Discrete Mathematics 312 (2012) 3357–3363

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Mixed unit interval graphs

Mitre C. Dourado [a], Van Bang Le [b], Fábio Protti [c], Dieter Rautenbach [d,*], Jayme L. Szwarcfiter [a]

[a] *Instituto de Matemática, NCE, and COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil*

[b] *Institut für Informatik, Universität Rostock, Rostock, Germany*

[c] *Instituto de Computação, Universidade Federal Fluminense, Niterói, RJ, Brazil*

[d] *Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany*

A R T I C L E   I N F O

Article history:
Received 15 March 2011
Received in revised form 26 July 2012
Accepted 29 July 2012
Available online 20 August 2012

Keywords:
Intersection graph
Interval graph
Proper interval graph
Unit interval graph

A B S T R A C T

The class of intersection graphs of unit intervals of the real line whose ends may be open or closed is a strict superclass of the well-known class of unit interval graphs. We pose a conjecture concerning characterizations of such mixed unit interval graphs, verify parts of it in general, and prove it completely for diamond-free graphs. In particular, we characterize diamond-free mixed unit interval graphs by means of an infinite family of forbidden induced subgraphs, and we show that a diamond-free graph is mixed unit interval if and only if it has intersection representations using unit intervals such that all ends of the intervals are integral.

1. Introduction

Interval graphs and subclasses like proper interval graphs and unit interval graphs have well-studied structural [2,6,9,17] as well as algorithmic [4,5,3,10,11] properties and occur in many applications such as scheduling [14], physical mapping of DNA [8,12], and archeology [13].

Interval graphs are the intersection graphs of intervals of the real line. An apparent weakness of this concise definition seems to be that it does not specify the type of interval that is allowed, that is, whether the ends of the intervals are open or closed. This ambiguity is acceptable because the class of graphs does actually not depend on this. Frankl and Maehara [7], for instance observed that using only open intervals or only closed intervals leads to the same class of graphs. As we show in Proposition 1, this is true even allowing all possible types of intervals. For unit interval graphs, which form a well-known subclass of the interval graphs, this is no longer true.[1] It is easy to see that the claw $K_{1,3}$, which is the only minimal interval graph that is not also a unit interval graph, has an intersection representation when both open intervals and closed intervals are allowed to be used. In [15] the last two authors proved that a graph is the intersection graph of open intervals and closed intervals of unit length if and only if it is an interval graph and does not contain one of five graphs as an induced subgraph. Furthermore, they also gave a structural characterization based on a suitable extension of proper interval graphs.

The aim of the present paper is to generalize the results of [15] to mixed unit interval graphs allowing all four distinct types of unit intervals. While finitely many forbidden induced subgraphs were sufficient when only fully open and fully closed intervals are allowed, the list of forbidden induced subgraphs for mixed unit interval graphs is infinite.

* Corresponding author.
E-mail addresses: mitre@nce.ufrj.br (M.C. Dourado), le@informatik.uni-rostock.de (V.B. Le), fabio@ic.uff.br (F. Protti), dieter.rautenbach@gmx.de, dieter.rautenbach@uni-ulm.de (D. Rautenbach), jayme@nce.ufrj.br (J.L. Szwarcfiter).

1 Tucker [18] made an analogous observation for unit circular-arc graphs.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2012.07.037

Our contributions are as follows. In Section 2 we collect basic terminology, definitions, and results. In Section 3 we pose a conjecture concerning characterizations of mixed unit interval graphs and verify parts of it. In Section 4 we verify our conjecture for diamond-free interval graphs.

2. Preliminaries

We consider only finite, simple, undirected graphs. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. For a vertex u of a graph G, the neighborhood of u in G is denoted $N_G(u)$ and the closed neighborhood of u in G is denoted $N_G[u]$. For a set U of vertices of a graph G, the subgraph of G induced by U is denoted $G[U]$. If $\mathcal{F}$ is a set of graphs and a graph G does not contain a graph in $\mathcal{F}$ as an induced subgraph, then G is *$\mathcal{F}$-free*. Two distinct vertices u and v of a graph G are *twins* if $N_G[u] = N_G[v]$. A graph without twins is *twin-free*. A *cut-vertex* of a graph is a vertex whose removal increases the number of components. A *block* of a graph is a maximal subgraph without a cut-vertex. An *end-block* of a graph G is a block that contains at most one cut-vertex of G. An *asteroidal triple* of a graph is a set of three pairwise non-adjacent vertices such that joining any two of them, there is a path avoiding the closed neighborhood of the third. It is well-known that interval graphs are *asteroidal triple-free* and *chordal*, where a chordal graph is one without induced cycles of length at least four.

