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Gibbs–Markov structures and limit laws for partially hyperbolic attractors with mostly expanding central direction ☆

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Abstract

We consider a partially hyperbolic set K on a Riemannian manifold M whose tangent space splits as $T_K M = E^{cu} \oplus E^s$, for which the center-unstable direction E^{cu} expands non-uniformly on some local unstable disk. We show that under these assumptions f induces a Gibbs–Markov structure. Moreover, the decay of the return time function can be controlled in terms of the time typical points need to achieve some uniform expanding behavior in the center-unstable direction. As an application of the main result we obtain certain rates for decay of correlations, large deviations, an almost sure invariance principle and the validity of the central limit theorem.

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Keywords: Partial hyperbolicity; Gibbs–Markov structure; Decay of correlations; Large deviations; Almost sure invariance principle; Central limit theorem

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1. Introduction

Remarkable advances in the study of dynamical systems, specially the statistical properties of those with chaotic behavior, have been achieved through the idea of *inducing*. Roughly speaking, this consists of replacing the initial dynamical system by another one whose dynamical features are easier to understand and from which one can recover much information on the initial system. This idea goes back to the 70's where Markov partitions have been used to study the statistical properties of uniformly hyperbolic dynamical systems via conjugations to shifts. Since then, a main goal in dynamical systems theory is to enlarge that strategy to wider classes of systems.

A main achievement in this direction has been attained by Young in [23,24]. In these works she developed an abstract framework, the *Young towers*, that proved usefulness in a systematic treatment of several classes of dynamical systems, including Axiom A attractors, piecewise hyperbolic maps, billiards with convex scatterers, logistic maps, intermittent maps and Hénon-type attractors. The latter case has actually been treated by Benedicks and Young in [9]. A preponderant role in this context is played by *Gibbs–Markov structures*, which may be understood as a generalization of the classical Markov partitions and are naturally associated to an inducing scheme that gives rise to a Young tower. Many statistical properties on the dynamics of these induced structures can be recovered from the *Gibbs–Markov map* that one obtains quotienting out by stable leaves. Gibbs–Markov maps constitute themselves object of great dynamical interest; see e.g. [2,3] for Aaronson and Denker contributions on the statistical properties of Gibbs–Markov maps, whose main ideas go back to Aaronson's book [1] on infinite ergodic theory.

A Gibbs–Markov structure is characterized by a suitable region of the phase space partitioned into subsets (possibly infinitely many) each of which with a given *return time*. Comparing to the classical Markov partitions, a main difference lies on the possibility of infinitely many return times, as long as the measure of points with larger and larger returns decays to zero. The flexibility conferred by the chance of arbitrarily large return times is a fundamental step towards applications to non-uniformly hyperbolic dynamical systems, where large waiting times are needed to reach good expansion rates in the center-unstable direction of some points. In certain cases we are able to determine the speed at which the return times decay, given in terms of the time that generic points need to achieve the good expansion rates.

Dynamical systems that do not fit the class of uniformly hyperbolic ones but combine non-uniform expanding/contracting central directions with other directions of uniformly hyperbolic behavior give rise to a wider class of partially hyperbolic dynamical systems. The case of partially hyperbolic diffeomorphisms for which the tangent bundle over some attracting set splits as $E^{cs} \oplus E^u$, the sum of one invariant sub-bundle with non-uniformly contracting behavior and another one with uniformly expanding behavior, has been treated in [12], where some Gibbs–Markov structures were obtained to deduce decay of correlations and the validity of the central limit theorem for the SRB measures which had been obtained in [10]; see also [11,21]. Limit theorems for partially hyperbolic systems of this type were also obtained in [13].

Our main goal in this work is to prove the existence of Gibbs–Markov structures in the case of partially hyperbolic diffeomorphisms with an attracting set over which the tangent bundle splits as $E^s \oplus E^{cu}$, the sum of a sub-bundle E^s having uniformly contracting behavior with another one E^{cu} having non-uniform expanding behavior. This kind of attracting set has been previously considered in [5], where the existence of *SRB measures* was established. The method used in [5] is based on the simple idea of iterating forward Lebesgue measure on some center-unstable disk and obtaining the absolute continuity with respect to Lebesgue measure on local unstable disks of weak* accumulation points. Due to the simplicity of the method, it gives not much information on the properties of such SRB measures. As a byproduct of the machinery that we develop here, we are also able to deduce the existence of the SRB measure. Our method uses deeper knowledge on the geometrical structure of the attractor, thus enabling us to prove the existence of a Gibbs–Markov structure inside it, leading to an inducing scheme. As an immediate consequence of our main result, combined with others from [8,18,20], we easily deduce some statistical properties of these dynamical systems, namely *decay of correlations*, *central limit theorem*, *large deviations* and a *multidimensional almost sure invariant principle*. Let us point out that the case $E^s \oplus E^{cu}$ that we consider here is, for our purposes, considerably more difficult to deal with than the dual case $E^{cs} \oplus E^u$, since the richest part of the dynamics in the neighborhood of an attracting set occurs in the unstable direction, where in our case the expansion is attained just asymptotically.

The final part of the proof of our main result follows the strategy used in [6] for non-uniformly expanding endomorphisms (non-invertible smooth dynamical systems). The argument gives no optimal conclusions outside the polynomial case. The lack of efficiency for exponential or subexponential decays essentially relies on [6, Proposition 6.1] which still has no suitable generalizations. A main achievement in this direction has been obtained in [14] by mean of a different geometrical construction, leading to exponential and subexponential decays of return times. The construction in [6] can be thought of as being local, in the sense that the partition is obtained by considering convenient returns of points in a small disk to itself. The construction performed in [14] uses instead a finite global partition of the whole attractor as a starting point. However, this strategy has no natural generalization to the present setting of partially hyperbolic diffeomorphisms, mostly due to the non-compactness of unstable manifolds.

1.1. Overview

This paper is organized as follows. In the remaining of this introduction we consider three subsections. In the first one we define the Gibbs–Markov structures. In the second subsection we introduce partial hyperbolicity and state our main result on the existence of Gibbs–Markov structures for certain partially hyperbolic attractors. In the final subsection we define some limit laws and state some statistical consequences of our main result. In Section 2 we recall some results on hyperbolic times and bounded distortion from [5]. The construction of Gibbs–Markov structures for partially hyperbolic attractors is performed in Section 3. We begin with some results on the recurrence of disks, and then present an algorithmic construction that gives rise to the product structure. Finally, in Section 3.5 we prove some results on the regularity of the stable and unstable foliations. This comprises the generalization of classical results for uniformly hyperbolic attractors to our setting, namely the Hölder continuity of the central-unstable direction and the absolute continuity of the stable foliation. Finally, the estimates on the decay of return times is obtained in Section 4.

1.2. Gibbs–Markov structures

Here we present the structures which have been introduced in [23] and constitute the main object of our interest. These structures comprise the dynamical and geometrical essence of the return map to the base of a Young tower.

Let $f : M \to M$ be a diffeomorphism of a Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. Let Leb denote the Lebesgue measure on the Borel sets of M associated to the Riemannian structure. Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the Lebesgue measure on γ induced by the restriction of the Riemannian structure to γ.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* if $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$ for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold* if $\mathrm{dist}(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for every $x, y \in \gamma$. Let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s, a unit disk D^u of some $\mathbb{R}^n$, and a map $\Phi^u : K^s \times D^u \to M$ such that

- $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
- Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
- $x \mapsto \Phi^u|(\{x\} \times D^u)$ is a continuous map from K^s to $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined similarly. We say that $\Lambda \subset M$ has a *product structure* if there exist a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

- $\Lambda = (\bigcup \gamma^u) \cap (\bigcup \gamma^s)$;
- $\dim \gamma^u + \dim \gamma^s = \dim M$;
- each γ^s meets each γ^u in exactly one point;
- stable and unstable manifolds meet transversally with angles bounded away from 0.

Let $\Lambda \subset M$ have a product structure whose associated defining families are Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if Λ_0 also has a hyperbolic product structure, and its defining families, Γ_0^s and Γ_0^u, can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined similarly.

Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. For each $n \geqslant 1$ we let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks and $\det D(f^n)^u$ denote the Jacobian of $D(f^n)^u$.

We require that the product structure satisfies several properties that we explicit below in (P_1)–(P_5). From here on we assume that $C > 0$ and $0 < \beta < 1$ are constants depending only on f and Λ.

(P_1) *Markov*: there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subset \Lambda$ such that

(a) $\mathrm{Leb}_\gamma((\Lambda \setminus \bigcup \Lambda_i) \cap \gamma) = 0$ on each $\gamma \in \Gamma^u$;

(b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}\big(\gamma^s(x)\big) \subset \gamma^s\big(f^{R_i}(x)\big) \quad \text{and} \quad f^{R_i}\big(\gamma^u(x)\big) \supset \gamma^u\big(f^{R_i}(x)\big).$$

This allows us to define the *return time* function $R : \Lambda \to \mathbb{N}$ as $R|\Lambda_i = R_i$.

