompact.”

**Definition 2.1.** A fuzzy subset \( \nu \in I^X \) of \((X, \text{lim}) \in [\text{FCS}]\) is said to be precompact (\( \omega \)-precompact) if \( \nu \in \mathfrak{I} \) implies that \( \sup_{x \in X} \lim \mathfrak{I}(x) = c(F) \) (\( \sup_{x \in X} \lim_\omega \mathfrak{I}(x) = 1 \), for each \( \mathfrak{I} \in \mathfrak{P}_p(X) \), respectively. Further, \((X, \text{lim})\) is called locally precompact (locally \( \omega \)-precompact) provided that each \( \mathfrak{I} \in \mathfrak{P}_p(X) \) contains a precompact (\( \omega \)-precompact) element when \( \lim \mathfrak{I} \neq 1_\mathfrak{I} \), respectively.

Equivalent forms of local precompactness (local \( \omega \)-precompactness) are listed below.

**Theorem 2.2.** Let \((X, \text{lim}) \in [\text{FCS}]\). Then TFAE:

(a) \((X, \text{lim})\) is locally precompact (locally \( \omega \)-precompact)

(b) each \( \mathfrak{I} \in \mathfrak{P}(X) \) contains a precompact (\( \omega \)-precompact) member provided \( \lim \mathfrak{I} \neq 1_\mathfrak{I} \).
(c) each \( \omega \mathcal{U} \) contains a precompact element when \( \mathcal{U} \) is an ultrafilter on \( X \) and \( \lim \omega \mathcal{U} \neq 1_\mathcal{A} \).

Proof. (a) \( \Rightarrow \) (b) Assume that \( \widehat{\mathcal{N}} \in P(X) \) satisfies \( \lim \widehat{\mathcal{N}} \neq 1_\mathcal{A} \). By Definition 1.4 (PST), \( \lim \widehat{\mathcal{N}} = \inf_{\mathcal{B} \in \mathcal{P}_s} \lim \mathcal{B} \) and thus \( \lim \mathcal{B} \neq 1_\mathcal{A} \) for each \( \mathcal{B} \in \mathcal{P}_s(\widehat{\mathcal{N}}) \). Then each \( \mathcal{B} \in \mathcal{P}_s(\widehat{\mathcal{N}}) \) contains a precompact member \( \nu_{\mathcal{B}} \) and thus by Theorem 1.3 (c), \( \nu = \sup_{1 \leq i \leq n} \nu_{\mathcal{B}_i} \in \widehat{\mathcal{N}} \) for some finite subcollection. If \( \widehat{\mathcal{S}} \in P_p(X) \) contains \( \nu \), then since \( \widehat{\mathcal{S}} \) is prime, \( \nu_{\mathcal{B}_i} \in \widehat{\mathcal{S}} \) for some \( i \), and thus \( \sup_{x \in X} \lim \mathcal{S}(x) = c(\widehat{\mathcal{S}}) \). Hence \( \nu \) is precompact.

(b) \( \Rightarrow \) (c) Clear.

(c) \( \Rightarrow \) (a) Let \( \widehat{\mathcal{N}} \in P_p(X) \) such that \( \lim (\mathcal{B} \neq 1_\mathcal{A} \cdot \lim \mathcal{B} \mathcal{N} \subseteq \widehat{\mathcal{N}} \mathcal{N} \). Then each \( \mathcal{B} \in \mathcal{P}_s(\widehat{\mathcal{N}}) \) contains a precompact member \( \nu_{\mathcal{B}} \) and thus by Theorem 1.3 (c), \( \nu = \sup_{1 \leq i \leq n} \nu_{\mathcal{B}_i} \in \widehat{\mathcal{N}} \) for some finite subcollection. If \( \mathcal{S} \in P_p(X) \) contains \( \nu \), then since \( \mathcal{S} \) is prime, \( \nu_{\mathcal{B}_i} \in \mathcal{S} \) for some \( i \), and thus \( \sup_{x \in X} \lim \mathcal{S}(x) = c(\mathcal{S}) \). Hence \( \nu \) is precompact.

