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Semilinear reaction–diffusion systems of several components

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Abstract

The homogeneous Dirichlet boundary value problem

$$u_{it} - \Delta u_i = \prod_{j=1}^{n} u_j^{p_{ij}}, \quad i = 1, 2, \ldots, n$$

in a bounded domain $\Omega \subset \mathbf{R}^N$ is considered, where $p_{ij} \geqslant 0$ $(1 \leqslant i, j \leqslant n)$ are constants. Denote by I the identity matrix and $P = (p_{ij})$, which is assumed to be irreducible. We find out that whether or not $I - P$ is a so-called M-matrix plays a fundamental role in the blow-up theorems.

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1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Denote $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and

$$f(\mathbf{u}) = \left(\prod_{j=1}^{n} u_j^{p_{1j}}, \prod_{j=1}^{n} u_j^{p_{2j}}, \ldots, \prod_{j=1}^{n} u_j^{p_{nj}} \right),$$

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where $p_{ij} \geqslant 0$ $(1 \leqslant i, j \leqslant n)$ are constants. We shall consider in this paper the following Dirichlet problem:

$$\begin{aligned} &\mathbf{u}_t = \Delta\mathbf{u} + f(\mathbf{u}), && x \in \Omega, \ t > 0, \\ &\mathbf{u}|_{\partial\Omega} = 0, && t > 0, \\ &\mathbf{u}(x,0) = \mathbf{u}_0(x), && x \in \Omega, \end{aligned} \tag{1}$$

where

$$\mathbf{u}_0(x) = (u_{10}(x), u_{20}(x), \ldots, u_{n0}(x)) \text{ is a continuous,}$$
$$\text{positive in } \Omega \text{ and bounded vector function.} \tag{2}$$

Such a system constitutes a simple example of a reaction–diffusion system exhibiting a nontrivial coupling on the unknowns $u_1(x,t), u_2(x,t), \ldots, u_n(x,t)$. These can be thought of as the temperatures of n substances which constitute a combustible mixture, where heat release is described by the power laws on the right-hand side of (1); see [2].

This paper is mainly concerned with the question of the lifespan of solutions of (1). To this end, we define

$$||\mathbf{u}(t)|| = \sum_{i=1}^{n} ||u_i(\cdot,t)||_\infty,$$

then we shall say that $\mathbf{u}$ blows up in a time $T < \infty$ if

$$\limsup_{t \to T} ||\mathbf{u}(t)|| = \infty.$$

When $n = 1$, (1) has been studied extensively by many authors; see [5,8,12,14,17,18,20] and the references therein.

For the reaction–diffusion systems involving two components, the questions about when, where and how blow-up occurs have all been investigated in the past two decades. Several interesting blow-up conditions are suggested; see [3,7–10,15,16,20,27] and the references therein. Friedman and Giga [13] also obtained a single-point blow-up for semilinear parabolic systems under some conditions. Recently, the blow-up rate is estimated by several authors assuming varies of conditions on the matrix P and initial data; see [1,6,26,28] and the references therein.

Parabolic systems involving n components have appeared in the literature; see [11,29].

In this paper we shall investigate the blow-up conditions of (1) with n components. To proceed further, we need some concepts from the theory of M-matrices. First we introduce some symbols, following the book [4]. $A \geqslant 0$ if each elements of the vector or matrix A is nonnegative. $A > 0$ if $A \geqslant 0$ and at least one element is positive. $A \gg 0$ if each element is positive. Similarly $A \gg B$ if $A - B \gg 0$.

Definition 1.1 (See Berman and Plemmons [4, Definition 1.2, pp. 27]). An $n \times n$ matrix A is reducible if for some permutation matrix Q

$$QAQ^t = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and C are square matrices and Q^t is the transpose of Q. Otherwise, A is irreducible.

Throughout this paper, $P = (p_{ij})$ is assumed to be irreducible, since if not the case, system (1) can be reduced to two subsystems with one being not coupled with the other.