(P_2) *Contraction on Γ^s*: for all $y \in \gamma^s(x)$ and $n \geqslant 1$

$$\mathrm{dist}\big(f^n(y), f^n(x)\big) \leqslant C\beta^n.$$

(P_3) *Backward contraction on Γ^u*: for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, and $0 \leqslant n < R_i$

$$\mathrm{dist}\big(f^n(y), f^n(x)\big) \leqslant C\beta^{R_i - n}\,\mathrm{dist}\big(f^{R_i}(x), f^{R_i}(y)\big).$$

(P_4) *Bounded distortion*: for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \leqslant C\,\mathrm{dist}\big(f^{R_i}(x), f^{R_i}(y)\big)^\eta.$$

(P_5) *Regularity of the foliations*:

(a) for all $y \in \gamma^s(x)$ and $n \geqslant 0$

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leqslant C\beta^n;$$

(b) given $\gamma, \gamma' \in \Gamma^u$, define $\phi : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ as $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and

$$\frac{d(\phi_*^{-1}\,\mathrm{Leb}_\gamma)}{d\,\mathrm{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

See Section 3.5 for a precise definition of absolute continuity. A set with a product structure for which properties (P_1)–(P_5) above hold will be called a *Gibbs–Markov structure*.

1.3. Partially hyperbolic attractors

Let $K \subset M$ be a compact *invariant* set for a C^1 diffeomorphism $f : M \to M$, meaning that $f(K) = K$. We say that K has a *dominated splitting* if there exists a continuous Df-invariant splitting $T_K M = E^{cs} \oplus E^{cu}$ and $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leqslant \lambda, \quad \text{for all } x \in K. \tag{1}$$

We call E^{cs} the *center-stable bundle* and E^{cu} the *center-unstable bundle*. We say that K is *partially hyperbolic* if it has a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$ for which E^{cs} is *uniformly contracting*: there is $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E^{cs}_x\| \leqslant \lambda, \quad \text{for all } x \in K. \tag{2}$$

Fixing some small number $c > 0$, we say that f is *non-uniformly expanding in the central-unstable direction* at a point $x \in K$ if

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log\|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c. \tag{NUE}$$

We define the *expansion time* function

$$\mathcal{E}(x) = \min\left\{N \geqslant 1:\ \frac{1}{n} \sum_{i=0}^{n-1} \log\|Df^{-1} \mid E^{cu}_{f^i(x)}\| < -c,\ \forall n \geqslant N\right\}. \tag{3}$$

Note that $\mathcal{E}(x)$ is finite for the points $x \in K$ satisfying (NUE).

In this work we consider partially hyperbolic sets of the same type of those considered in [5], for which the center-stable direction is uniformly contracting and the central-unstable direction is non-uniformly expanding. To highlight the uniform contraction in the center-stable direction we shall write E^s instead of E^{cs}.

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a transitive partially hyperbolic set such that $T_K M = E^s \oplus E^{cu}$. Assume that for some local unstable disk $D \subset K$ and $\tau > 0$ one has $\mathrm{Leb}_D\{\mathcal{E} > n\} = O(n^{-\tau})$. Then there exists $\Lambda \subset K$ with a Gibbs–Markov structure. Moreover, $\mathrm{Leb}_\gamma\{R > n\} = O(n^{-\tau})$ for any $\gamma \in \Gamma^u$.*

Remark 1.1. The existence of a disk D for which (NUE) is satisfied Lebesgue almost everywhere can actually hold under very mild conditions. Indeed, as shown in [7, Theorem A], if K attracts a positive Lebesgue measure set of points for which (NUE) holds, then K contains some local unstable disk D for which (NUE) holds for Leb_D almost every point.

Remark 1.2. Under the assumptions of Theorem A we are able to say more about the set Λ with the product structure. Actually, our construction gives that the set Λ itself coincides with the union of the leaves in Γ^u. This is not always the case, e.g. for Hénon attractors Λ is a Cantor set; see [9].

An open class of diffeomorphisms for which $K = M$ is partially hyperbolic and satisfies the assumptions of Theorem A can be found in [5, Appendix A]. The calculations in [5] give that for such diffeomorphisms one has $\mathrm{Leb}_D\{\mathcal{E} > n\}$ decaying exponentially fast, which then implies that

$\mathrm{Leb}_\gamma\{R > n\}$ has *super-polynomial decay*, meaning that it decays faster than any polynomial. Transitivity of the diffeomorphisms in that class has been proved in [22].

1.4. Limit laws

An f-invariant Borel probability μ in M is called an *SRB measure* if, for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi\big(f^j(x)\big) = \int \varphi\, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \tag{4}$$

Defining the *correlation function* of observables $\varphi, \psi : M \to \mathbb{R}$ as

$$\mathcal{C}_n(\varphi, \psi) = \left|\int \big(\varphi \circ f^n\big)\psi\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu\right|,$$

it is sometimes possible to obtain specific rates for which $C_n(\varphi, \psi)$ decays to 0 as n tends to infinity, at least for certain classes of observables with some regularity. Note that taking the observables as characteristic functions of Borel sets we are led to the classical definition of *mixing*.

The next two corollaries follow from Theorem A together with [8, Theorem B] and [8, Theorem C]; see also [8, Remark 2.4].

Corollary B *(Decay of correlations). Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a transitive partially hyperbolic set such that $T_K M = E^s \oplus E^{cu}$. Assume that there is a local unstable disk $D \subset K$ and $\tau > 1$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = O(n^{-\tau})$. Then some power of f has an SRB measure μ and $\mathcal{C}_n(\varphi, \psi) = O(n^{-\tau+1})$ for Hölder continuous $\varphi, \psi : M \to \mathbb{R}$.*

The existence of the SRB measure for f has already been proved in [5, Theorem A]. In general, we can only assure that the correlation decay holds for some power of f. However, if the return times associated to the elements of the Gibbs–Markov structure given by Theorem A are relatively prime, i.e. $\gcd\{R_i\} = 1$, then the same conclusion holds with respect to f. *For simplicity, from here on we assume that* $\gcd\{R_i\} = 1$. Otherwise, the same conclusions hold for some power of f.

In the next result we obtain conditions for the validity of the *Central Limit Theorem* (*CLT*), which states that the deviation of the average values of an observable along an orbit from the asymptotic average is given by a Normal Distribution: given any Hölder continuous $\varphi : M \to \mathbb{R}$ which is not a *coboundary* ($\varphi \neq \psi \circ f - \psi$ for any $\psi \in L^2$) there exists $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\varphi \circ f^j - \int \varphi\, d\mu\right) \xrightarrow{\text{distr}} N(0, \sigma), \quad \text{as } n \to \infty.$$

Corollary C *(Central limit theorem). Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a transitive partially hyperbolic set with $T_K M = E^s \oplus E^{cu}$. Assume that there is a local unstable disk $D \subset K$ and $\tau > 2$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = O(n^{-\tau})$. Then CLT holds for any Hölder continuous $\varphi : M \to \mathbb{R}$ which is not a coboundary.*

Given a Hölder continuous observable $\varphi : M \to \mathbb{R}$ and $\epsilon > 0$, we define the *large deviation* of the time average with respect to the mean of φ as

$$\mathcal{D}_n(\varphi, \epsilon) = \mu\left(\left\{x \in M\colon \left|\frac{1}{n}\sum_{j=0}^{n-1}\varphi\big(f^j(x)\big) - \int \varphi\, d\mu\right| > \epsilon\right\}\right).$$

The next result is a consequence of Theorem A together with [19, Theorem 4.2] and [18, Theorem 1.2].

Corollary D *(Large deviations). Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a transitive partially hyperbolic set with $T_K M = E^s \oplus E^{cu}$. Assume that there is a local unstable disk $D \subset K$ and $\tau > 2$ such that* $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(n^{-\tau})$*. Then, for any Hölder continuous $\varphi : M \to \mathbb{R}$ and any $\epsilon > 0$, we have $\mathcal{D}_n(\varphi, \epsilon) = O(n^{-\tau+1})$.*

Given $d \geqslant 1$ and a Hölder continuous $\varphi : M \to \mathbb{R}^d$, we denote $\varphi_n = \sum_{j=0}^{n-1}(\varphi \circ f^j - \int \varphi\, d\mu)$, for each $n \geqslant 1$. We say that the sequence $\{\varphi_n\}_n$ satisfies a *d-dimensional almost sure invariance principle* (*ASIP*) if there exists $\lambda > 0$ and a probability space supporting a sequence of random variables $\{\varphi_n^*\}_n$ and a d-dimensional Brownian motion $W(t)$ such that

1. $\{\varphi_n\}_n$ and $\{\varphi_n^*\}_n$ are equally distributed;
2. $\varphi_n^* = W(n) + O(n^{1/2-\lambda})$, as $n \to \infty$, almost everywhere.

The ASIP is said to be *non-degenerate* if the Brownian motion $W(t)$ has non-singular covariance matrix Σ. For the dynamical systems considered in this paper, there is a closed subspace Z of infinite codimension in the space of all (piecewise) Hölder $\varphi : M \to \mathbb{R}^d$ such that Σ is non-singular whenever $\varphi \notin Z$; see [20, Remark 1.2] and [16, Section 4.3].

The next result follows from Theorem A and [20, Theorem 1.6], and it strengthens Corollary C above.

Corollary E *(Almost sure invariance principle). Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a transitive partially hyperbolic set with $T_K M = E^s \oplus E^{cu}$. Assume that there is a local unstable disk $D \subset K$ and $\tau > 2$ such that* $\mathrm{Leb}_D\{\mathcal{E} > n\} = O(n^{-\tau})$*. Then $\{\varphi_n\}_n$ satisfies an ASIP for any Hölder continuous $\varphi \notin Z$.*

2. Preliminaries

With the only exception of Lemma 2.7, the material contained in this section comes from [5]. Our aim is to recall some results on the Hölder continuity of the tangent direction of center-unstable submanifolds, to introduce hyperbolic times and recall their main properties.