Only minor changes above are needed to verify the \( \omega \)-precompact case.

Let \( (X, q) \in \mathcal{CONV} \), and define \( \lim_q : P(X) \rightarrow I^X \) as follows:

(a) \( \lim_q \widehat{\mathcal{N}} = c(\widehat{\mathcal{N}}) \cdot 1_{\{x \in X : q(x) \}} \), when \( \widehat{\mathcal{N}} \in P_p(X) \)

(b) (PST) otherwise (see Definition 1.4).

It is easily checked that \( (X, \lim_q) \in \mathcal{FCS} \); moreover, Lowen et al. [6, Theorem 3.13] show that \( F(X, q) = (X, \lim_q) \), \( F(f) = f \), defines a functor that embeds \( \mathcal{CONV} \) simultaneously as a coreflective and bireflective subcategory of \( \mathcal{FCS} \). A property \( P \) is said to have a good extension if \( (X, q \in \mathcal{CONV} \) possesses property \( P \) iff \( (X, \lim_q) \in \mathcal{FCS} \) possesses property \( P \).

**Theorem 2.3.** (a) A locally precompact object in \( \mathcal{FCS} \) is locally \( \omega \)-precompact.

(b) Local precompactness possesses a good extension.

(c) \( (X, q) \in \mathcal{CONV} \) is locally precompact iff \( (X, \lim_q) \) is locally \( \omega \)-precompact.

Proof. (a) Assume that \( (X, \lim) \in \mathcal{FCS} \) is locally precompact and \( \widehat{\mathcal{N}} \in P_p(X) \) such that \( \lim \widehat{\mathcal{N}} \neq 1_\mathcal{A} \); then \( \omega \mathcal{N} \neq 1_\mathcal{A} \) and thus there exists \( 1_\mathcal{A} \in \omega \mathcal{N} \subseteq \widehat{\mathcal{N}} \) for which \( 1_\mathcal{A} \) is precompact. It easily follows that \( 1_\mathcal{A} \) is also \( \omega \)-precompact and thus \( (X, \lim) \) is locally \( \omega \)-precompact.

(b) Suppose that \( (X, q) \in \mathcal{CONV} \) is locally precompact and \( \mathcal{U} \) is an ultrafilter on \( X \) satisfying \( \lim_q \omega \mathcal{U} \neq 1_\mathcal{A} \). It follows that \( \lim \omega \mathcal{U} = 1_{\{x : \omega \mathcal{U} \neq 1_\mathcal{A} \}} = 1_{\{x : \omega \mathcal{U} q x \}}, \) and thus \( \mathcal{U} \rightarrow x_0 \) for some \( x_0 \). Then \( \mathcal{U} \) contains a
q-precompact element $U$ and hence $1_U \in \omega\mathcal{U}$ is \(q\)-precompact. By Theorem 2.2 (c), \((X, \lim_q)\) is locally precompact.

Conversely, assume that \((X, \lim_q)\) is locally precompact and \(\mathcal{U}\) is an ultrafilter on \(X\) such that \(\mathcal{U} \to x_0\). Then \(\lim_q \omega\mathcal{U}(x_0) = 1\) and thus there exists a \(\lim_q\)-precompact element \(1_U\) for some \(U \in \mathcal{U}\). Hence \(U\) is \(q\)-precompact and thus \((X, q)\) is locally precompact.

(c) The “only if” part follows from (a) and (b) above, whereas the “if” part proceeds as in part (b).

The converse of Theorem 2.3 (a) is shown to be false in Example 3.6 below.