The so-called M-matrices play a fundamental role in the blow-up theorems. Following [4], we give its definition.

Definition 1.2 (See Berman and Plemmons [4, Definition 1.2, pp. 133]). Any matrix A is called an M-matrix if A can be expressed in the form

$$A = sI - B, \quad s > 0, \ B \geqslant 0 \tag{3}$$

with $s \geqslant \rho(B)$, the spectral radius of B. If $s > \rho(B)$, A is called a nonsingular M-matrix.

Let I be the identity matrix. Our main result is the following theorem.

Theorem 1.1. (i) *If $I - P$ is an M-matrix, then all solutions of* (1) *are global.*
(ii) *If $I - P$ is not an M-matrix, then there exist both nontrivial global solutions and nonglobal solutions of* (1).

We would like to point out that, by introducing the M-matrix, the blow-up theorem of the semilinear parabolic system has a consistent and concise form.

In [9], the authors studied the blow-up conditions for the system (1) with $n = 2$. They proved that:

(a) If $p_{11} > 1$ or $p_{22} > 1$ or $p_{12}p_{21} > (1 - p_{11})(1 - p_{22})$, (1) admits both global and nonglobal solutions.
(b) If $p_{11} \leqslant 1$, $p_{22} \leqslant 1$ and $p_{12}p_{21} \leqslant (1 - p_{11})(1 - p_{22})$, all solutions of (1) are global.

In fact, by Theorem 2.3 in the next section, under the conditions of (b), $I - P$ is an M-matrix. So our theorem recovers their results.

Concerning the blow-up properties of the solutions of (1) with $n = 2$, generally, the eigenfunction method is applied; see [7,10,27] and the references therein. In [9,27], a sub-solution technique is present, but their methods seemed to work only for systems with two components.

As to the general problem (1), it seems difficult for the eigenfunction method to work. In this paper we use the classical super- and sub-solution technique combining with the basic properties of M-matrices to give a complete picture of the blow-up conditions. The sub-solution we construct is some powers of the function

$$v = \frac{w(r)}{T - w^{\sigma}(r)t}, \tag{4}$$

where

$$w(r) = \frac{1}{2}\left(1 + \cos\frac{\pi r}{R}\right).$$

For the local existence, uniqueness and comparison principle we refer to [9, Theorem 1.1, Remark 2.6; 10, Lemma 2.2] and [19, Theorem VII.7.1]. For the sake of convenience we state the comparison theorem here.

Lemma 1.1. *Let* $\bar{\mathbf{u}}$ *and* $\underline{\mathbf{u}}$ *be a pair of super- and sub-solution in a cylinder* $\Omega_T = \Omega \times (0, T)$ *with* $T > 0$ *and satisfy*

$$\begin{aligned} &\bar{\mathbf{u}}_t - \Delta\bar{\mathbf{u}} - f(\bar{\mathbf{u}}) \geqslant 0 \geqslant \underline{\mathbf{u}}_t - \Delta\underline{\mathbf{u}} - f(\underline{\mathbf{u}}), && (x,t) \in \Omega_T, \\ &\bar{\mathbf{u}}|_{\partial\Omega} \geqslant \underline{\mathbf{u}}|_{\partial\Omega} \geqslant 0, && 0 < t < T, \\ &\bar{\mathbf{u}}(x,0) \geqslant \underline{\mathbf{u}}(x,0) \geqslant 0, && x \in \Omega. \end{aligned}$$

Then

$$\bar{\mathbf{u}}(x,t) \geqslant \underline{\mathbf{u}}(x,t), \quad (x,t) \in \Omega_T.$$

In the next section, we collect some properties of M-matrices as required. The last section proves our theorem.

2. Several results of M-matrices

M-matrices have important applications, for instance, in iterative methods in numerical analysis, in input–output analysis in economics and in the analysis of Markov chains. For a detailed discussion, see Chapter 7–10 in the book by Berman and Plemmons [4], also see [22,23]. In this paper we will demonstrate the relationship between M-matrices and the blow-up properties for the general reaction–diffusion systems.

