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On Finite Groups with $\mathfrak{F}$-Sets

UDO PREISER

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Let $\mathfrak{F}$ be an isomorphism class of groups and G a finite group. Following Fischer, an $\mathfrak{F}$-set of G is a collection Ω of subgroups such that Ω is normalized by G, generates G, and such that any pair of distinct members of Ω generates a subgroup in $\mathfrak{F}$.

The case that Ω is a conjugate class of subgroups of order 2 has been studied by Aschbacher, Fischer, and Timmesfeld. Also, various results have been obtained in the case that Ω is a conjugate class of subgroups of order p, p an odd prime.

The purpose of this paper is to prove the following:

THEOREM. *Let G be a finite group with Ω a conjugate class of subgroups of order* 11 *such that Ω is a $\{PSL_2(11), J_1\}$-set of G. Then G is isomorphic to $PSL_2(11)$ or J_1.*

J_1 denotes the simple group of order 175, 560 given by Janko [6].

1. NOTATION

The notation is standard and taken from [5]. Moreover the reader is assumed to be familiar with the properties of $PGL_2(11)$ and J_1 (see [5] and [6]).

2. PRELIMINARY RESULTS

LEMMA 2.1. *Let G be a simple group, and assume that G contains a nontrivial Abelian 2-subgroup which is strongly closed in a Sylow 2-subgroup of G. Then G is one of the following:*

(i) $PSL_2(q)$, $q \equiv 0, 3, 5 \pmod 8$;

(ii) $Sz(2^{2n+1})$;

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(iii) $PSU_3(2^n)$;

(iv) *a group of Janko–Ree type.*

Proof. See [3].

LEMMA 2.2. *Let G be* $PGL_2(11)$ *or* J_1. *Then a Sylow-11-normalizer is Frobenius of order* $11 \cdot 10$. *An element of order 5 is centralized by exactly one involution; in particular, a Sylow 11-subgroup of G is normalized by each involution which centralizes an element of order 5 in the normalizer of this Sylow 11-subgroup.*

Proof. See [5] and [6].

LEMMA 2.3. *Let G be a finite group and P a Sylow p-subgroup of G. Denote by* $\mathfrak{I}$ *the family of all pairs* $(H, N_G(H))$ *where H is a subgroup of P such that there exists a Sylow p-subgroup Q of G with* $H = P \cap Q$ *a tame intersection and with* $C_P(H) \leqslant H$. *Then* $\mathfrak{I}$ *is a weak conjugation family.*

Proof. See [1].

LEMMA 2.4. *Let G be a finite group, P a Sylow p-subgroup of G, and S a strongly closed p-subgroup contained in P. Then*

$$(G' \cap P)S = (N_G(S)' \cap P)S.$$

Proof. See [4].

3. PROOF OF THE THEOREM

Let G be a minimal counterexample to the Theorem. Then G is simple (see [7]). If any pair of distinct members of Ω generates a subgroup isomorphic to $PSL_2(11)$, then G must be $PSL_2(11)$ (see [2]). Hence, we may assume there are distinct members A and B of Ω with $\langle A, B\rangle \cong J_1$. Consequently, the set D of all involutions of G which act as inversion on some element of Ω is not empty moreover G contains an elementary Abelian subgroup of order 8 whose involutions belong to D. We first note:

3.1. Let y be an element of G centralizing some member A of Ω. Then y centralizes each member of Ω normalized by y.

Proof. Assume there is some member B in Ω which is normalized but not centralized by y. Then y acts nontrivially on $\langle A, B\rangle$, contradicting 2.2.

As a consequence we have:

3.2. If y has order 4, then y^2 does not belong to D.

Proof. If not, pick a member A of Ω with $A^y = A$. By 3.1 y^2 does not centralize A and $A^y \neq A$. Then y acts nontrivially on $\langle A, A^y \rangle$, contradicting 2.2.

The same kind of argument yields:

3.3. Let A be a member of Ω and y an element of order 5 acting nontrivially on A. Then each involution in $C_G(y)$ fixes A.

3.4. Let A be a member of Ω. Then $C_G(A)$ has odd order.

Proof. Assume the order of $C_G(A)$ even; we try to derive a contradiction.

3.4.1. Let i be an involution in $C_G(A)$. Then there exists a member B of Ω distinct from A and fixed by i.

Proof. Otherwise $|\Omega|$ is odd and $N_G(A)$ contains a Sylow 2-subgroup of G. We remarked that Ω contains distinct members A and B with $\langle A, B \rangle \cong J_1$. Hence G contains a four group V whose involutions belong to D. We may assume $V \leqslant N_G(A)$. But then $V \cap C_G(A) \neq \langle 1 \rangle$, contradicting 3.1.