Let $\mathcal{M}$ be a family of sets. An *$\mathcal{M}$-intersection representation of a graph G* is a function $M\colon V(G) \to \mathcal{M}$ such that for any two distinct vertices u and v of G, we have $uv \in E(G)$ if and only if $M(u) \cap M(v) \neq \emptyset$. A graph is an *$\mathcal{M}$-graph* if it has an $\mathcal{M}$-intersection representation.

For two real numbers x and y, the *open interval* (x, y) is $\{z \in \mathbb{R} : x < z < y\}$, the *closed interval* $[x, y]$ is $\{z \in \mathbb{R} : x \le z \le y\}$, the *open–closed interval* $(x, y]$ is $\{z \in \mathbb{R} : x < z \le y\}$, and the *closed–open interval* $[x, y)$ is $\{z \in \mathbb{R} : x \le z < y\}$. Let

$$\mathcal{I}^{--} = \{(x, y) : x, y \in \mathbb{R}, x < y\}, \qquad \mathcal{U}^{--} = \{(x, x+1) : x \in \mathbb{R}\},$$
$$\mathcal{I}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \le y\}, \qquad \mathcal{U}^{++} = \{[x, x+1] : x \in \mathbb{R}\},$$
$$\mathcal{I}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x < y\}, \qquad \mathcal{U}^{-+} = \{(x, x+1] : x \in \mathbb{R}\},$$
$$\mathcal{I}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x < y\}, \qquad \mathcal{U}^{+-} = \{[x, x+1) : x \in \mathbb{R}\},$$
$$\mathcal{I}^{\pm} = \mathcal{I}^{++} \cup \mathcal{I}^{--}, \qquad \mathcal{U}^{\pm} = \mathcal{U}^{++} \cup \mathcal{U}^{--},$$
$$\mathcal{I} = \mathcal{I}^{++} \cup \mathcal{I}^{--} \cup \mathcal{I}^{-+} \cup \mathcal{I}^{+-}, \qquad \mathcal{U} = \mathcal{U}^{++} \cup \mathcal{U}^{--} \cup \mathcal{U}^{-+} \cup \mathcal{U}^{+-}.$$

We allow arithmetic operations with intervals, that is, for an interval I in $\mathcal{I}$ and two real numbers x and y, we have $xI + y = \{xz + y : z \in I\}$. For an interval I in $\mathcal{I}$, let $\ell(I) = \inf(I)$ and $r(I) = \sup(I)$. The points $\ell(I)$ and $r(I)$ are called the *ends* of I.

Our first result confirms the claim from the introduction that the class of interval graphs does not depend on the type of interval used in the intersection representation.

Proposition 1. *The classes of $\mathcal{I}^{--}$-graphs, $\mathcal{I}^{++}$-graphs, $\mathcal{I}^{\pm}$-graphs, $\mathcal{I}^{-+}$-graphs, $\mathcal{I}^{+-}$-graphs, and $\mathcal{I}$-graphs are the same.*

Proof. By Proposition 1 in [15], the classes of $\mathcal{I}^{--}$-graphs, $\mathcal{I}^{++}$-graphs, and $\mathcal{I}^{\pm}$-graph are the same. Clearly, $\mathcal{I}^{++}$-graphs, $\mathcal{I}^{-+}$-graphs, and $\mathcal{I}^{+-}$-graphs are $\mathcal{I}$-graphs. By symmetry, the classes of $\mathcal{I}^{-+}$-graphs and $\mathcal{I}^{+-}$-graphs are the same. We complete the proof by showing that connected $\mathcal{I}$-graphs of order at least 2 are $\mathcal{I}^{++}$-graphs and that $\mathcal{I}^{++}$-graphs are $\mathcal{I}^{-+}$-graphs.