3. LOCALLY PRECOMPACT COREFLECTIONS

Let \((X, \lim) \in |\text{FCS}|\); it is shown in Theorem 3.2 below that there exists a coarsest locally precompact (locally \(\omega\)-precompact) object in FCS which is finer than \((X, \lim)\) called the \textit{locally precompact (locally \(\omega\)-precompact) modification} of \((X, \lim)\), respectively. In preparation, given \((X, q) \in |\text{CONV}|\), define

\[
\mathcal{F} \to X \quad \text{iff } \mathcal{F} \to X \quad \text{and } \mathcal{F} \text{ contains a } q\text{-precompact element.}
\]

It can be shown that \((X, q^*)\) is the coarsest locally precompact object in CONV which is finer than \((X, q)\). Let \(F: \text{CONV} \to \text{FCS}\) denote the embedding functor defined prior to Theorem 2.3.

**Theorem 3.1.** If \((X, q) \in |\text{CONV}|\), then \((X, \lim_{q^*})\) is the coarsest locally precompact (locally \(\omega\)-precompact) object in FCS which is finer than \((X, \lim_q)\) and, moreover, the full subcategory of FCS consisting of all objects of the form \((X, \lim_{q^*})\), where \((X, q) \in |\text{CONV}|\), is coreflective in \(F(\text{CONV})\).

**Proof.** Since \((X, q^*)\) is a locally precompact object in CONV, it follows from Theorem 2.3 (b) that \((X, \lim_{q^*})\) is a locally precompact object in FCS. Assume that \((X, \lim)\) is any locally precompact object in FCS satisfying \(\lim \leq \lim_{q^*}\); it remains to show that \(\lim \leq \lim_{q^*}\). Suppose that \(\hat{\alpha} \in P_\omega(X)\) such that \(\lim \hat{\alpha}(x_0) > 0\); then \(\omega\alpha \hat{\alpha} \geq \lim \hat{\alpha} \neq 1_\emptyset\) and thus \(\omega\alpha \hat{\alpha}\) contains a \(\lim\)-precompact element \(1_A\), for some \(A \in \omega \hat{\alpha}\). It follows that \(1_A\) is also \(\lim_{q^*}\)-precompact, and thus \(A\) is \(q\)-precompact. Moreover, \(\lim \hat{\alpha}(x_0)\)
> 0 implies that \( \lim_{q} x_0 > 0 \) and thus \( \iota_{\mathcal{N}}^{<q} \rightarrow x_0 \). Since \( A \in \iota_{\mathcal{N}} \) is \( q \)-precompact, \( \iota_{\mathcal{N}}^{<q} \rightarrow x_0 \) and hence \( \lim_{q} \mathcal{N}(x_0) = c(\mathcal{N}) \). Therefore \( \lim \mathcal{N}(x_0) \leq \lim_{q} \mathcal{N}(x_0) \) and thus \( \lim \leq \lim_{q} \). Also, it follows from Theorem 2.3 (c) that \((X, \lim_{q})\) is locally \( \omega \)-precompact. Likewise, it is the coarsest such object in FCS which is finer than \((X, \lim_{q})\). The coreflective part follows immediately.

Theorem 3.2 below extends Theorem 3.1 from objects of the form \((X, \lim_{q})\) to all of FCS. To this end, define

\[
\lim^* \mathcal{N} = \begin{cases} 
\lim \mathcal{N}, & \text{provided each } \emptyset \in P_p(X), \emptyset \subseteq \mathcal{N}, \\
1_\emptyset, & \text{otherwise}
\end{cases}
\]

when \( \mathcal{N} \in P_p(X) \), and by (PST) otherwise

\[
\lim_* \mathcal{N} = \begin{cases} 
\lim \mathcal{N}, & \text{when each } \emptyset \in P_p(X), \emptyset \subseteq \mathcal{N}, \\
1_\emptyset, & \text{otherwise}
\end{cases}
\]

provided \( \mathcal{N} \in P_p(X) \), and by (PST) elsewhere.

Let LPFCS (L\( \omega \)PFCS) denote the full subcategory of FCS whose objects are locally precompact (locally \( \omega \)-precompact), respectively.