Part 3.4.1 yields the existence of a simple subgroup K of G generated by members of Ω such that $|C_G(K)|$ is even. Among the subgroups with these properties we choose K such that $|K|$ is as large as possible. Let $H = N_G(K)$, $C = C_G(K)$, S a Sylow 2-subgroup of H, $S_1 = S \cap C$, and $S_2 = S \cap K$.

3.4.2. $N_G(C) = H$ is a maximal subgroup of G. If $g \in G \backslash H$ then $C \cap C^g$ has odd order.

Proof. Assume H is properly contained in a maximal subgroup M of G. Let $L = \langle K^M \rangle$ be the normal closure of K in M. Then L is well known by minimality of G. Since K is a proper subgroup of L we conclude that L is isomorphic to J_1 and $C_G(L)$ has odd order. Consequently an involution from S_1 acts on L and centralizes $K \cong PSL_2(11)$, a contradiction. Hence H is a maximal subgroup of G and $N_G(C) = H$. The choice of K implies $|C \cap C^g|$ odd for $g \in G \backslash H$.

3.4.3. The order of S_2 is 4; the order of S is 2^4 or 2^5. S does not contain a four group whose involutions belong to D.

Proof. Suppose K is isomorphic to J_1. Then $S = S_1 \times S_2$. From 3.1 S_1 is strongly closed in S. Hence it follows from 3.4.2 that S is a Sylow 2-subgroup of G. If Q is a subgroup of S with $C_S(Q) \leqslant Q$, then $S_1 \cap Q$ is a nontrivial, in Q strongly closed subgroup of Q and $N_G(Q) \leqslant N_G(S_1 \cap Q) \leqslant H$. Hence H contains a weak conjugation family (see 2.3). Therefore H controls fusion in S; in particular, S_2 is an Abelian strongly closed 2-subgroup of G. This contradicts 2.1. We conclude that K is isomorphic to $PSL_2(11)$, and S_2 has order 4.

Now pick a member A of $\Omega \cap K$. We know that S_1 is a Sylow 2-subgroup of $C_H(A)$. If R is a 2-subgroup of $C_G(A)$ which contains S_1 as a normal subgroup we conclude from 3.4.2 that R is contained in H and $R = S_1$. In particular, S_1 is a Sylow 2-subgroup of $C_G(A)$. Now let R be a Sylow 2-subgroup of $N_G(A)$ containing S_1. As D is not empty, S_1 has index 2 in R. Because of 3.4.2 R is contained in H. Without loss, we may assume that $N_S(A)$ is a Sylow 2-subgroup of $N_G(A)$. We see that S contains an involution of D which inverts a member of $\Omega \cap K$. In particular, $S_1 \times S_2$ is properly contained in S and S/S_1 is dihedral.

If an elementary-Abelian subgroup of order 8, whose involutions belong to D, were contained in S, it would intersect S_1 nontrivially, contradicting 3.1. Hence S is properly contained in a Sylow 2-subgroup T of G. Let T_0 be the normalizer of S in T and fix an element t of $T_0 \backslash S$. If i is an involution in S_1 we conclude from 3.1 and 3.4.2 that $i^t \in S_1 \times S_2 \backslash S_1$ and $S_1 \cap S_1{}^t = \langle 1 \rangle$. Consequently the order of S_1 is at most 4. As $Z(S)$ is contained in $S_1 \times S_2$ and $Z(S) \cap S_2$ has order 2, the action of t on $\Omega_1(Z(S))$ implies that $\Omega_1(Z(S))$ has order 4. If x were an element of order 4 in $Z(S)$, x^2 would have to centralize K and therefore $\langle x^2 \rangle = \Omega_1(Z(S)) \cap S_1$; this means that x^2 is the only involution in $Z(S)$ which is a square in $Z(S)$. Then $(x^2)^t = x^2$, a contradiction. We conclude that $Z(S)$ is elementary Abelian of order 4.

We note that $Z(T) = Z(T_0)$ is contained in $Z(S)$ and intersects S_1 trivially. Furthermore the index of S in T_0 is 2.

Assume S_1 cyclic of order 4. Let $S_1 = \langle x \rangle$ and d be an involution in $S \backslash S_1 \times S_2$. We know $x^d = x^{-1}$. Pick involutions z and j of S_2 such that $z^d = z$ and $j^d = jz$. We have $Z(S) = \langle x^2, z \rangle$. As the involutions x^2 and z are the only squares in S, they are conjugate under t by 3.4.2 and do not belong to D by 3.1. Hence $\Omega_1(S_1 \times S_2)$ is generated by the involutions of S which do not belong to D. Thus $\Omega_1(S_1 \times S_2)$ is invariant under $\langle t \rangle$. In particular, $\{x^2 j, x^2 jz\}$ is invariant under $\langle t \rangle$. Hence $z^t = z$, a contradiction.