Let G be a connected $\mathcal{I}$-graph of order at least 2. Let I be an $\mathcal{I}$-intersection representation of G. For every edge uv of G, select some element $x(uv)$ in the intersection of $I(u)$ and $I(v)$. For every vertex u of G, let $I'(u) = [\min\{x(uv) : v \in N_G(u)\}, \max\{x(uv) : v \in N_G(u)\}]$. By definition, $I'(u)$ is a closed interval and a subset of $I(u)$. Hence if $I'(u)$ and $I'(v)$ intersect, then $I(u)$ and $I(v)$ intersect. Furthermore, if $I(u)$ and $I(v)$ intersect, then uv is an edge of G and $x(uv)$ belongs to $I'(u)$ as well as $I'(v)$, that is, $I'(u)$ and $I'(v)$ intersect. Therefore, $I'\colon V(G) \to \mathcal{I}^{++}$, $u \mapsto I'(u)$ is an $\mathcal{I}^{++}$-intersection representation of G and G is an $\mathcal{I}^{++}$-graph.

Let G be an $\mathcal{I}^{++}$-graph, and let I be an $\mathcal{I}^{++}$-intersection representation of G. Since disjoint closed intervals have a positive distance and G is finite, there is some $\epsilon > 0$ such that $I''\colon V(G) \to \mathcal{I}^{-+}$ with $I''(u) = (\ell(I(u)) - \epsilon, r(I(u))]$ for $u \in V(G)$ is an $\mathcal{I}^{-+}$-intersection representation of G and G is an $\mathcal{I}^{-+}$-graph. $\square$

The following proposition extends a result of Frankl and Maehara [7], who proved that a graph is a $\mathcal{U}^{++}$-graph if and only if it is a $\mathcal{U}^{--}$-graph.

Proposition 2. *The classes of $\mathcal{U}^{--}$-graphs, $\mathcal{U}^{++}$-graphs, $\mathcal{U}^{-+}$-graphs, $\mathcal{U}^{+-}$-graphs, and $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$-graphs are the same.*

Proof. It is easy to see that these equivalences follow using similar arguments as in the proof of Proposition 1. Therefore, we only give details for the proofs that (i) $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$-graphs are $\mathcal{U}^{++}$-graphs and that (ii) $\mathcal{U}^{++}$-graphs are $\mathcal{U}^{+-}$-graphs. Clearly, it suffices to consider connected graphs that are not complete.

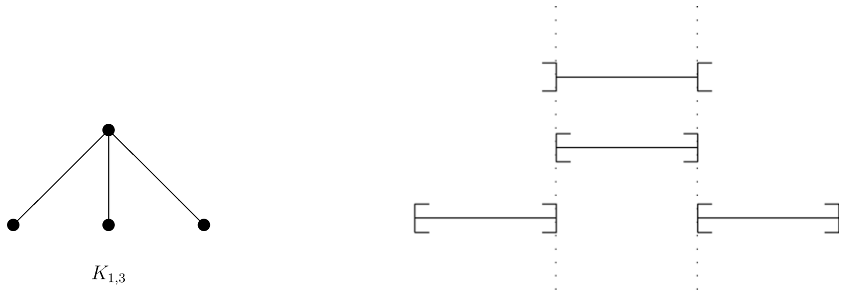

Fig. 1. The claw $K_{1,3}$ and one of its $\mathcal{U}$-intersection representations.

(i) Let G be a connected $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$-graph that is not complete, and let I be a $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$-intersection representation of G. For every edge uv of G, select some element $x(uv)$ in the intersection of $I(u)$ and $I(v)$. Let

$$\epsilon_l = \min\{x(uv) - \ell(I(u)) : u \in V(G), v \in N_G(u), \text{ and } I(u) \in \mathcal{U}^{-+}\},$$
$$\epsilon_r = \min\{r(I(u)) - x(uv) : u \in V(G), v \in N_G(u), \text{ and } I(u) \in \mathcal{U}^{+-}\}, \quad \text{and}$$
$$\epsilon = \min\{\epsilon_l, \epsilon_r\}.$$

Since G is finite, $\epsilon > 0$. For every vertex u of G, let

$$I'(u) = \begin{cases} \frac{1}{1-\epsilon}[\ell(I(u)) + \epsilon, r(I(u))], & I(u) \in \mathcal{U}^{-+} \quad \text{and} \\ \frac{1}{1-\epsilon}[\ell(I(u)), r(I(u)) - \epsilon], & I(u) \in \mathcal{U}^{+-}. \end{cases}$$

It follows similarly as in the proof of Proposition 1 that $I' \colon V(G) \to \mathcal{I}^{++}, u \mapsto I'(u)$ is a $\mathcal{U}^{++}$-intersection representation of G.