**Theorem 3.2.** Let \((X, \lim) \in \text{FCS}\); then \((X, \lim^*) ((X, \lim_*))\) is its locally precompact (locally \( \omega \)-precompact) modification, respectively, and more importantly, LPFCS (L\( \omega \)PFCS) is a coreflective subcategory of FCS.

**Proof.** It is shown here that \((X, \lim^*)\) is the locally precompact modification of \((X, \lim)\); details involving \((X, \lim_* )\) are similar and thus omitted. It is straightforward to verify that \((X, \lim^*) \in \text{FCS}\). First, let us show that \((X, \lim^*) \in \text{LPFCS}\). Assume that \( \mathcal{N} \in P_p(X) \) such that \( \lim^* \mathcal{N} \neq 1_\emptyset \); then \( \lim^* \omega \iota \mathcal{N} \neq 1_\emptyset \) and thus \( \lim^* \omega \iota \mathcal{N} = \lim \omega \iota \mathcal{N} \). Since \( \omega \iota \mathcal{N} \in P_p(X) \), there exists a \( \lim \)-precompact element \( 1_A \in \omega \iota \mathcal{N} \), where \( A \in \iota \mathcal{N} \). Moreover, \( 1_A \) is \( \lim^* \)-precompact. Indeed, suppose that \( 1_A \in \mathcal{N} \in P_p(X) \). Since \( 1_A \) is \( \lim^* \)-precompact, \( \sup_{x \in X} \lim \mathcal{N}(x) = c(\mathcal{N}) \). It remains to verify that \( \lim^* \mathcal{N} = \lim \mathcal{N} \). Let \( \emptyset \subseteq \mathcal{N}, \emptyset \in P_p(X) \); then \( \iota(\emptyset) = \iota \mathcal{N} \) and thus \( 1_A \in \omega \iota \mathcal{N} = \omega \iota(\emptyset) \subseteq (\emptyset) \). It follows from the definition that \( \lim^* \mathcal{N} = \lim \mathcal{N} \), and hence \( \sup_{x \in X} \lim^* \mathcal{N} = c(\mathcal{N}) \). Therefore \( 1_A \in \omega \iota \mathcal{N} \subseteq \mathcal{N} \) is \( \lim^* \)-precompact and thus \((X, \lim^*) \in \text{LPFCS}\).
Since \( \lim^* \leq \lim \), the identity map \( 1_{X'} : (X, \lim^*) \to (X, \lim) \) is continuous. The remainder of the theorem will follow by proving that if \( f : (Y, \lim^*_Y) \to (X, \lim) \) is continuous, then \( f : (Y, \lim^*_Y) \to (X, \lim^*) \) is also continuous. Assume that \( \widehat{\gamma} \in P_p(Y) \) and \( y_0 \in Y \) for which \( \lim^*_Y \widehat{\gamma}(y_0) > 0 \); it remains to show that \( \lim^*_Y \widehat{\gamma}(y_0) \leq \lim^* f \widehat{\gamma}(f(y_0)) \). Since \( \lim^*_Y \widehat{\gamma} \neq 1_{A}, \lim^*_Y \widehat{\gamma} = \lim_Y \widehat{\gamma} \) and thus by hypothesis, \( \lim^*_Y \widehat{\gamma}(y_0) \leq \lim f \widehat{\gamma}(f(y_0)) \); hence, it suffices to show that \( \lim^* f \widehat{\gamma}(f(y_0)) = \lim f \widehat{\gamma}(f(y_0)) \). Suppose that \( \delta \in P_p(X) \) such that \( \delta \subseteq f \widehat{\gamma} \); then \( f^{-1} \delta \subseteq \widehat{\gamma} \) and, using Theorem 1.3 (d), it is straightforward to show that there exists \( \delta_l \in P_m(f^{-1} \delta) \) such that \( \delta_l \subseteq \widehat{\gamma} \). It follows from [5, Lemma 1.2] that \( f \delta_l = \delta_l \). Since \( \delta_l \subseteq \widehat{\gamma} \) and \( \lim^*_Y \widehat{\gamma} = \lim_Y \widehat{\gamma} \), \( \delta_l \) contains a \( \lim_Y \) precompact element \( \nu \). According to [2, Proposition 2.8], \( f(\nu) \in \delta \) is \( \lim \) precompact and thus by definition, \( \lim^* f \widehat{\gamma} = \lim f \widehat{\gamma} \). Hence \( f : (Y, \lim^*_Y) \to (X, \lim^*) \) is continuous.