Thus S_1 is elementary Abelian of order 2 or 4. In each case S is uniquely determined. Easy calculations show that S does not contain a four group whose involutions belong to D.

3.4.4. *H contains a Sylow 5-subgroup of G. If $\langle y \rangle$ is a subgroup of order 5 in H then H contains a Sylow 2-subgroup of $N_G(\langle y \rangle)$.*

Proof. Suppose first that $\langle y \rangle$ is a subgroup of order 5 in K. Let i be some involution in S_1 and A a member of Ω normalized by $\langle y \rangle$. By 3.1 $\langle y \rangle$ acts nontrivially on A. Now 3.3 implies that i normalizes A and hence i centralizes A by 3.1. Therefore A must be a subgroup of K. As $N_G(\langle y \rangle)$ permutes the elements of Ω fixed by $\langle y \rangle$, we see $N_G(\langle y \rangle) \leqslant H$.

Let $g \in G \backslash H$, and suppose $C \cap C^g \neq \langle 1 \rangle$. As G is simple, $\langle K, K^g \rangle$ is a proper subgroup of G. By minimality of G, $\langle K, K^g \rangle$ is isomorphic to J_1. Hence an element of order 5 in K fixes an element of Ω not contained in K, a contradiction. Thus C is a TI-subgroup.

Let P be a Sylow 5-subgroup of H, $P_1 = P \cap C$ and $P_2 = P \cap K$. If Q is a Sylow 5-subgroup of G containing P we conclude that $Z(Q)$ is contained in P. If P_2 is contained in $Z(Q)$ we have $P = Q$. Therefore we may assume $Z(Q) \cap P_2 = \langle 1 \rangle$; in particular, $P_1 \neq \langle 1 \rangle$. As P_1 is a nontrivial strongly closed subgroup in P and C is a TI-subgroup we have $N_G(P) \leqslant N_G(P_1) \leqslant H$. In each case P is a Sylow 5-subgroup. As Ω contains two elements which generate a subgroup isomorhic to J_1, P contains an element of order 5 which fixes more than two members of Ω; this implies $P_1 \neq \langle 1 \rangle$.

If Q is a subgroup of P with $C_P(Q)$ contained in Q, then $P_1 \cap Q$ is a nontrivial strongly closed subgroup of Q; therefore $N_G(Q) \leqslant N_G(P_1 \cap Q) \leqslant H$, as C is a TI-subgroup. From 2.3 H controls fusion in P.

Now let $\langle y \rangle$ be a subgroup of order 5 in P. We claim that H contains a Sylow 2-subgroup of $N_G(\langle y \rangle)$. If $y \in P_1$, then $N_G(\langle y \rangle) \leqslant H$. Thus we may assume that $\langle y \rangle$ acts nontrivially on K. Let A and B be two Sylow 11-subgroups of K normalized by y. Each involution in $C_G(y)$ normalizes A and B by 3.3 and lies in $N_G(\langle A, B \rangle) = H$. Let x be some element of order 4 in $C_G(y)$. The involution x^2 normalizes A and B, and from 3.2 one infers that x^2 centralizes A and B. Hence $x^2 \in C$ and $x \in H$. If w were an element of order 8 in $C_G(y)$, we could conclude $w^4 \in C$ and $w \in H$; however, H does not contain an element of order 8. We proved that H contains each Sylow 2-subgroup of $C_G(y)$. As H controls fusion in P we conclude that some Sylow 2-subgroup of $N_G(\langle y \rangle)$ lies in H. This completes the proof of 3.4.4.

Parts 3.4.3 and 3.4.4 contradict each other: By hypothesis G contains a subgroup isomorphic to J_1 and thus a subgroup $\langle y \rangle$ of order 5 which is normalized by a four group V whose involutions belong to D. Replacing $\langle y \rangle$ and V by suitable conjugates, we assume by 3.4.4 that $\langle y \rangle$ and V lie in H. This contradicts 3.4.3 and completes the proof of 3.4. In particular, D is a conjugate class of involutions.

Finally, we need some information allowing us to decide whether some involution belongs to D or not.

3.5. Let $\langle y \rangle$ be a subgroup of order 5. Suppose $\langle y \rangle$ acts nontrivially on some element A of Ω and $\langle y \rangle$ is inverted by some involution. Then each involution in $N_G(\langle y \rangle)$ belongs to D.

Proof. Suppose $C_G(y)$ has even order. Let i be an involution in $C_G(y)$. By 3.3 and 3.4 i belongs to D. By 3.2 a Sylow 2-subgroup of $C_G(y)$ has order 2.