M-matrices have close relation to the nonnegative matrices. First, we state the classical Perron–Frobenius theorem for nonnegative matrices.

Theorem 2.1 (See Berman and Plemmons [4, Theorem 1.4, p. 27]). (a) *If* A *is a positive matrix, then* $\rho(A)$, *the spectral radius of* A, *is a simple eigenvalue, greater than the magnitude of any other eigenvalue.*

(b) *If* $A \geqslant 0$ *is irreducible, then* $\rho(A)$ *is a simple eigenvalue, and* A *has a positive eigenvector* x *corresponding to* $\rho(A)$.

In [4], the authors collected 50 equivalent conditions of nonsingular M-matrices. Here are some of them.

Theorem 2.2 (See Berman and Plemmons [4, A_1, p. 134; I_{27}, p. 136]). *The following three statements are equivalent*:

(a) A *is a nonsingular* M*-matrix.*
(b) *All of the principal minors of* A *is positive.*
(c) A *is semi-positive*; *that is, there exists* $x \gg 0$ *such that* $Ax \gg 0$.

For the general M-matrices we also have similar equivalent conditions.

Theorem 2.3 (See Berman and Plemmons [4, A_1, p. 149]). *The following two statements are equivalent*:

(a) A *is an* M*-matrix.*
(b) *All of the principal minors of* A *is nonnegative.*

The singular, irreducible M-matrix plays a critical role in the proof of Theorem 1.1. The following theorem concerns with such matrices.

Theorem 2.4 (See Berman and Plemmons [4, Theorem 4.16, p. 156]). *Let* A *is a singular*, *irreducible* M*-matrix of order* n. *Then*

(a) A *has rank* $n-1$.
(b) *There exists a vector* $x \gg 0$ *such that* $Ax = 0$.
(c) *Each principal sub-matrix of* A *other than* A *itself is a nonsingular* M*-matrix.*

Since $I - P$ is not always an M-matrix, the properties of non$M$-matrices of the form (3) are used in our argument.

Theorem 2.5. *Assume that* A *is an irreducible non*M*-matrix of the form* (3). *Then*

(a) $s < \rho(B)$.
(b) *There exists a vector* $x \gg 0$ *such that* $Ax \ll 0$.

Proof. (a) follows directly from the definition of M-matrices. Now we prove (b). Since that A is irreducible is equivalent to that B is irreducible, by Theorem 2.1(b), there exists a positive vector x such that $Bx = \rho(B)x$. Hence $Ax = (sI - B)x = (s - \rho(B))x \ll 0$. $\square$

3. Proof of Theorem 1.1

We divide the argument into three lemmata. In the following lemmata the eigenvalue problem

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega_1; \quad \varphi|_{\partial\Omega_1} = 0 \tag{5}$$

plays a crucial role, where the bounded domain $\Omega_1 \supset\supset \Omega$. Denote by λ_1 the first eigenvalue and by $\varphi(x)$ the corresponding eigenfunction.

Lemma 3.1. *If $I - P$ is a nonsingular M-matrix, then all solutions of* (1) *are globally bounded.*

Proof. Let $\varphi(x)$ be the first eigenfunction of (5) with $\min_{\bar{\Omega}} \varphi(x) = 1$. Denote $\Phi = \max_{\overline{\Omega_1}} \varphi(x)$. Let $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ be the positive vector asserted in Theorem 2.2(c). Clearly we can choose $\ell_i \leqslant 1$ for all $1 \leqslant i \leqslant n$. And we have

$$(I - P)\ell \gg 0. \tag{6}$$

Put

$$\bar{\mathbf{u}} = ((k\varphi)^{\ell_1}, (k\varphi)^{\ell_2}, \dots, (k\varphi)^{\ell_n}) \quad \text{in } \Omega, \tag{7}$$