Example 3.6 below shows that, in general, the two modifications of Theorem 3.2 are distinct. Given \( (X, \lim) \in \text{FCS} \), define

\[
\mathcal{T} \lim_{q} X \quad \text{iff for each ultrafilter } \mathcal{U} \supseteq \mathcal{T}, \lim \omega \mathcal{U}(x) = 1.
\]

It is straightforward to show that \( (X, q_{\lim}) \in \text{CONV} \). Basic properties needed for further development are listed below.

**Lemma 3.3.** Assume that \( (X, \lim) \in \text{FCS}, (X, q_{\lim}) \) is as defined above, and \( A \subseteq X \). Then

(a) \( 1_A \) is \( \omega \)-precompact and \( A \) is \( q_{\lim} \)-precompact

(b) \( A \) is \( q_{\lim} \)-precompact when \( 1_A \) is \( \omega \)-precompact and \( (X, \lim) \) is \( T_2 \)

(c) \( 1_{\mathcal{T}_{\lim}} \leq 1_A \) provided \( A \) is \( q_{\lim} \)-precompact and, moreover, equality is valid when \( (X, \lim) \) is \( T_2 \)

(d) \( \lim^* \widehat{\gamma} = \lim \widehat{\gamma} \) when \( 1_A \in \widehat{\gamma} \in P_p(X) \) is \( \omega \)-precompact

(e) \( \overline{\lim}^* = \overline{\lim} \) when \( 1_A \in \widehat{\gamma} \in P_p(X) \) is \( \omega \)-precompact, where \("*" denotes the closure in \((X, \lim^*)\)).

**Proof.** (c) Let \( \nu \in \widehat{\gamma} \) for which \( \nu \leq 1_A \). Then, using (d),

\[
\overline{\nu}^* = \sup \{ \lim^* (\delta) : \nu \in \delta \in P_p(X) \} = \sup \{ \lim (\delta) : \nu \in \delta \in P_p(X) \} = \overline{\nu}.
\]

Since \( \{ \nu \in \widehat{\gamma} : \nu \leq 1_A \} \) is a base for \( \widehat{\gamma} \), it follows that \( \overline{\lim}^* = \overline{\lim} \) when \( 1_A \in \widehat{\gamma} \) is \( \omega \)-precompact.

Next, the question of invariance of regularity in FCS under the process of taking locally precompact (locally \( \omega \)-precompact) modifications is considered. As mentioned in Section 1, the notions of regularity and \( \omega \)-regu-
larity in the category FCS have been investigated by Minkler et al. [9], where it is shown that regularity implies \( \omega \)-regularity. Moreover, the following result is needed.

**Lemma 3.4** [9]. Let \( (X, \text{lim}) \in [\text{FCS}] \) be \( \omega \)-regular, then \( (X, q_{\text{lim}}) \) is a regular convergence space.

**Theorem 3.5.** Assume that \( (X, \text{lim}) \in [\text{FCS}] \) is \( T_2 \) and \( \omega \)-regular. Then its locally \( \omega \)-precompact modification \( (X, \text{lim}_p) \) is also \( T_2 \) and \( \omega \)-regular.