We fix continuous extensions of the two subbundles E^s and E^{cu} to some neighborhood U of K that we denote by $\tilde{E}^s$ and $\tilde{E}^{cu}$. We do not require these extensions to be invariant under Df. Given $0 < a < 1$, we define the *center-unstable cone field $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ of width a* by

$$C_a^{cu}(x) = \big\{v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_1\| \leqslant a\|v_2\|\big\}. \tag{5}$$

We define the *center-stable cone field $C_a^s = (C_a^s(x))_{x\in U}$ of width a* in a similar way, simply reversing the roles of the subbundles in (5). We fix $a > 0$ and U small enough so that, up to slightly increasing $\lambda < 1$, the domination condition (1) remains valid for any pair of vectors in the two cone fields:

$$\left\| Df(x)v^s \right\| \cdot \left\| Df^{-1}\big(f(x)\big)v^{cu} \right\| \leqslant \lambda \left\| v^s \right\| \left\| v^{cu} \right\|$$

for every $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$, and any point $x \in U \cap f^{-1}(U)$. Note that the center-unstable cone field is positively invariant:

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}\big(f(x)\big), \quad \text{whenever } x, f(x) \in U.$$

Actually, the domination property together with the invariance of $E^{cu} = \tilde{E}^{cu}|K$ imply that

$$Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}\big(f(x)\big) \subset C_a^{cu}\big(f(x)\big),$$

for every $x \in K$, and this extends to any $x \in U \cap f^{-1}(U)$ just by continuity.

We say that an embedded C^1 submanifold $N \subset U$ is *tangent to the center-unstable cone field*, if the tangent subspace to N at each point $x \in N$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(N)$ is also tangent to the center-unstable cone field, if it is contained in U, by the domination property. The tangent bundle of N is said to be *Hölder continuous* if $x \mapsto T_xN$ defines a Hölder continuous section from N to the corresponding Grassman bundle of M. Given a C^1 submanifold $N \subset U$, we define

$$\kappa(N) = \inf\big\{C > 0\colon \text{ the tangent bundle of } N \text{ is } (C, \zeta)\text{-Hölder}\big\}. \tag{6}$$

The next result contains all the information we need on the Hölder control of the tangent direction and its proof can be found in [5, Corollary 2.4].

Proposition 2.1. *There exists $C_1 > 0$ such that, given any C^1 submanifold $N \subset U$ tangent to the center-unstable cone field, then*

1. *there exists $n_0 \geqslant 1$ such that $\kappa(f^n(N)) \leqslant C_1$ for every $n \geqslant n_0$ such that $f^k(N) \subset U$ for all $0 \leqslant k \leqslant n$;*
2. *if $\kappa(N) \leqslant C_1$, then $\kappa(f^n(N)) \leqslant C_1$ for every $n \geqslant 1$ such that $f^k(N) \subset U$ for all $0 \leqslant k \leqslant n$;*
3. *if N and n are as in the previous item, then the functions*

$$J_k : f^k(N) \ni x \mapsto \log\big|\det\big(Df \mid T_x f^k(N)\big)\big|, \quad 0 \leqslant k \leqslant n,$$

are (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f.

The notion we introduce next allows us to derive uniform expansion and bounded distortion estimates from the non-uniform expansion assumption in the center-unstable direction.

Definition 2.2. Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \left\| Df^{-1} \mid E_{f^j(x)}^{cu} \right\| \leqslant \sigma^k, \quad \text{for all } 1 \leqslant k \leqslant n.$$

In particular, if n is a σ-hyperbolic time for x, then $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for every $1 \leqslant k \leqslant n$:

$$\left\| Df^{-k} \mid E^{cu}_{f^n(x)} \right\| \leqslant \prod_{j=n-k+1}^{n} \left\| Df^{-1} \mid E^{cu}_{f^j(x)} \right\| \leqslant \sigma^k. \tag{7}$$

If $a > 0$ is sufficiently small and we choose $\delta_1 > 0$ also small so that the δ_1-neighborhood of K is contained in U, then, by continuity,

$$\left\| Df^{-1}\big(f(y)\big)v \right\| \leqslant \frac{1}{\sqrt{\sigma}} \left\| Df^{-1} \mid E^{cu}_{f(x)} \right\| \|v\|, \tag{8}$$

whenever $x \in K$, $\operatorname{dist}(x, y) \leqslant \delta_1$, and $v \in C^{cu}_a(y)$.

Given any disk $\Delta \subset M$, we use $\operatorname{dist}_\Delta(x, y)$ to denote the distance between $x, y \in \Delta$, measured along Δ. The distance from a point $x \in \Delta$ to the boundary of Δ is $\operatorname{dist}_\Delta(x, \partial\Delta) = \inf_{y \in \partial\Delta} \operatorname{dist}_\Delta(x, y)$. The next result has essentially been proved in [5, Lemma 2.7]; see [7, Lemma 4.2] for a detailed proof.

Lemma 2.3. *Let $0 < \delta < \delta_1$ and $\Delta \subset U$ be a C^1 disk of radius δ tangent to the center-unstable cone field. Then, there is $n_0 \geqslant 1$ such that for $x \in \Delta \cap K$ with $\operatorname{dist}_\Delta(x, \partial\Delta) \geqslant \delta/2$ and $n \geqslant n_0$ a σ-hyperbolic time for x there is a neighborhood V_n of x in Δ such that:*

1. *f^n maps V_n diffeomorphically onto a center-unstable disk of radius δ_1 around $f^n(x)$;*
2. *for every $1 \leqslant k \leqslant n$ and $y, z \in V_n$,*

$$\operatorname{dist}_{f^{n-k}(V_n)}\big(f^{n-k}(y), f^{n-k}(z)\big) \leqslant \sigma^{k/2} \operatorname{dist}_{f^n(V_n)}\big(f^n(y), f^n(z)\big).$$

We call the sets V_n *hyperbolic pre-balls* and their images $f^n(V_n)$ *hyperbolic balls*. Notice that the latter are indeed center-unstable balls of radius δ_1. The next result follows from Proposition 2.1 and Lemma 2.3 above exactly as in the proof of [5, Proposition 2.8].

Corollary 2.4. *There exists $C_2 > 1$ such that given a disk Δ as in Lemma 2.3 with $\kappa(\Delta) \leqslant C_1$, and given any hyperbolic pre-ball $V_n \subset \Delta$ with $n \geqslant n_0$, then for all $y, z \in V_n$*

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \leqslant C_2 \operatorname{dist}_{f^n(D)}\big(f^n(y), f^n(y)\big)^\zeta.$$

The next result states the existence of σ-hyperbolic times for points satisfying (NUE) and its proof can be found in [5, Lemma 3.1, Corollary 3.2].

Proposition 2.5. *There are $\theta > 0$ and $\sigma > 0$ such that for every $x \in K$ with $\mathcal{E}(x) \leqslant n$ there exist σ-hyperbolic times $1 \leqslant n_1 < \cdots < n_l \leqslant n$ for x with $l \geqslant \theta n$.*

Given $n \geqslant 1$, we define

$$H_n = \{x \in K\colon n \text{ is a } \sigma\text{-hyperbolic time for } x\}.$$

The next result follows from the proposition above as in [5, Proposition 3.5] or [6, Corollary 2.3]. It plays important role in the metric estimates of Section 4.

Corollary 2.6. *Let D be a local unstable disk for which* (NUE) *holds* Leb_D *almost everywhere. Given $n \geqslant 1$ and $A \subset D \setminus \{\mathcal{E} > n\}$ with* $\mathrm{Leb}_D(A) > 0$ *we have*

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\mathrm{Leb}_D(A\cap H_j)}{\mathrm{Leb}_D(A)} \geqslant \theta.$$

We finish this section with a technical lemma which will be useful in the proof of Proposition 4.3. We take $\delta_s > 0$ sufficiently small so that local stable manifolds $W^s_{\delta_s}(x)$ are defined for all points $x \in K$ and

$$\left|\log\left\|Df^{-1} \mid E^{cu}_x\right\| - \log\left\|Df^{-1} \mid \tilde{E}^{cu}_y\right\|\right| < \frac{c}{4}, \tag{9}$$

for all $x \in K$ and $y \in W^s_{\delta_s}(x)$, where $c > 0$ is given by (NUE).