Next suppose the order of $C_G(A)$ is not divisible by 5. By hypothesis there exist two members of Ω which generate a subgroup L isomorphic to J_1. By assumption we may assume that y lies in L. In L there exists a four group V such that the involutions in V belong to D and V normalizes $\langle y \rangle$. As we already know that a Sylow 2-subgroup of $C_G(y)$ has order 2, we infer from 3.2 that V is a Sylow 2-subgroup of $N_G(\langle y \rangle)$.

To prove the statement under 3.5 we must therefore assume that the order of $C_G(A)$ is divisible by 5. Let P be a $\langle y\rangle$-invariant Sylow 5-subgroup of $C_G(A)$. Note that P is strongly closed in $P\langle y\rangle$ with respect to G. If $N_G(P)$ were contained in $N_G(A)$, $P\langle y\rangle$ would be a Sylow 5-subgroup of G and $N_G(P)' \cap P\langle y\rangle \leqslant P$; this contradicts 2.4. Hence P centralizes a conjugate of A distinct from A. By minimality G contains a simple subgroup K generated by members of Ω such that the order of $C_G(K)$ is divisible by 5. Among all subgroups of G with these properties we choose K such that the order of K is as large as possible. We set $H = N_G(K)$ and $C = C_G(K)$. The choice of K implies that for each $g \in G\backslash H$ the order of $C \cap C^g$ is not divisible by 5.

Suppose H is properly contained in a maximal subgroup M of G. Then K is a proper subgroup of $\langle K^M\rangle$ which is isomorphic to J_1 by minimality of G. The maximality of K yields that $C_G(\langle K^M\rangle)$ is a $5'$-group, and a Sylow 5-subgroup has order 5, a contradiction. Hence H is a maximal subgroup of G and $H = N_G(C)$.

Let Q be a Sylow 5-subgroup of H. Then $Q \cap C$ is strongly closed in Q with respect to G. Hence $N_G(Q) \leqslant N_G(Q \cap C) \leqslant H$ and Q is a Sylow 5-subgroup of G. Moreover H controls fusion in Q by 2.3. We choose notation such that $P = Q \cap C$ and that $\langle y\rangle$ is a subgroup of Q.

Suppose H/C is isomorphic to $PGL_2(11)$. Thus a Sylow 2-subgroup of H is dihedral of order 8. We know that G contains a subgroup $\langle z\rangle$ of order 5 such that a Sylow 2-subgroup of $N_G(\langle z\rangle)$ is a four group whose involutions belong to D. Replacing $\langle z\rangle$ by a suitable conjugate we may assume that z lies in Q and acts nontrivially on K. From 3.3 we obtain $V \cap C_G(z) \leqslant H$. Since H controls fusion in Q, a conjugate of V lies in H. This contradicts 3.2.

We conclude $H = C \times K$. Now $Q \cap K$ is the only subgroup of order 5 in Q inverted by some involution. Hence $\langle y\rangle = Q \cap K$ and the result follows easily.

3.6. A Sylow 2-subgroup of G is elementary Abelian.

Proof. Let d be an element of D and i some involution in $C_G(d)$ different from d. Let A be a member of Ω with $A^d = A$. From 3.4 $A^i \neq A$. If $\langle A, A^i\rangle$ were isomorphic to $PSL_2(11)$, the involution d would induce an outer automorphism on $\langle A, A^i\rangle$ and by 3.2 the involutions of $\langle A, A^i\rangle$ would not lie in D; however, this contradicts 3.5. Now $\langle A, A^i\rangle \cong J_1$ implies that i lies in $\langle A, A^i\rangle$ and belongs to D. Consequently all involutions of G belong to D. From 3.2 G does not contain an element of order 4.

Because of the simplicity of G and 3.6 the fusion result 2.1 shows that G is not a counterexample after all. This contradiction concludes the proof of the theorem.

References

1. J. L. Alperin, Sylow intersections and fusion, *J. Algebra* **6** (1967), 222–241.
2. M. Aschbacher, $\mathfrak{F}$-sets and permutation groups, *J. Algebra* **30** (1974), 400–416.

3. D. Goldschmidt, 2-Fusion in finite groups, *Ann. of Math.* **99** (1974), 70–117.
4. D. Goldschmidt, Strongly closed 2-subgroups of finite groups, *Ann. of Math* **102** (1975), 475–489.
5. B. Huppert, "Endliche Gruppen," I, Springer-Verlag, Berlin, 1967.
6. Z. Janko, A new finite simple group with Abelian Sylow 2-subgroups and its characterisation, *J. Algebra* **3** (1966), 147–186.
7. U. Preiser, Eine Charakterisierung einiger endlicher einfacher Gruppen durch $\mathfrak{F}$-Mengen, *Arch. Math.* **28** (1977), 369–373.