**Proof.** Verification involves the use of properties of the associated convergence space \( (X, q_{\text{lim}}) \) listed in Lemmas 3.3 and 3.4. For the sake of brevity, denote \( p = q_{\text{lim}} \). Clearly \( (X, \text{lim}^*) \) is \( T_2 \) since \( \text{lim}^* \leq \text{lim} \). Assume that \( \bar{\gamma} \in P_p(X) \); it must be shown that \( \text{lim}^* \omega \text{lim}^* \bar{\gamma} \geq \text{lim}^* \omega \gamma \). Suppose that \( \text{lim}^* \omega \gamma \neq 1_{\bar{\gamma}} \); then \( \text{lim}^* \omega \gamma = \text{lim} \omega \gamma \) and thus there exists an \( \omega \)-precompact element \( 1_{\gamma} \in \omega \gamma \subseteq \gamma \). It follows from Lemma 3.3 that \( \bar{\gamma} \in P_p(\omega \gamma) \); then \( \bar{\gamma} \in \gamma \) and thus \( 1_{\gamma} \in \omega \gamma \subseteq \gamma \). By Lemmas 3.3 (b) and 3.4, \( A \) is \( p \)-precompact and \( (X, p) \) is regular. It follows from the regularity of \( (X, p) \) and [3, Lemma 2.1] that \( A^p \) is also \( p \)-precompact. Then by Lemma 3.3, \( 1_{\gamma} \) is \( \omega \)-precompact and \( \text{lim}^* \omega \gamma = \text{lim} \omega \gamma \). Hence \( \text{lim}^* \omega \gamma \geq \text{lim} \omega \gamma \) since \( (X, \text{lim}) \) is \( \omega \)-regular. Therefore \( (X, \text{lim}^*) \) is \( \omega \)-regular. \( \blacksquare \)

An elementary example of a locally \( \omega \)-precompact object in FCS which fails to be locally precompact is given below; moreover, its locally precompact modification is found.

**Example 3.6.** Let \( X \) denote an infinite set and define \( \text{lim}: P(X) \to I^X \) as follows:

\[
\text{lim} \bar{\gamma} = \begin{cases} 
\{ c(\bar{\gamma}) \}, & \text{if } c(\bar{\gamma}) = 1 \\
\frac{c(\bar{\gamma})}{2}, & \text{if } c(\bar{\gamma}) < 1
\end{cases}
\]

provided \( \iota \bar{\gamma} \) is a free ultrafilter

(b) \( \text{lim} \bar{\gamma} = c(\bar{\gamma}) \cdot 1_x \) when \( \iota \bar{\gamma} = x \)

(c) (PST) otherwise.

Recall from Theorem 1.3 (c) that \( \bar{\gamma} \in P_p(X) \) iff \( \iota \bar{\gamma} \) is an ultrafilter. It is straightforward to show that \( (X, \text{lim}) \in [\text{FCS}] \). In order to show that \( (X, \text{lim}) \) is locally \( \omega \)-precompact, it suffices to verify that \( 1_x \) is \( \omega \)-precompact. To this end, let \( 1_x \in \bar{\gamma} \in P_p(X) \). Suppose that \( \iota \bar{\gamma} \) is a free ultrafilter; then \( \text{lim} \omega \bar{\gamma} = c(\omega \bar{\gamma}) = 1 \). If \( \iota \bar{\gamma} = x \), then \( \omega \bar{\gamma} = 1_x \) and \( \text{lim} \omega \bar{\gamma} \)
= 1_\xi. In either case, \( \sup_{y \in X} \lim \omega \hat{\mathcal{H}}(y) = 1 \), and thus 1_\xi is \( \omega \)-precompact.

It is shown that \((X, \lim)\) fails to be locally precompact. Let \( \mathcal{U} \) be a free ultrafilter on \( X \) and define \( \hat{\mathcal{H}} \) to be the prefilter on \( X \) whose base is \( \{ \frac{1}{2} 1_U : U \in \mathcal{U} \} \). Since \( \mathcal{H} = \mathcal{U} \), according to (a), \( \lim \hat{\mathcal{H}} = \frac{c(\mathcal{U})}{2} = \frac{1}{2} \). It follows that \( \mathcal{H} \) fails to contain a precompact element, and thus \((X, \lim)\) is not locally precompact.