Lemma 2.7. *Given $0 < \varepsilon < \delta_1$, there exists $N_\varepsilon > 0$ such that for every $n \geqslant n_0$ and every $x \in D \cap H_n$, the ball of radius ε centered at $f^n(z)$ inside the hyperbolic ball $f^n(V_n(x))$ contains a hyperbolic pre-ball $V_k(z)$ with $k \leqslant N_\varepsilon$.*

Proof. Given $n \geqslant n_0$ and $x \in D \cap H_n$, let $B^n_x = f^n(V_n(x))$ be the hyperbolic ball associated to x with hyperbolic time n. Recall that B^n_x is a center-unstable ball of radius δ_1 around $f^n(x)$. We define the cylinder

$$C^n_x = \bigcup_{y\in B^n_x} W^s_{\delta_s}(y).$$

Since (NUE) holds for Leb_D almost every point and it remains valid by forward iteration, it follows that $\mathrm{Leb}_{B^n_x}$ almost every point in B^n_x also satisfies (NUE). Given $z \in C^n_x$, we define

$$\tilde{\mathcal{E}}(z) = \min\left\{N \geqslant 1\colon\ \frac{1}{n}\sum_{i=0}^{n-1}\log\left\|Df^{-1} \mid \tilde{E}^{cu}_{f^i(w)}\right\| < -\frac{3c}{4},\ \forall n \geqslant N\right\}.$$

The fact that (NUE) holds for $\mathrm{Leb}_{B^n_x}$ almost every point in B^n_x together with (9) imply that $\tilde{\mathcal{E}}(z)$ is well defined for Leb almost every point $z \in C^n_x$. Hence $\tilde{\mathcal{E}}(z)$ is well defined for Leb almost every point z belonging to the set

$$C = \bigcup_{\substack{n\geqslant n_0\\ x\in D\cap H_n}} C^n_x.$$

Using Lemma 2.3 we may choose n_ε large enough so that any hyperbolic pre-ball of σ-hyperbolic time $n \geqslant n_\varepsilon$ will have diameter not exceeding $\varepsilon/2$. Let now $B_x^n(\varepsilon/2)$ denote the ball of radius $\varepsilon/2$ around $f^n(x)$ inside B_x^n, and take

$$v_\varepsilon = \min_{\substack{n\geqslant n_0 \\ x\in D\cap H_n}} \left\{ \mathrm{Leb}\left(\bigcup_{y\in B_x^n(\varepsilon/2)} W_{\delta_s}^s(y) \right) \right\}.$$

Since the sizes and tangent directions of hyperbolic balls and stable disks are uniformly controlled, this minimum v_ε must be strictly positive. Hence, as

$$\mathrm{Leb}\left\{z \in C\colon \tilde{\mathcal{E}}(z) > n\right\} \to 0, \quad \text{when } n \to \infty,$$

it is possible to choose $N_\varepsilon \in \mathbb{N}$ large enough so that

$$\mathrm{Leb}\left\{z \in C\colon \tilde{\mathcal{E}}(z) > N_\varepsilon\right\} \leqslant v_\varepsilon. \tag{10}$$

We take N_ε also satisfying $\theta N_\varepsilon > n_\varepsilon$. By the choice of v_ε, given $n \geqslant n_0$ and $x \in D \cap H_n$ there must be some $z \in W_{\delta_s}^s(y)$ with $y \in B_x^n(\varepsilon/2)$ such that $\tilde{\mathcal{E}}(z) \leqslant N_\varepsilon$. Using (9) we easily deduce that $\mathcal{E}(z) \leqslant N_\varepsilon$; recall (3). Hence, by Proposition 2.5 there exists some σ-hyperbolic time h for y with $\theta N_\varepsilon < h \leqslant N_\varepsilon$. Since we have taken $\theta N_\varepsilon > n_\varepsilon$, then by the choice of n_ε we are done. □

3. Gibbs–Markov structure

In this section we describe the geometric construction of the product structure. This will be made in three steps. In the first one we prove the existence of a center-unstable disk (a reference disk) for which forward iterates come back to a neighborhood of itself, and whose projection along stable leaves cover the disk completely. In the next step we use these returns to define a partition on the reference disk. This part of the construction follows ideas from [6, Section 3]. Finally we use the partition on the reference leaf and the returns to define the product structure.

3.1. Returning disks

Let D be a local unstable disk as in Theorem A. Diminishing $\delta_1 > 0$, if necessary, we may assume that D has radius δ_1. Take $0 < \delta_s < \delta_1/2$ such that points in K have local stable manifolds of radius δ_s. In particular, these local stable manifolds are contained in U; recall (8).

Lemma 3.1. *There are $N_0 \geqslant 1$ and $q \in K$ such that:*

1. *$W_{\delta_s/2}^s(q)$ intersects D in a point p with $\mathrm{dist}_D(p, \partial D) > \delta_1/2$;*
2. *for each center-unstable disk γ_1^u of radius δ_1 centered at a point in K there are $0 \leqslant j \leqslant N_0$ and a disk $\gamma_2^u \subset \gamma_1^u$ of radius $\delta_1/2$ centered at a point $z \in W_{\delta_s/4}^s(f^{-j}(q))$.*

Proof. We start by observing that there is a constant $\alpha = \alpha(\rho) > 0$ with $\alpha \to 0$ as $\rho \to 0$ for which the following holds: *given $x \in K$, $\rho > 0$ and $y \in K$ with* $\mathrm{dist}(x, y) < \rho$ *having a local unstable disk of radius δ_1 centered at y, then $W_{\delta_s}^s(x)$ intersects $W_{\delta_1}^u(y)$ in a point z with*

$$\mathrm{dist}_{W_{\delta_s}^s(x)}(z, x) < \alpha \quad \textit{and} \quad \mathrm{dist}_{W_{\delta_1}^u(y)}(z, y) < \delta_1/2.$$

In particular, such a point z has a neighborhood of radius $\delta_1/2$ inside $W^u_{\delta_1}(y)$.

Take $\rho > 0$ small so that $4\alpha < \delta_s$. Since we are assuming $f|K$ transitive, we may fix $q \in K$ and $N_0 \in \mathbb{N}$ such that both (i) $W^s_{\delta_s/2}(q)$ intersects D in a point p with $\mathrm{dist}_D(p, \partial D) > \delta_1/2$, and (ii) $\{f^{-N_0}(q), \dots, f^{-1}(q), q\}$ is ρ-dense in K. By ρ-dense we mean that any other point in K has one of the points in the set above at a distance less than ρ.

Hence, given any center-unstable disk γ^u_1 of radius δ_1 centered at a point $y \in K$, there is $0 \leqslant j \leqslant N_0$ such that $\mathrm{dist}(f^{-j}(q), y) < \rho$. Then, by the choice of α and ρ, we have that $W^s_{\delta_s}(f^{-j}(q))$ intersects γ^u_1 in a point z with $\mathrm{dist}_{W^s_{\delta_s}(f^{-j}(q))}(z, f^{-j}(q)) < \alpha < \delta_s/4$ and $\mathrm{dist}_{\gamma^u_1}(z, y) < \delta_1/2$. □

Lemma 3.2. *There is $\delta_2 > 0$ such that if γ^u is a center-unstable disk of radius $\delta_1/2$ centered at a point $z \in W^s_{\delta_s}(w)$ with $w \in K$, then $f^j(\gamma^u)$ contains a center-unstable disk of radius δ_2 centered at $f^j(z)$, for each $1 \leqslant j \leqslant N_0$.*

Proof. Let us first prove the result for $j = 1$. Let $f(y)$ be a point in $\partial f(\gamma^u)$ minimizing the distance from $f(z)$ to $\partial f(\gamma^u)$, and let η_1 be a curve of minimal length in $f(\gamma^u)$ connecting $f(z)$ to $f(y)$. Define $\eta_0 = f^{-1}(\eta_1)$. Denote by $\dot{\eta}_1(x)$ the tangent vector to the curve η_1 at the point x. Then,

$$\left\| Df^{-1}(w)\dot{\eta}_1(x)\right\| \leqslant C\left\|\dot{\eta}_1(x)\right\|,$$

where

$$C = \max_{x\in M}\left\{\left\|Df^{-1}(x)\right\|\right\} \geqslant 1.$$

Hence,

$$\mathrm{length}(\eta_0) \leqslant C\,\mathrm{length}(\eta_1).$$

Noting that η_0 is a curve connecting z to $y \in \partial\gamma^u$, this implies that $\mathrm{length}(\eta_0) \geqslant \delta_1/2$. Hence

$$\mathrm{length}(\eta_1) \geqslant C^{-1}\,\mathrm{length}(\eta_0) \geqslant C^{-1}\delta_1/2.$$

Thus $f(\gamma^u)$ contains the disk γ^u_1 of radius $C^{-1}\delta_1/2$ around $f(z)$. Moreover,

$$\mathrm{dist}\big(f(z), f(w)\big) \leqslant \lambda\delta_s < \delta_s,$$

and so, by the choice of δ_s, we have that γ^u_1 is also a center-unstable disk. Making now γ^u_1 play the role of γ^u and $f^2(z)$ play the role of $f(z)$ we prove that:

(a) $f(\gamma^u_1)$ contains a center-unstable disk of radius $C^{-2}\delta_1/2^2$ centered at $f^2(z)$;
(b) $\mathrm{dist}(f^2(z), f^2(w)) \leqslant \lambda^2\delta_s < \delta_s$.

Item (a) gives in particular that $f^2(\gamma^u)$ contains a center-unstable disk of radius $C^{-2}\delta_1/2^2$ centered at $f^2(z)$. Arguing inductively we are able to prove that $f^j(\gamma^u)$ contains a disk of radius $C^{-j}\delta_1/2^j \geqslant C^{-N_0}\delta_1/2^{N_0}$ around $f^j(z)$, for each $1 \leqslant j \leqslant N_0$. Hence, we just have to take $\delta_2 = C^{-N_0}\delta_1/2^{N_0}$. □

3.2. Partition on the reference leaf

The construction that we are going to explain below requires the use several constants. First we take $\delta_1 > 0$ as in (8), and $0 < \delta_2 < \delta_1$ as in Lemma 3.2. Then we take $\delta_0 > 0$ and $\varepsilon > 0$ so that

$$\delta_0 \ll \delta_2 \quad \text{and} \quad \varepsilon \ll \delta_0.$$

Next we describe the construction of the (m_D mod 0) partition $\mathcal{P}$ of the unstable disk of radius δ_0 centered at p contained in D. For that we consider the following neighborhoods of p in D

$$\Delta_0^0 = B^u(p, \delta_0), \quad \Delta_0^1 = B^u(p, 2\delta_0), \quad \Delta_0^2 = B^u(p, \sqrt{\delta_0}) \quad \text{and} \quad \Delta_0^3 = B^u(p, 2\sqrt{\delta_0}),$$

and the cylinders over these sets,

$$\mathcal{C}^i = \bigcup_{x \in \Delta_0^i} W^s_{\delta_s}(x), \quad \text{for } i = 0, 1, 2, 3.$$

Letting π denote the projection from $\mathcal{C}^3$ onto Δ_0^3 along local stable leaves, we have

$$\pi\big(\mathcal{C}^i\big) = \Delta_0^i, \quad \text{for } i = 0, 1, 2, 3.$$

We say that a center-unstable disk γ^u *u-crosses* $\mathcal{C}^i$ if $\pi(\gamma^u) = \Delta_0^i$.