(ii) Let G be a connected $\mathcal{U}^{++}$-graph that is not complete, and let I be a $\mathcal{U}^{++}$-intersection representation of G. Since G is finite,

$$\epsilon = \min\{\ell(I(v)) - r(I(u)) : u, v \in V(G) \text{ and } r(I(u)) < \ell(I(v))\}$$

is positive. For $u \in V(G)$, let $I''(u) = \frac{1}{1+\epsilon/3}[\ell(I(u)), r(I(u)) + \epsilon/3)$. By definition, $I''(u) \in \mathcal{U}^{+-}$ and $I(u) \subseteq I''(u)$. Hence if $I(u)$ and $I(v)$ intersect, then $I'(u)$ and $I'(v)$ intersect. Furthermore, if $I(u)$ and $I(v)$ do not intersect, then, by the definition of ϵ, the intervals $I''(u)$ and $I''(v)$ do not intersect. Therefore, $I'' \colon V(G) \to \mathcal{U}^{+-}, u \mapsto I''(u)$ is a $\mathcal{U}^{+-}$-intersection representation of G and G is a $\mathcal{U}^{+-}$-graph. □

Interval graphs coincide with $\mathcal{I}^{++}$-graphs and *unit interval graphs* coincide with $\mathcal{U}^{++}$-graphs. A graph G is a *proper interval graph* if it has an $\mathcal{I}^{++}$-intersection representation $I \colon V(G) \to \mathcal{I}^{++}$ for which there are no two vertices u and v of G such that $I(u)$ is a proper subset of $I(v)$. In this case I is a *proper interval representation of* G. The most important result relating these three classes of interval graphs is due to Roberts.

Theorem 3 (*Roberts [16]*)**.** *The classes of unit interval graphs, proper interval graphs, and $K_{1,3}$-free interval graphs are the same.*

By this result, the claw is the only minimal interval graph that is not also a unit interval graph. In fact, the reason why all classes of graphs considered in Proposition 2 coincide is that none of them contains the claw.

As observed in the introduction, the claw has $\mathcal{U}$-intersection representations; cf. Fig. 1. Our next result shows that these are essentially unique.

Proposition 4. *The claw $K_{1,3}$ has a $\mathcal{U}$-intersection representation. Also, for every $\mathcal{U}$-intersection representation $I \colon V(K_{1,3}) \to \mathcal{U}$ of the claw $K_{1,3}$, there is some $x \in \mathbb{R}$ such that $I(V(K_{1,3}))$ contains*

- $[x, x+1]$ *and* $(x, x+1)$,
- *either* $(x-1, x]$ *or* $[x-1, x]$, *and*
- *either* $[x+1, x+2)$ *or* $[x+1, x+2]$.

That is, up to translation there are four different representations.

Proof. The right part of Fig. 1 illustrates a $\mathcal{U}^{\pm}$-intersection representation of the claw. Now let $V(K_{1,3}) = \{a, b, c, d\}$ and $E(K_{1,3}) = \{ad, bd, cd\}$. Let $I \colon V(K_{1,3}) \to \mathcal{U}$ be a $\mathcal{U}$-intersection representation of the claw. Let $x(a) \in I(a) \cap I(d), x(b) \in I(b) \cap I(d)$, and $x(c) \in I(c) \cap I(d)$. Since $I(a), I(b)$, and $I(c)$ are disjoint, we may assume that $x(a) < x(b) < x(c)$. Since $x(a), x(c) \in I(d) \in \mathcal{U}$, we have $x(c) - x(a) \leq 1$. Since $I(a), I(b)$, and $I(c)$ are disjoint, the interval $I(b)$ must be a proper subset of $(x(a), x(c))$. Since $I(b) \in \mathcal{U}$, this implies that $x(c) = x(a)+1$, $I(b) = (x(a), x(a)+1)$, $I(d) = [x(a), x(a)+1]$, $I(a) \in \{[x(a)-1, x(a)], (x(a)-1, x(a)]\}$, and $I(c) \in \{[x(a)+1, x(a)+2], [x(a)+1, x(a)+2)\}$, which completes the proof. □