where k is a large constant to be fixed later. Then a routine computation yields

$$\begin{aligned} &\Delta\bar{u}_i + \prod_{j=1}^{n} \bar{u}_j^{p_{ij}} \\ &\leqslant -\lambda_1 \ell_i k^{\ell_i} + k^{\sum_{j=1}^{n} p_{ij}\ell_j} \Phi^{\sum_{j=1}^{n} p_{ij}\ell_j} \end{aligned} \tag{8}$$

for all $1 \leqslant i \leqslant n$. It follows from (2) and(6) that k can be chosen so large that

$$\Delta\bar{\mathbf{u}} + f(\bar{\mathbf{u}}) \leqslant 0 \quad \text{in } \Omega$$

and

$$\bar{\mathbf{u}}(x) \gg \mathbf{u}_0(x) \quad \text{in } \Omega.$$

The comparison principle implies that $\mathbf{u}(x,t) \leqslant \bar{\mathbf{u}}(x)$ for all $x \in \Omega$ and $t > 0$. $\square$

The following lemma deals with the case where $I - P$ is a singular M-matrix.

Lemma 3.2. *If $I-P$ is a singular M-matrix, also all solutions of* (1) *exist globally.*

Proof. Let ℓ be the positive vector in Theorem 2.4(b). Then

$$(I-P)\ell = 0.$$

Also let $\ell_i \leqslant 1$ for all $1 \leqslant i \leqslant n$. Put

$$\bar{\mathbf{u}} = ((ke^{\rho t}\varphi)^{\ell_1}, (ke^{\rho t}\varphi)^{\ell_2}, \dots, (ke^{\rho t}\varphi)^{\ell_n}),$$

where $\rho = \max_i\{1/\ell_i\}$ and $\varphi(x)$ is the first eigenfunction of (5). Then we have

$$\bar{u}_{it} - \Delta\bar{u}_i - \prod_{j=1}^{n} \bar{u}_j^{p_{ij}} \geqslant \lambda_1 \ell_i k^{\ell_i} e^{\ell_i t}\varphi^{\ell_i} > 0 \quad \text{in } \Omega \times (0, \infty)$$

for all $1 \leqslant i \leqslant n$. Choose k sufficiently large that $\bar{\mathbf{u}}(x,0) \geqslant \mathbf{u}_0(x)$ in Ω. Thus the comparison principle implies this lemma. □

Finally, we consider the case where $I-P$ is not an M-matrix.

Lemma 3.3. *Assume that $I-P$ is not an M-matrix. Then*

(a) *Solutions of* (1) *with small initial data exist globally.*
(b) *Solutions of* (1) *with large initial data become infinite in finite time.*

Proof. For (a) to hold, the proof is very similar to that of Lemma 3.1. But here we choose ℓ to be the positive vector in Theorem 2.5(b) with $\ell_i \leqslant 1$ for all $1 \leqslant i \leqslant n$ and we have

$$(I-P)\ell \ll 0. \tag{9}$$

Choose $\bar{\mathbf{u}}$ in (7) and hence (8) holds. Now from (9), k can be chosen so small that

$$\Delta\bar{\mathbf{u}} + f(\bar{\mathbf{u}}) \leqslant 0 \quad \text{in } \Omega.$$

For such a small constant $k > 0$, let $\mathbf{u}_0(x)$ be small that $\bar{\mathbf{u}}(x) \geqslant \mathbf{u}_0(x)$ in Ω. Then $\bar{\mathbf{u}}(x)$ is a super-solution of (1).