Remark 3.3. To simplify the exposition we shall pretend that each center-unstable disk γ^u u-crossing $\mathcal{C}^i$ is still a disk centered at a point in $W^s_{\delta_s}(p)$ with the same radius of Δ_0^i. Actually, the radius of such a disk γ^u is proportional to the radius of Δ_0^i, with the proportionality depending only on the height of the cylinder and the angles of the two fiber bundles in the dominated splitting.

Let $\partial^u \mathcal{C}_1^3$ denote the *top* and *bottom* components of $\partial\mathcal{C}^3$, i.e. the set of points $z \in \partial\mathcal{C}^3$ such that $z \in \partial W^s_{\delta_s}(x)$ for some $x \in \Delta_0^3$. By the domination property, we may take $\delta_0 > 0$ small so that no center-unstable disk contained in $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/2}(p)$ can reach $\partial^u\mathcal{C}^3$. For $0 < \sigma < 1$ given by Lemma 2.5, let

$$I_k = \big\{x \in \Delta_0^1\colon\ \delta_0\big(1 + \sigma^{k/2}\big) < \operatorname{dist}_D(x, p) < \delta_0\big(1 + \sigma^{(k-1)/2}\big)\big\}, \quad k \geqslant 1,$$

be a partition (Leb_D mod 0) into countably many rings of $\Delta_0^1 \setminus \Delta_0^0$.

We are now able to start with the construction of the partition $\mathcal{P}$ of Δ_0. The construction requires that we introduce inductively several objects. In particular, we will consider sequences of sets (Δ_n), (A_n), and (B_n). For each $n \geqslant 0$, the set Δ_n is that part of Δ_0 that has not yet been partitioned up to time n. The set Δ_n is the disjoint union of A_n and B_n, where A_n is essentially the part of Δ_n where new elements of partition may be appear in the next step of the construction, and B_n is some protection that we put around the sets previously constructed in order to avoid overlaps. For technical reasons, a small neighborhood A_n^ε of each A_n will also be considered.

3.2.1. First step of induction

We fix R_0 some large integer, and we ignore any dynamics occurring up to time R_0. Let $k \geqslant R_0 + 1$ be the first time that $\Delta_0 \cap H_k \neq \emptyset$. For $j < k$ we define formally the objects

$$A_j = A_j^{\varepsilon} = \Delta_j = \Delta_0 \quad \text{and} \quad B_j = \emptyset.$$

Let $(\omega_{k,j}^3)_j$ be all the center-unstable disks in A_{k-1}^{ε} contained in hyperbolic pre-balls V_m, with $k - N_0 \leqslant m \leqslant k$, which are mapped by f^k onto a center-unstable disk u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Then we let

$$\omega_{k,j}^i = \omega_{k,j}^3 \cap f^{-k}\big(\mathcal{C}^i\big), \quad i = 0, 1, 2$$

and set $R(x) = k$ for $x \in \omega_{k,j}^0$. We take

$$\Delta_k = \Delta_{k-1} \setminus \{R = k\},$$

and define a function $t_k : \Delta_k \to \mathbb{N}$ by

$$t_k(x) = \begin{cases} s, & \text{if } x \in \omega_{k,j}^1 \text{ and } \pi(f^k(x)) \in I_s \text{ for some } j; \\ 0, & \text{otherwise.} \end{cases}$$

Finally let

$$A_k = \big\{x \in \Delta_k \colon t_k(x) = 0\big\}, \qquad B_k = \big\{x \in \Delta_k \colon t_k(x) > 0\big\}$$

and

$$A_k^{\varepsilon} = \big\{x \in \Delta_k \colon \operatorname{dist}_{f^{k+1}(D)}\big(f^{k+1}(x), f^{k+1}(A_k)\big) < \varepsilon\big\}.$$

General step of induction

The general step of the construction follows the ideas above with minor modifications. Assume that the sets Δ_i, A_i, A_i^{ε} B_i, $\{R = i\}$ and functions $t_i : \Delta_i \to \mathbb{N}$ are defined for each $i \leqslant n - 1$. Let $(\omega_{n,j}^3)_j$ be all the center-unstable disks in A_{n-1}^{ε} contained in hyperbolic pre-balls V_m, with $n - N_0 \leqslant m \leqslant n$, which are mapped by f^n onto a center-unstable disk u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Take

$$\omega_{n,j}^i = \omega_{n,j}^3 \cap f^{-n}\big(\mathcal{C}_1^i\big), \quad i = 0, 1, 2 \tag{11}$$

set $R(x) = n$ for $x \in \omega_{n,j}^0$, and let

$$\Delta_n = \Delta_{n-1} \setminus \{R = n\}.$$

The definition of the function $t_n : \Delta_n \to \mathbb{N}$ is slightly different in the general case:

$$t_n(x) = \begin{cases} x, & \text{if } x \in \omega_{n,j}^1 \setminus \omega_{n,j}^0 \text{ and } f^n(x) \in I_s \text{ for some } j, \\ 0, & \text{if } x \in A_{n-1} \setminus \bigcup_j \omega_{n,j}^1, \\ t_{n-1}(x) - 1, & \text{if } x \in B_{n-1} \setminus \bigcup_j \omega_{n,j}^1. \end{cases}$$

Finally, we let

$$A_n = \{x \in \Delta_n \colon t_n(x) = 0\}, \qquad B_n = \{x \in \Delta_n \colon t_n(x) > 0\}$$

and

$$A_n^\varepsilon = \{x \in \Delta_n \colon \operatorname{dist}_{f^{n+1}(D)}(f^{n+1}(x), f^{n+1}(A_n)) < \varepsilon\}.$$

At this point we have described the construction of the sets A_n, A_n^ε, B_n and $\{R = n\}$.

Since the components of $\{R = n\}$ are taken in A_{n-1}^ε, it could happen that these new components intersect B_{n-1}. The next lemma shows that this is not the case as long as $\epsilon > 0$ is taken small enough. For notational simplicity we will drop the index j in the elements defined at (11).

Lemma 3.4. *If $\epsilon > 0$ is small, then $\omega_n^1 \cap \{t_{n-1} \geqslant 1\} = \emptyset$ for all $n \geqslant 1$.*

Proof. Take $k \geqslant 1$ and let ω_{n-k}^0 be a component of $\{R = n - k\}$. Let Q_k be the part of ω_{n-k}^1 that is mapped by $\pi \circ f^{n-k}$ onto I_k, and assume that Q_k intersects some ω_n^3. Recall that, by construction, Q_k is precisely that part of ω_{n-k}^1 on which $t_{n-1} = 1$, and ω_n^3 is contained in a hyperbolic pre-ball V_m with $n - N_0 \leqslant m \leqslant n$.

Let q_1 and q_2 be any two points in distinct components (inner and outer, respectively) of the boundary of Q_k. If we assume that $q_1, q_2 \in \omega_n^3$, then $q_1, q_2 \in V_m$, and so by Lemma 2.3 we have

$$\operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(q_1), f^{n-k}(q_2)) \leqslant C_0 \sigma^{k/2} \operatorname{dist}_{f^n(D)}(f^n(q_1), f^n(q_2)) \tag{12}$$

for some C_0 depending on N_0. We also have for some $C_1 > 0$ depending on the angle of the stable and center-unstable spaces over K_∞

$$\begin{aligned} \operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(q_1), f^{n-k}(q_2)) &\geqslant C_1 \delta_0 (1 + \sigma^{(k-1)/2}) - \delta_0 (1 + \sigma^{k/2}) \\ &= C_1 \delta_0 \sigma^{k/2} (\sigma^{-1/2} - 1), \end{aligned}$$

which combined with (12) gives

$$\operatorname{dist}_{f^n(D)}(f^n(q_1), f^n(q_2)) \geqslant \frac{C_1}{C_0} \delta_0 (\sigma^{-1/2} - 1).$$

On the other hand, since $\omega_n^3 \subset A_{n-1}^\varepsilon$ by construction of ω_n^3, taking

$$\varepsilon < \frac{C_1}{C_0} \delta_0 (\sigma^{-1/2} - 1) \tag{13}$$

we have $\omega_n^3 \cap \{t_{n-1} > 1\} = \emptyset$. This implies $\omega_n^1 \cap \{t_{n-1} \geqslant 1\} = \emptyset$. □

3.3. Product structure

Consider the center-unstable disk $\Delta_0 \subset D$ and the partition $\mathcal{P}$ of Δ_0 (Leb_D mod 0) defined in Section 3.2. We shall use the elements of $\mathcal{P}$ to define the s-subsets that give rise to the hyperbolic structure. Given an arbitrary element $\omega \in \mathcal{P}$, we have by construction some $R(\omega) \in \mathbb{N}$ such that $f^{R(\omega)}(\omega)$ is a center-unstable disk u-crossing $\mathcal{C}^0$. We define $\mathcal{C}_\omega$ as the cylinder made by the stable leaves passing through the points in ω, i.e.

$$\mathcal{C}_\omega = \bigcup_{x\in\omega} W^s_{\delta_s}(x).$$

The sets $\mathcal{C}_\omega$, with $\omega \in \mathcal{P}$, are by definition the pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots$ which define the Markovian structure.