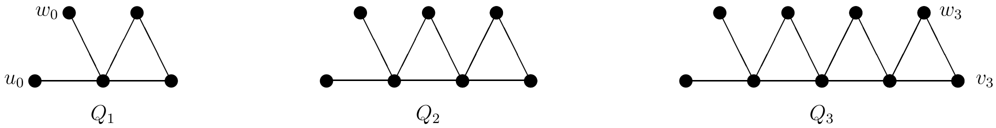

Fig. 2. The graphs Q_1, Q_2, and Q_3.

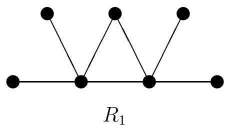

Fig. 3. The graphs R_1 and R_2.

Fig. 4. The graphs R_0 and G_0.

3. A conjecture for mixed unit interval graphs

In this section we pose a conjecture concerning $\mathcal{U}$-graphs and verify parts of it. We begin with the definition of some important infinite sequences of graphs.

For $k \in \mathbb{N}$, let the graph Q_k arise from k disjoint triangles with vertex sets $\{u_1, v_1, w_1\}, \ldots, \{u_k, v_k, w_k\}$ and two further vertices u_0 and w_0 by identifying v_i with u_{i+1} for $1 \leq i \leq k-1$ and adding the two edges u_0u_1 and w_0u_1. See Fig. 2 for an illustration. The two vertices v_k and w_k are the *special vertices* of Q_k.

The graphs Q_k lead to the following infinite sequence of forbidden induced subgraphs of $\mathcal{U}$-graphs.

For $k \in \mathbb{N}$, let R_k arise from Q_k by adding two vertices v_{k+1} and w_{k+1} and two edges v_kv_{k+1} and v_kw_{k+1}. See Fig. 3 for an illustration.

In addition to this infinite sequence there are some further forbidden induced subgraphs shown in Fig. 4. Note that the graph R_0 shares structural features with the graphs R_k for $k \geq 1$.

The next lemma collects essential properties of these graphs.

Lemma 5. *Let $k \in \mathbb{N}$.*

(a) *Every injective $\mathcal{U}$-intersection representation of Q_k arises by translation (replacing I by $I + x$ for some $x \in \mathbb{R}$) and inversion (replacing I by $-I$) from one of two injective $\mathcal{U}$-intersection representations $I: V(Q_k) \to \mathcal{U}$ of Q_k, where $I(V(Q_k))$ consists of the following intervals*
- *either* $(0, 1]$ *or* $[0, 1]$,
- $[1, 2]$ *and* $(1, 2)$, *and*
- $[i, i+1]$ *and* $[i, i+1)$ *for* $2 \leq i \leq k+1$.

(b) *R_k is a minimal forbidden induced subgraph for the class of $\mathcal{U}$-graphs.*
(c) *R_0 and G_0 are minimal forbidden induced subgraphs for the class of $\mathcal{U}$-graphs.*

Proof. (a) The statement follows by induction on k. For $k = 1$, deleting one of the two special vertices, say w_1, of Q_1 results in a claw. Therefore, the restriction of I to $V(Q_1) \setminus \{w_1\}$ is as described in Proposition 4, which implies the desired statement.

For $k \geq 2$, Q_k arises from Q_{k-1} by attaching the two special vertices v_k and w_k of Q_k to the special vertex v_{k-1} of Q_{k-1}. By the induction hypothesis, the restriction of I to $V(Q_k) \setminus \{v_k, w_k\}$ is almost uniquely determined up to translation and inversion. Furthermore, for the two special vertices v_{k-1} and w_{k-1} of Q_{k-1}, we may assume, by the induction hypothesis, that $\{I(v_{k-1}), I(w_{k-1})\} = \{[k, k+1), [k, k+1]\}$. Since I is injective, this implies $I(v_{k-1}) = [k, k+1]$, $I(w_{k-1}) = [k, k+1)$, and $\{I(v_k), I(w_k)\} = \{[k+1, k+2), [k+1, k+2]\}$, which completes the proof of (a).

(b) Clearly, every $\mathcal{U}$-intersection representation of a twin-free graph is injective. Since R_k is twin-free and contains Q_k as an induced subgraph, part (a) easily implies the desired result.