Next, the locally precompact modification is found. Assume that \( \mathcal{H} \in P_\mathcal{F}(X) \) such that \( \mathcal{H} \) is a free ultrafilter on \( X \) and \( c(\mathcal{H}) = 1 \); then \( \lim \mathcal{H} = c(\mathcal{H}) \). Let \( \nu \in \mathcal{H} \) and define \( \mathcal{G} \) to be the prefilter on \( X \) whose base is \( \{ \mu \wedge \frac{1}{2} : \mu \in \mathcal{H} \} \); then \( \mathcal{G} = \nu \mathcal{H} \) and thus \( \nu \in P_\mathcal{F}(X) \). Clearly \( c(\mathcal{G}) \leq \frac{1}{2} \), and thus \( \lim \mathcal{G} = \frac{c(\mathcal{G})}{2} \). Then \( \sup_{x \in X} \lim \mathcal{G}(x) < c(\mathcal{G}) \), \( \nu \in \mathcal{G} \), and thus \( \nu \) is not precompact. Hence \( \lim^* \mathcal{H} = 1_\phi \). In case \( \mathcal{H} = \hat{x} \), \( \mathcal{G} \in P_\mathcal{F}(X) \), and \( \mathcal{G} \subseteq \mathcal{H} \), it follows that \( \mathcal{G} = \mathcal{H} = \hat{x} \) and thus \( 1_x \in \mathcal{G} \). Since \( 1_x \) is precompact, it follows that \( \lim^* \mathcal{H} = \lim \mathcal{H} = c(\mathcal{H})1_x \). Hence the locally precompact modification of \((X, \lim)\) is given by

\[
\lim^* \mathcal{H} = \begin{cases} 1_\phi, & \text{if } \mathcal{H} \text{ is a free ultrafilter} \\ c(\mathcal{H})1_x, & \text{if } \mathcal{H} = \hat{x} \end{cases}
\]

and by (PST) otherwise.

4. CONCLUDING REMARKS

The authors were unable to prove Theorem 3.5 without use of the \( T_2 \) axiom. Moreover, is Theorem 3.5 valid when the assumption of \( \omega \)-regularity is replaced by regularity?

LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CONV</td>
<td>category of convergence spaces (Choquet or pseudotopological convergence spaces)</td>
</tr>
<tr>
<td>FCS</td>
<td>category of fuzzy convergence spaces</td>
</tr>
<tr>
<td>LPFCS</td>
<td>category of locally precompact spaces</td>
</tr>
<tr>
<td>L_\omega PFCS</td>
<td>category of locally ( \omega )-precompact spaces</td>
</tr>
<tr>
<td>\mathcal{G}, \mathcal{F}</td>
<td>filters</td>
</tr>
<tr>
<td>\mathcal{H}, \mathcal{G}</td>
<td>prefilters (fuzzy filters)</td>
</tr>
<tr>
<td>F(X)</td>
<td>set of all filters on ( X )</td>
</tr>
</tbody>
</table>
\( P(X) \) set of all prefilters on \( X \)
\( P_p(X) \) set of all prime prefilters on \( X \)
\( P_m(\mathfrak{F}) \) set of all minimal prime prefilters finer than \( \mathfrak{F} \)
\( I^X \) set of all fuzzy subsets of \( X \)
\( \overline{\mu} \) closure of \( \mu \)
\( \mathfrak{F} \) prefilter whose base is \( \{ \overline{\mu} : \mu \in \mathfrak{F} \} \)
\( \tau_\mu \) \( \{ x : \mu(x) > 0 \} \)
\( \iota_\mathfrak{F} \) filter \( \{ \tau_\mu : \mu \in \mathfrak{F} \} \)
\( 1_F \) characteristic function of \( F \)
\( \omega \mathfrak{F} \) prefilter whose base is \( \{ 1_F : F \in \mathfrak{F} \} \)
\( c(\mathfrak{F}) \) \( \inf_{\mu \in \mathfrak{F}} \sup_{x \in X} \mu(x) \), characteristic value of \( \mathfrak{F} \)

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REFERENCES