Now we define inductively some sets of center-unstable manifolds u-crossing $\mathcal{C}^0$ that will give rise to the family Γ^u. The first one is

$$\Gamma_0 = \{\Delta_0\}.$$

Having defined Γ_j, for some $j \geqslant 0$, we define

$$\Gamma_{j+1} = \bigl\{ f^{R(\omega)}(\mathcal{C}_\omega \cap \gamma)\colon\ \omega \in \mathcal{P} \text{ and } \gamma \in \Gamma_j \bigr\}.$$

Observe that each element of Γ_j is equal to an iterate of a subset of Δ_0. In particular, the elements of each Γ_j are unstable manifolds. Moreover, since by construction $f^{R(\omega)}(\omega)$ intersects $W^s_{\delta_s/4}(p)$, then according to the choice of δ_0 and the invariance of the stable foliation, we have that each element of Γ_j must u-cross $\mathcal{C}^0$.

Since the union of the leaves of the sets Γ_j, with $j \geqslant 0$, is not necessarily compact, we still need to take accumulation points of that union. Let

$$\Delta_\infty = \overline{\bigcup_{j\geqslant 0} \bigcup_{\gamma_j\in\Gamma_j} \gamma_j}.$$

Given $x \in \Delta_\infty$, there are $(j_k)_k \to \infty$, disks $\gamma_{j_k} \in \Gamma_{j_k}$ and points $x_k \in \gamma_{j_k}$ converging to x as $k \to \infty$. Using the domination property and Ascoli–Arzela theorem we conclude that the disks γ_{j_k} converge to a disk γ_∞ containing x. Since the disk γ_∞ is accumulated by disks u-crossing $\mathcal{C}^0$ then it also must u-cross $\mathcal{C}^0$. We define Γ_∞ as the set of all these accumulation disks. Finally, we take

$$\Gamma^u = \bigcup_{j\geqslant 0} \Gamma_j \cup \Gamma_\infty.$$

3.4. Backward contraction and bounded distortion

The backward contraction property (P_3) follows from Lemma 3.5 below. Bounded distortion is typically a consequence of backward contraction together with some Hölder control of $\log|\det Df^u|$. Property (P_4) follows naturally from Proposition 2.1 together with Lemma 3.5 exactly as in the proof of [5, Proposition 2.8].

Lemma 3.5. *There is $C > 0$ such that, given $\omega \in \mathcal{P}$ and $\gamma \in \Gamma^u$, we have for all $1 \leqslant k \leqslant R(\omega)$ and all $x, y \in \mathcal{C}_\omega \cap \gamma$*

$$\operatorname{dist}_{f^{R(\omega)-k}(\mathcal{C}_\omega\cap\gamma)}\big(f^{R(\omega)-k}(x), f^{R(\omega)-k}(y)\big) \leqslant C\sigma^{k/2}\operatorname{dist}_{f^{R(\omega)}(\mathcal{C}_\omega\cap\gamma)}\big(f^{R(\omega)}(x), f^{R(\omega)}(y)\big).$$

Proof. Recall that, by construction, for each $\omega \in \mathcal{P}$ there is a hyperbolic pre-ball $V_{n(\omega)}(x)$ containing ω associated to some point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \leqslant n(\omega) \leqslant R(\omega)$. Taking $\delta_s, \delta_0 < \delta_1/2$, it follows from (8) that $n(\omega)$ is a $\sqrt{\sigma}$-hyperbolic time for every point in $\mathcal{C}_\omega \cap \gamma$. Then, recall (7), this implies that for all $1 \leqslant k \leqslant n(\omega)$ and all $x, y \in \mathcal{C}_\omega \cap \gamma$ we have

$$\operatorname{dist}_{f^{n(\omega)-k}(\mathcal{C}_\omega\cap\gamma)}\big(f^{n(\omega)-k}(x), f^{n(\omega)-k}(y)\big) \leqslant \sigma^{k/2}\operatorname{dist}_{f^{n(\omega)}(\mathcal{C}_\omega\cap\gamma)}\big(f^{n(\omega)}(x), f^{n(\omega)}(y)\big).$$

Since the difference between $R(\omega)$ and $n(\omega)$ is at most N_0, the result follows with C depending only on N_0 and the derivative of f. □

3.5. Regularity of the foliations

Here we prove property (P_5). This is standard for uniformly hyperbolic attractors, and we shall adapt classical ideas to our setting. We begin with the statement of a useful result on vector bundles whose proof can be found in [15, Theorem 6.1].

Lemma 3.6. *Let $p : Y \to X$ be a vector bundle over a metric space X endowed with an admissible metric. Let $D \subset Y$ be the unit ball bundle, and $F : D \to D$ a map covering a homeomorphism $f : X \to X$. Suppose $0 \leqslant \kappa < 1$ and that for each $x \in X$, the restriction $F_x : D_x \to D_x$ has Lipschitz constant $\operatorname{Lip}(F_x) \leqslant \kappa$. Then*

1. *there is a unique section $\sigma_0 : X \to D$ whose image is invariant under F;*
2. *let $\operatorname{Lip}(f) = c < \infty$ and $0 < \alpha \leqslant 1$ be such that $\kappa c^\alpha < 1$. Then σ_0 satisfies a Hölder condition of exponent α.*

For the sake of completeness, let us mention that a metric d on E is *admissible* if there is a complementary bundle E' over X, and an isomorphism $h : E \oplus E' \to X \times A$ to a product bundle, where A is a Banach space, such that d is induced from the product metric on $X \times A$.

Theorem 3.7. *Let $f : M \to M$ be a C^1 diffeomorphism and let $K \subset M$ be a compact invariant set with a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$. Then the fiber bundles E^{cs} and E^{cu} are Hölder continuous on K.*

Proof. We consider the center-unstable bundle, the other one is similar. For each $x \in K$ let L_x be the space of bounded linear maps $L(E^{cu}_x, E^{cs}_x)$. For each $x \in K$, let $L_x(1)$ denote the unit ball around $0 \in L_x$, and define $\Gamma_x : L_x(1) \to L_{f(x)}(1)$ as the graph transform induced by $Df(x)$:

$$\Gamma_x(\mu_x) = \big(Df \mid E^{cs}_x\big) \cdot \mu_x \cdot \big(Df^{-1} \mid E^{cu}_{f(x)}\big).$$

Consider $L(E^{cu}, E^{cs})$ the vector bundle over K whose fiber over $x \in K$ is L_x, and let D be its unit ball bundle. Then $\Gamma : D \to D$ is a bundle map covering $f|K$ with

$$\mathrm{Lip}(\Gamma_x) \leqslant \| Df \mid E^{cs}_x \| \cdot \| Df^{-1} \mid E^{cu}_{f(x)} \| \leqslant \lambda < 1.$$

Let c be a Lipschitz constant for $|K$, and choose $0 < \alpha \leqslant 1$ small so that $\lambda c^\alpha < 1$. By Lemma 3.6 there exists a unique section $\sigma_0 : X \to D$ whose image is invariant under F and it satisfies a Hölder condition of exponent α. This unique section is necessarily the null section. □

The next result gives precisely (P_5)(a).

Corollary 3.8. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \geqslant 0$*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leqslant C\beta^n.$$

Proof. As we are assuming that Df is Hölder continuous, it follows from Theorem 3.7 that $\log|\det Df^u|$ is Hölder continuous. The conclusion is then an immediate consequence of the uniform contraction on stable leaves. □

Now we are going to prove (P_5)(b). We start by introducing some useful notions. We say that $\phi : N \to P$, where N and P are submanifolds of M, is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$, called the *Jacobian* of ϕ, such that

$$\mathrm{Leb}_P\big(\phi(A)\big) = \int_A J \, d\,\mathrm{Leb}_N .$$

Property (P_5)(b) can be restated in the following terms:

Proposition 3.9. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi : \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

One can easily deduce from Corollary 3.8 that this infinite product converges uniformly. The remaining of this section is devoted to the proof of Proposition 3.9. We start with a general result about the convergence of Jacobians whose proof is given in [17, Theorem 3.3].

Lemma 3.10. *Let N and P be manifolds, P with finite volume, and for each $n \geqslant 1$, $\phi_n : N \to P$ an absolutely continuous map with Jacobian J_n. Assume that*

1. *ϕ_n converges uniformly to an injective continuous map $\phi : N \to P$;*
2. *J_n converges uniformly to an integrable function $J : N \to \mathbb{R}$.*

Then ϕ is absolutely continuous with Jacobian J.

For the sake of completeness, let us mention that there is a slight difference in our definition of absolute continuity of maps. Contrarily to [17], and for reasons that will become clear in later, we do not impose continuity of the maps ϕ_n. However, the proof of [17, Theorem 3.3] uses only the continuity of the limit function ϕ, and so it still works in our case.