(c) Note that R_0 and G_0 contain $K_{1,3}$ as an induced subgraph. Now Proposition 4 easily implies the desired result. □

A graph G is a *mixed proper interval graph* if it has an $\mathcal{I}$-intersection representation $I: V(G) \to \mathcal{I}$ such that

- there are no two distinct vertices u and v of G with $I(u), I(v) \in \mathcal{I}^{++}$, $I(u) \subseteq I(v)$, and $I(v) \not\subseteq I(u)$, and
- for every vertex u of G with $I(u) \notin \mathcal{I}^{++}$, there is a vertex v of G with $I(v) \in \mathcal{I}^{++}$, $\ell(I(u)) = \ell(I(v))$, and $r(I(u)) = r(I(v))$,

that is, no closed interval properly contains another closed interval and for every non-closed interval, there is a closed interval with the same ends. In this case I is a *mixed proper interval intersection representation of* G.

We are now in a position to phrase our conjecture.

Let $\mathcal{R} = \{R_k : k \in \mathbb{N}_0\} = \{R_0, R_1, R_2, \ldots\}$.

Conjecture 6. *For a graph G, the following statements are equivalent.*

(1) • *G is a $\{G_0\} \cup \mathcal{R}$-free interval graph and*
  • *for every induced subgraph H of G that is isomorphic to Q_k and every vertex $u^* \in V(G) \setminus V(H)$ such that u^* is adjacent to exactly one of the two special vertices of H, the vertex u^* has exactly one neighbor in $V(H)$.*

(2) *G is a mixed proper interval graph.*

(3) *G is a mixed unit interval graph.*

It is obvious that the technical condition in (1) of Conjecture 6 concerning H and u^* can be equivalently expressed by further forbidden induced subgraphs. However, we leave open the problem of finding the complete list of these graphs.

The last two results in this section confirm Conjecture 6 partly.

Proposition 7. *The implication* (3) $\Rightarrow$ (1) *of Conjecture 6 is true.*

Proof. Let G by a $\mathcal{U}$-graph, and let I be a $\mathcal{U}$-intersection representation of G. Since the graphs in $\{G_0\} \cup \mathcal{R}$ are twin-free, Lemma 5(b) and (c) imply that G is a $\{G_0\} \cup \mathcal{R}$-free interval graph.

Now let H be an induced subgraph of G that is isomorphic to Q_k. Let the vertices in H be denoted as in the definition of Q_k; that is, the two special vertices are v_k and w_k. Let $u^* \in V(G) \setminus V(H)$ be such that u^* is adjacent to v_k but not to w_k. By Lemma 5(a), we may assume that there is some $x \in \mathbb{R}$ such that $I(v_k) = [x, x+1]$, $I(w_k) = [x, x+1)$, and $r(I(u)) \le x$ for every $u \in V(H) \setminus \{v_k, w_k\}$. This implies that $I(u^*) \in \{[x+1, x+2), [x+1, x+2]\}$ and hence $N_G(u^*) \cap V(H) = \{v_k\}$, which completes the proof. □

Theorem 8. *A graph is a mixed proper interval graph if and only if it is a $\mathcal{U}$-graph; that is, statements* (2) *and* (3) *of Conjecture 6 are equivalent.*

Proof. The "only if" part can be proved exactly as in [15] using a recent argument due to Bogart and West [1]. For the sake of completeness, we give details.

Let G be a mixed proper interval graph. Let I be a mixed proper interval intersection representation of G. Let U denote the set of vertices u of G such that $I(u) \in \mathcal{U}^{++}$. By the definition of mixed proper interval graphs, the subgraph $G[U]$ of G induced by U is a proper interval graph and the restriction I_U of I to U is a proper interval representation of $G[U]$.

In [1] Bogart and West gave a constructive proof of the fact that every proper interval graph is a unit interval graph. Their proof starts with a proper interval representation I_1 and produces a unit interval representation I_2 gradually converting the intervals into unit intervals by means of successive contractions, dilations, and translations. A property maintained by this constructive procedure is that two intervals intersect in a single point before the modifications if and only if they intersect in a single point after the modifications. More specifically, for every two vertices u and v, we have

- $r(I_1(u)) = \ell(I_1(v))$ if and only if $r(I_2(u)) = \ell(I_2(v))$, and
- $I_1(u) \cap I_1(v)$ is a set of positive measure if and only if $I_2(u) \cap I_2(v)$ is a set of positive measure.