That (b) holds requires some more details. We shall construct an unbounded self-similar sub-solution to complete the proof. For the detailed discussion of self-similar sub-solutions, we refer to [24,25]. But here the self-similar sub-solution has a different form; see [21]. Without loss of generality, we may assume that $0 \in \Omega$, then there exists a ball $B_R(0) \subset\subset \Omega$. Let

$$w(r) = \frac{1}{2}\left(1 + \cos\frac{\pi r}{R}\right), \quad 0 \leqslant r < R.$$

Then

$$-\Delta w = \frac{\pi}{2R}\left(\frac{\pi}{R}\cos\frac{\pi r}{R} + \frac{n-1}{r}\sin\frac{\pi r}{R}\right),$$

hence there exists a unique $r_0 \in (0, R)$ such that

$$-\Delta w \leqslant 0 \quad \text{for } r_0 \leqslant r < R,$$

$$0 \leqslant -\Delta w \leqslant \frac{n\pi^2}{2R^2} \text{ and } w \geqslant \cos^2\frac{\pi r_0}{2R} \text{ for } 0 \leqslant r < r_0. \tag{10}$$

Let

$$V(r,t) = T - w^{\sigma}(r)t, \quad 0 < t < T,$$

where $T \in (0,1)$ is small and $\sigma > 0$ is large to be fixed later. Choose ℓ as in (a) with $\ell_i \geqslant 1$ and $\sum_{j=1}^n p_{ij}\ell_i > \ell_i + 1$ for all $1 \leqslant i \leqslant n$. This can be done according to (9). Put

$$\underline{\mathbf{u}} = \left(\left(\frac{w}{V}\right)^{\ell_1}, \left(\frac{w}{V}\right)^{\ell_2}, \dots, \left(\frac{w}{V}\right)^{\ell_n}\right) \quad \text{in } B_R(0) \times (0,T).$$

Then a routine computation yields

$$\begin{aligned} &\underline{u}_{it} - \Delta\underline{u}_i - \prod_{j=1}^n \underline{u}_j^{p_{ij}} \\ &\quad \leqslant \frac{\ell_i w^{\ell_i+\sigma}}{V^{1+\ell_i}} + \frac{(1+\sigma)\ell_i(-\Delta w)_+}{V^{1+\ell_i}} - \frac{w^{\sum_{j=1}^n p_{ij}\ell_j}}{V^{\sum_{j=1}^n p_{ij}\ell_j}}, \end{aligned}$$

where $h_+ = \max\{h, 0\}$. Fix σ such that $\ell_i + \sigma > \sum_{j=1}^n p_{ij}\ell_j$ for all $1 \leqslant i \leqslant n$. From (10), for $r_0 \leqslant r < R$, we have

$$\begin{aligned} &\underline{u}_{it} - \Delta\underline{u}_i - \prod_{j=1}^n \underline{u}_j^{p_{ij}} \\ &\quad \leqslant \frac{\ell_i w^{\ell_i+\sigma}}{V^{1+\ell_i}} - \frac{w^{\sum_{j=1}^n p_{ij}\ell_j}}{V^{\sum_{j=1}^n p_{ij}\ell_j}} \leqslant 0 \end{aligned}$$

for all $1 \leqslant i \leqslant n$ if $T < 1$ is sufficiently small. For $0 \leqslant r < r_0$, we have

$$\begin{aligned} &\underline{u}_{it} - \Delta\underline{u}_i - \prod_{j=1}^n \underline{u}_j^{p_{ij}} \\ &\quad \leqslant \left(1 + (1+\sigma)\frac{n\pi^2}{2R^2}\right)\frac{\ell_i}{V^{1+\ell_i}} - \frac{(\cos^2\frac{\pi r_0}{2R})^{\sum_{j=1}^n p_{ij}\ell_j}}{V^{\sum_{j=1}^n p_{ij}\ell_j}} \leqslant 0 \end{aligned}$$

for all $1 \leqslant i \leqslant n$ if also $T < 1$ is sufficiently small. Choose $\mathbf{u}_0(x)$ so large that $\mathbf{u}_0(x) \geqslant \underline{\mathbf{u}}(x, 0)$ in $B_R(0)$. Since $\mathbf{u}(x,t) \geqslant 0$ in $B_R(0) \times (0, T)$, applying the comparison theorem, $\mathbf{u}(x,t)$ must become infinite before T. Thus the proof is completed. □

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