Consider now $\gamma, \gamma' \in \Gamma^u$ and $\phi : \gamma' \to \gamma$ as in Proposition 3.9. The proof of the next lemma is given in [17, Lemma 3.4] for uniformly hyperbolic diffeomorphisms. Nevertheless, one can easily see that it is obtained as a consequence of [17, Lemma 3.8] whose proof uses only the existence of a dominated splitting.

Lemma 3.11. *For each $n \geqslant 1$, there is an absolutely continuous $\pi_n : f^n(\gamma) \to f^n(\gamma')$ with Jacobian G_n satisfying*

1. $\lim_{n\to\infty} \sup_{x\in\gamma}\{\mathrm{dist}_{f^n(\gamma')}(\pi_n(f^n(x)), f^n(\phi(x)))\} = 0$;
2. $\lim_{n\to\infty} \sup_{x\in f^n(\gamma)}\{|1 - G_n(x)|\} = 0$.

We consider the sequence of consecutive return times for points in Λ,

$$s_1 = R \quad \text{and} \quad s_{n+1} = s_n + R \circ f^{s_n}, \quad \text{for } n \geqslant 1.$$

Notice that these return time functions are defined Leb_γ almost everywhere on each $\gamma \in \Gamma^u$ and are piecewise constant.

Remark 3.12. Using the sequence of return times one can easily construct a sequence of (Leb_γ mod 0 on each $\gamma \in \Gamma^u$) partitions $(\mathcal{P}_n)_n$ by s-subsets of Λ with s_n constant on each element of $\mathcal{P}_n$, for which (P_1)–(P_5) hold when we take s_n playing the role of R and the elements of $\mathcal{P}_n$ playing the role of the s-subsets. Moreover, the constants $C > 0$ and $0 < \beta < 1$ can be chosen not depending on n.

We define, for each $n \geqslant 1$, the map $\phi_n : \gamma \to \gamma'$ as

$$\phi_n = f^{-s_n} \pi_{s_n} f^{s_n}. \tag{14}$$

It is straightforward to check that ϕ_n is absolutely continuous with Jacobian

$$J_n(x) = \frac{|\det(Df^{s_n})^u(x)|}{|\det(Df^{s_n})^u(\phi_n(x))|} \cdot G_{s_n}\big(f^{s_n}(x)\big). \tag{15}$$

Observe that these functions are defined Leb_γ almost everywhere. So, we may find a Borel set $F \subset \gamma$ with full Leb_γ measure on which they are all defined. We extend ϕ_n to γ simply by considering $\phi_n(x) = \phi(x)$ and $J_n(x) = J(x)$ for all $n \geqslant 1$ and $x \in \gamma \setminus F$. Since F has zero Leb_γ measure one still has that J_n is the Jacobian of ϕ_n.

Proposition 3.9 is and immediate consequence of Lemma 3.10 together with the next one.

Lemma 3.13. *The maps ϕ_n converge uniformly to ϕ, and the Jacobians J_n converge uniformly to J.*

Proof. It is enough to prove the convergence of each sequence restricted to F, i.e. restricted to the set of points where the expressions of ϕ_n and J_n are given by (14) and (15) respectively.

Let us prove first that ϕ_n converges uniformly to ϕ. Using the backward contraction on unstable leaves given by (P_3), recall Remark 3.12, we may write for $x \in \gamma$

$$\begin{aligned}\operatorname{dist}_{\gamma'}\big(\phi_n(x), \phi(x)\big) &= \operatorname{dist}_{\gamma'}\big(f^{-s_n}\pi_{s_n} f^{s_n}(x),\, f^{-s_n} f^{s_n}\phi(x)\big)\\ &\leqslant C\beta^{s_n} \operatorname{dist}_{f^{s_n}(\gamma')}\big(\pi_{s_n} f^{s_n}(x),\, f^{s_n}\phi(x)\big).\end{aligned}$$

Since $s_n \to \infty$ as $n \to \infty$ and $\operatorname{dist}_{f^{s_n}(\gamma')}(\pi_{s_n} f^{s_n}(x), f^{s_n}\phi(x))$ is bounded by Lemma 3.11, we have the uniform convergence of ϕ_n to ϕ.

By (15) we have

$$J_n(x) = \frac{|\det(Df^{s_n})^u(x)|}{|\det(Df^{s_n})^u(\phi(x))|} \cdot \frac{|\det(Df^{s_n})^u(\phi(x))|}{|\det(Df^{s_n})^u(\phi_n(x))|} \cdot G_{s_n}\big(f^{s_n}(x)\big).$$

Using the chain rule and Corollary 3.8 it easily follows that the first term in the product above converges uniformly to $J(x)$. Moreover, the third term converges uniformly to one, by Lemma 3.11. It remains to see that the middle term also converges uniformly to one. Recalling Remark 3.12, by bounded distortion we have

$$\begin{aligned}\frac{|\det(Df^{s_n})^u(\phi(x))|}{|\det(Df^{s_n})^u(\phi_n(x))|} &\leqslant \exp\big(C \operatorname{dist}_{f^{s_n}(\gamma')}\big(f^{s_n}\big(\phi(x)\big), f^{s_n}\big(\phi_n(x)\big)\big)^\eta\big)\\ &= \exp\big(C \operatorname{dist}_{f^{s_n}(\gamma')}\big(f^{s_n}\big(\phi(x)\big), \pi_{s_n}\big(f^{s_n}(x)\big)\big)^\eta\big).\end{aligned}$$

Similarly we obtain

$$\frac{|\det(Df^{s_n})^u(\phi(x))|}{|\det(Df^{s_n})^u(\phi_n(x))|} \geqslant \exp\big(-C \operatorname{dist}_{f^{s_n}(\gamma')}\big(f^{s_n}\big(\phi(x)\big), \pi_{s_n}\big(f^{s_n}(x)\big)\big)^\eta\big).$$

The conclusion then follows from Lemma 3.11. □

4. Decay estimates

In this section we obtain the metric estimates on the decay of $\mathrm{Leb}_D\{R > n\}$ of Theorem A. These estimates are an adaptation of similar ones from [6, Section 5] to our setting. We start by proving estimates arising directly from the construction preformed in Section 3.2. In the final part of the argument we use some results that have been put into an abstract setting in [4, Section 4.5.2] and get the desired conclusion.

Lemma 4.1. *There exists $a_0 > 0$ such that*

$$\mathrm{Leb}_D(B_{n-1} \cap A_n) \geqslant a_0 \,\mathrm{Leb}_D(B_{n-1}),$$

for all $n \geqslant 1$.

Proof. It is enough to see this for each component of B_{n-1}. Let C be a component of B_{n-1} and let Q be its outer ring, corresponding to $t_{n-1}=1$. Observe that by Lemma 3.4 we have $Q=C\cap A_n$. Moreover, there must be some $k<n$ and a component ω_k^0 of $\{R=k\}$ such that $\pi\circ f^k$ maps C diffeomorphically onto $\bigcup_{i=k}^{\infty} I_i$ and Q onto I_k, both with uniform bounded distortion as in Corollary 2.4. Thus, it is sufficient to compare the Lebesgue measures of $\bigcup_{i=k}^{\infty} I_i$ and I_k. We have

$$\frac{\mathrm{Leb}_D(I_k)}{\mathrm{Leb}_D(\bigcup_{i=k}^{\infty} I_i)}\approx\frac{[\delta_0(1+\sigma^{(k-1)/2})]^u-[\delta_0(1+\sigma^{k/2})]^u}{[\delta_0(1+\sigma^{(k-1)/2})]^u-\delta_0^u}\approx 1-\sigma^{1/2},$$

where u is the dimension of E^{cu}. □

Lemma 4.2. *There exist $b_0, c_0>0$ with $b_0, c_0\to 0$ as $\delta_0\to 0$, such that*

1. $\mathrm{Leb}_D(A_{n-1}\cap B_n)\leqslant b_0\,\mathrm{Leb}_D(A_{n-1})$,
2. $\mathrm{Leb}_D(A_{n-1}\cap\{R=n\})\leqslant c_0\,\mathrm{Leb}_D(A_{n-1})$,

for all $n\geqslant 1$.

Proof. It is enough to prove this for each neighborhood of a component ω_n^0 of $\{R=n\}$. Observe that by construction we have $\omega_n^3\subset A_{n-1}^{\varepsilon}$, which means that $\omega_n^2\subset A_{n-1}$, because we are taking $\varepsilon<\delta_0$. Using the uniform bounded distortion of Corollary 2.4 we obtain

$$\frac{\mathrm{Leb}_D(\omega_n^1\setminus\omega_n^0)}{\mathrm{Leb}_D(\omega_n^2\setminus\omega_n^1)}\approx\frac{\mathrm{Leb}_D(\Delta_0^1\setminus\Delta_0^0)}{\mathrm{Leb}_D(\Delta_0^2\setminus\Delta_0^1)}\approx\frac{\delta_0^d}{\delta_0^{d/2}}\ll 1,$$

which gives the first estimate. Moreover,

$$\frac{\mathrm{Leb}_D(\omega_n^0)}{\mathrm{Leb}_D(\omega_n^2\setminus\omega_n^1)}\approx\frac{\mathrm{Leb}_D(\Delta_0^0)}{\mathrm{Leb}_D(\Delta_0^2\setminus\Delta_0^1)}\approx\frac{\delta_0^d}{\delta_0^{d/2}}\ll 1,$$

and this gives the second one. □

Proposition 4.3. *There exist $c_1>0$ and a positive integer $N=N(\varepsilon)$ such that*

$$\mathrm{Leb}_D\left(\bigcup_{i=0}^{N}\{R=n+i\}\right)\geqslant c_1\,\mathrm{Leb}_D(A_{n-1}\cap H_n),$$

for all $n\geqslant 1$.