This implies that reinserting the mixed intervals corresponding to the vertices in $V(G) \setminus U$ as mixed copies of the corresponding closed intervals results in a $\mathcal{U}$-intersection representation of G, which completes the proof of the "only if" part.

Now we prove the "if" part. Let G be a $\mathcal{U}$-graph. Let $I\colon V(G) \to \mathcal{U}$ be a $\mathcal{U}$-intersection representation of G such that

$$f(I) = 2|\{u \in V(G) : I(u) \in \mathcal{U}^{--}\}| + |\{u \in V(G) : I(u) \in \mathcal{U}^{-+} \cup \mathcal{U}^{+-}\}|$$

is the smallest possible. For contradiction, we may assume that the vertex u^* of G is such that $I(u^*) \notin \mathcal{U}^{++}$ and there is no vertex v of G with $\ell(I(u^*)) = \ell(I(v))$ and $I(v) \in \mathcal{U}^{++}$. By symmetry, we may assume that $r(I(u^*)) \notin I(u^*)$.

For a vertex v of G, let

$$\Delta(v) = \min\{|\ell(I(v)) - (\ell(I(u^*)) + x)| : x \in \mathbb{Z}\} \quad \text{and}$$
$$\epsilon = \min\left(\{1\} \cup \{\Delta(v) : v \in V(G) \text{ and } \Delta(v) \neq 0\}\right).$$

Since G is finite, $\epsilon > 0$. Let

$$U = \{v \in V(G) : \ell(v) > \ell(u^*), \Delta(v) = 0\} \cup \{v \in V(G) : \ell(v) = \ell(u^*) \text{ and } I(v) \in \mathcal{U}^{-+}\}.$$

Let $I'\colon V(G) \to \mathcal{U}$ be such that

$$I'(v) = \begin{cases} I(u^*) \cup \{r(I(u^*))\}, & v = u^*, \\ I(v) + \dfrac{\epsilon}{2}, & v \in U, \quad \text{and} \\ I(v), & v \in V(G) \setminus (\{u^*\} \cup U). \end{cases}$$

It is straightforward to verify that I' is a $\mathcal{U}$-intersection representation of G with $f(I') < f(I)$, which is a contradiction and completes the proof. □

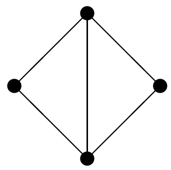

Fig. 5. The diamond.

4. Diamond-free mixed unit interval graphs

A $\mathcal{U}$-intersection representation I of a graph G is *integral* if $\{\ell(I(u)) : u \in V(G)\} \subseteq \mathbb{Z}$, that is, all ends of intervals are integers. Note that the diamond (see Fig. 5) has an integral $\mathcal{U}$-intersection representation using the intervals $(0, 1)$, $(0, 1]$, $[0, 1]$, and $[1, 2)$.

In this final section we verify Conjecture 6 for diamond-free graphs. Note that the graph G_0 contains a diamond. It turns out as a by-product that a diamond-free graph has a $\mathcal{U}$-intersection representation if and only if it has an integral $\mathcal{U}$-intersection representation.

Theorem 9. *For a diamond-free graph G, the following statements are equivalent.*

(1) *G is a $\mathcal{R}$-free interval graph.*
(2) *G has an integral $\mathcal{U}$-intersection representation.*
(3) *G is a $\mathcal{U}$-graph.*

Proof. (1) $\Rightarrow$ (2): Let G satisfy (1). Clearly, we may assume that G is connected and twin-free. Since G is diamond-free and chordal, G is a block graph, that is, all blocks of G are complete subgraphs. The next claim collects several properties of the blocks of G.

Claim 1.