Proof. Let $K_0=\max_{x\in\Lambda}\|Df^{-1}\|$ and take $r=5\delta_0K_0^{N_0}$, where N_0 is given by Lemma 3.1. Recall that by Lemma 2.3, for each $z\in f^n(A_{n-1}\cap H_n)$ there is $x\in H_n$ and a σ-hyperbolic pre-ball $V_n(x)\subset D$ which is sent diffeomorphically onto the center-unstable ball of radius δ_1 around z. Let $\{z_j\}$ be a maximal set in $f^n(A_{n-1}\cap H_n)$ with the property that the sets $B_r(z_j)$ are pairwise disjoint, where each $B_r(z_j)$ is the ball of radius r centered at z_j inside the hyperbolic ball around z_j. By maximality we have

$$\bigcup_j B_{2r}(z_j) \supset f^n(A_{n-1} \cap H_n). \tag{16}$$

For each j let $x_j \in H_n$ be the point such that $f^n(x_j) = z_j$.

Claim 1. *There is $0 \leqslant k \leqslant N_\epsilon + N_0$ such that t_{n+k} is not identically zero in $f^{-n}(B_\epsilon(z))$.*

Assume, by contradiction, that $t_{n+k}|f^{-n}(B_\epsilon(z)) = 0$ for all $0 \leqslant k \leqslant N_\epsilon + N_0$. This implies that $f^{-n}(B_\epsilon(z)) \subset A_{n+k}$ for all $0 \leqslant k \leqslant N_\epsilon + N_0$. Using Lemma 2.7 we may find a hyperbolic pre-ball $V_m \subset B_\epsilon(z)$ with σ-hyperbolic time $m \leqslant N_\epsilon$. Now, since $f^m(V_m)$ is a center-unstable disk of radius δ_1, it follows from Lemma 3.1 and Lemma 3.2 that there are $V \subset f^m(V_m)$ and $m' \leqslant N_0$ such that u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Thus, taking $k = m + m'$ we have that $0 \leqslant k \leqslant N_\epsilon + N_0$ and $f^{-n}(V_m)$ contains an element of $\{R = n + k\}$ inside $f^{-n}(B_\epsilon(z))$. This contradicts the fact that $t_{n+k}|f^{-n}(B_\epsilon(z)) = 0$ for all $0 \leqslant k \leqslant N_\epsilon + N_0$.

Claim 2. *$f^{-n}(B_{\delta_1/4}(z))$ contains a component of $\{R = n + k\}$ with $0 \leqslant k \leqslant N_\epsilon + N_0$.*

Let k be the smallest integer $0 \leqslant k \leqslant N_\epsilon + N_0$ for which t_{n+k} is not identically zero in $f^{-n}(B_\epsilon(z))$. Since $f^{-n}(B_\epsilon(z)) \subset A^\epsilon_{n-1} \subset \{t_{n-1} \leqslant 1\}$, there must be some component ω^0_{n+k} of $\{R = n + k\}$ for which $f^{-n}(B_\epsilon(z)) \cap \omega^1_{n+k} \neq \emptyset$. Recall that, by definition, f^{n+k} sends ω^1_{n+k} diffeomorphically onto a center-unstable disk (of radius $2\delta_0$) u-crossing $\mathcal{C}^1$ and intersecting $W^s_{\delta_s/4}(p)$. Thus, the diameter of $f^n(\omega^1_{n+k})$ is at most $4\delta_0 K_0^{N_0}$. Since $B_\epsilon(z)$ intersects $f^n(\omega^1_{n+k})$ and $\epsilon < \delta_0 < \delta_0 K_0^{N_0}$, we have $f^{-n}(B_{\delta_1/4}(z)) \supset \omega^0_{n+k}$, as long as we take $\delta_0 > 0$ small so that $5\delta_0 K_0^{N_0} < \delta_1/4$. Hence, we have shown that $f^{-n}(B_{\delta_1/4}(z))$ contains some component of $\{R = n + k\}$ with $0 \leqslant k \leqslant N_\epsilon + N_0$, and so we have proved the claim.

Since n is a hyperbolic time for x_j, we have by the distortion control given by Corollary 2.4 that there is some constant C only depending on C_2 and δ_1 for which

$$\frac{\mathrm{Leb}_D(f^{-n}(B_{2r}(z_j)))}{\mathrm{Leb}_D(f^{-n}(B_r(z_j)))} \leqslant C \frac{\mathrm{Leb}_{f^n(D)}(B_{2r}(z_j))}{\mathrm{Leb}_{f^n(D)}(B_r(z_j))} \tag{17}$$

and

$$\frac{\mathrm{Leb}_D(f^{-n}(B_r(z_j)))}{\mathrm{Leb}_D(\omega^0_{n+k})} \leqslant C \frac{\mathrm{Leb}_{f^n(D)}(B_r(z_j))}{\mathrm{Leb}_{f^n(D)}(f^n(\omega^0_{n+k}))}. \tag{18}$$

Recalling that from time n up to $n + k$ we have at most N_0 iterates, from (17) and (17) we easily deduce that there is some positive constant, that we still denote by C, for which

$$\mathrm{Leb}_D\big(f^{-n}\big(B_{2r}(z_j)\big)\big) \leqslant C\,\mathrm{Leb}_D\big(f^{-n}\big(B_r(z_j)\big)\big)$$

and

$$\mathrm{Leb}_D\big(f^{-n}\big(B_r(z_j)\big)\big) \leqslant C\,\mathrm{Leb}_D\big(\omega^0_{n+k_j}\big).$$

Finally, let us compare the Lebesgue measure of the sets $\bigcup_{i=0}^{N}\{R=n+i\}$ and $A_{n-1}\cap H_n$. By (16) we have

$$\mathrm{Leb}_D(A_{n-1}\cap H_n)\leqslant\sum_j \mathrm{Leb}_D\big(f^{-n}\big(B_{2r}(z_j)\big)\big)\leqslant C\sum_j \mathrm{Leb}_D\big(f^{-n}\big(B_r(z_j)\big)\big).$$

On the other hand, by the disjointness of the balls $B_r(z_j)$ we have

$$\sum_j \mathrm{Leb}_D\big(f^{-n}\big(B_r(z_j)\big)\big)\leqslant C\sum_j \mathrm{Leb}_D\big(\omega^0_{n+k}\big)\leqslant C\,\mathrm{Leb}_D\left(\bigcup_{i=0}^{N}\{R=n+i\}\right).$$

We just have to take $c_1=C^{-2}$. □

For completing the proof of Theorem A, it is enough to show that

$$\mathrm{Leb}_D\{\mathcal{E}>n\}=O\big(n^{-\tau}\big)\quad\Rightarrow\quad \mathrm{Leb}_D\{R>n\}=O\big(n^{-\tau}\big).$$

Recall that we have defined H_n, for $n\geqslant 1$, as the set of points for which n is a σ-hyperbolic time. In Corollary 2.6 we obtained the following estimate:

(m_1) *There is* $\theta>0$ *such that for all* $n\geqslant 1$ *and* $A\subset M\setminus\{\mathcal{E}>n\}$ *with* $\mathrm{Leb}_D(A)>0$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\mathrm{Leb}_D(A\cap H_j)}{\mathrm{Leb}_D(A)}\geqslant\theta.$$

In the construction of the Markov structure have taken a disk Δ of radius $\delta_0>0$ and defined inductively the subsets A_n, B_n, $\{R=n\}$ and Δ_n related in the following way:

$$\Delta_n=\Delta\setminus\{R\leqslant n\}=A_n\,\dot{\cup}\,B_n.$$

Moreover, we have proved in Lemma 4.1, Lemma 4.2 and Proposition 4.3 that the following metric relations hold:

(m_2) *There is* $a_0>0$ *(bounded away from 0 for all* δ_0*) such that for all* $n\geqslant 1$

$$\mathrm{Leb}_D(B_{n-1}\cap A_n)\geqslant a_0\mathrm{Leb}_D(B_{n-1}).$$

(m_3) *There are* $b_0,c_0>0$ *with* $b_0,c_0\to 0$ *as* $\delta_0\to 0$, *such that for all* $n\geqslant 1$

$$\frac{\mathrm{Leb}_D(A_{n-1}\cap B_n)}{\mathrm{Leb}_D(A_{n-1})}\leqslant b_0\quad\text{and}\quad\frac{\mathrm{Leb}_D(A_{n-1}\cap\{R=n\})}{\mathrm{Leb}_D(A_{n-1})}\leqslant c_0.$$

(m_4) *There is* $r_0>0$ *and an integer* $N\geqslant 0$ *such that for all* $n\geqslant 1$

$$\mathrm{Leb}_D\left(\bigcup_{i=0}^{N}\{R=n+i\}\right)\geqslant r_0\,\mathrm{Leb}_D(A_{n-1}\cap H_n).$$

Estimates (m_1)–(m_4) are enough to use the results of [4, Section 4.5.2] and obtain the decay of $\mathrm{Leb}_\gamma\{R > n\}$ as in the conclusion of Theorem A.

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