(i) *Every block of G is an edge or a triangle.*
(ii) *Every end-block of G is an edge.*
(iii) *Every triangle of G contains exactly two cut-vertices of G.*
(iv) *Every cut-vertex of G belongs to at most three blocks. Moreover, if a cut-vertex belongs to exactly three blocks, then at least one of these blocks is an end-block, which, by* (ii) *, is an edge.*
(v) *The subgraph P of G induced by the cut-vertices of G is a path.*

Proof of Claim 1. (i) Since G is a block graph, we may assume that $G[\{a, b, c, d\}]$ is a complete subgraph of G. Since G is twin-free and a block graph, there are vertices a', b', and c' of G such that $N_G(a') \cap \{a, b, c\} = \{a\}$, $N_G(b') \cap \{a, b, c\} = \{b\}$, and $N_G(c') \cap \{a, b, c\} = \{c\}$. Now $\{a', b', c'\}$ is an asteroidal triple, which is a contradiction.

(ii) This follows from the twin-freeness of G.

(iii) This follows from the twin-freeness of G using the same argument as for (i).

(iv) If a cut-vertex belongs to four blocks, then G has an induced R_0. If a cut-vertex is in three triangles, then, since G is twin-free, G would have an asteroidal triple.

(v) As G is a connected block graph, P is a tree. Moreover, P has maximum degree at most 2; otherwise, G would have an asteroidal triple. $\square$

By (v), let $a_1, \ldots, a_s$ be the cut-vertices of G such that $P : a_1 \cdots a_s$ is an induced path in G with edges $a_i a_{i+1}$ for $1 \le i < s$. By the claim, $G \setminus V(P)$ is edgeless.

Let $a_0, a_{s+1} \in V(G) \setminus V(P)$ be such that the edges a_0a_1 and a_sa_{s+1} are end-blocks of G. Note that a_0 and a_{s+1} exist, because a_1 and a_s are cut-vertices of G.

For each i with $0 \le i \le s$, let B_i be the block of G containing the edge a_ia_{i+1}. Note that $B_0 = a_0a_1$ and $B_s = a_sa_{s+1}$. For $1 \le i < s$, (i) implies that the block B_i is either the edge a_ia_{i+1} or a triangle $a_ia_{i+1}b_i$ for a unique vertex $b_i \in V(G) \setminus \{a_0, \ldots, a_{s+1}\}$.

Let $t = |X|$, where $X = V(G) \setminus \bigcup_{i=0}^{s} V(B_i)$. Note that $0 \le t \le s$. Also, the vertices in X can be ordered as $x_{i_1}, \ldots, x_{i_t}$ with $1 \le i_1 < \cdots < i_t \le s$ such that $a_{i_1}x_{i_1}, \ldots, a_{i_t}x_{i_t}$ are end-blocks of G. Thus, the vertices $a_{i_1}, \ldots, a_{i_t}$ are exactly those cut-vertices of G that belong to three blocks, and if $t = 0$, then every vertex of G belongs to at most two blocks.

Furthermore, since G is R_k-free for $k \in \mathbb{N}$, we have the following "gap" property of the blocks B_i between two blocks $a_{i_j}x_{i_j}$ and $a_{i_{j+1}}x_{i_{j+1}}$, which will be crucial for our construction below:

$$\forall j \in \{1, \ldots, t-1\} \exists g_j : i_j \le g_j < i_{j+1} \quad \text{and} \quad B_{g_j} = a_{g_j}a_{g_j+1}.$$

We define an interval representation $I : V(G) \to \mathcal{U}$ as follows; for convenience we may assume that $B_i = a_ia_{i+1}b_i$ for all $1 \le i < s$ with $i \notin \{g_1, \ldots, g_{t-1}\}$. While the intervals assigned to the a_i and x_{i_j} are defined in a uniform way, the intervals assigned to the b_i will depend on their position. We distinguish the b_i that are either "left" of x_{i_1}, or "between" x_{i_j} and the "gap" at g_j, or "between" the "gap" at g_j and $x_{i_{j+1}}$, or "right" of x_{i_t}.

- For $0 \le i \le s+1$, let $I(a_i) = [i, i+1]$,
- for $1 \le j \le t$, let $I(x_{i_j}) = (i_j, i_j + 1)$ (thus, $I(x_{i_j}) = I(a_{i_j}) \setminus \{\ell(I(a_{i_j})), r(I(a_{i_j}))\}$),

- for $1 \leq i \leq i_1 - 1$, let $I(b_i) = (i, i + 1]$,
- for $1 \leq j \leq t - 1$ and $i_j \leq i \leq g_j - 1$, let $I(b_i) = [i + 1, i + 2)$,
- for $1 \leq j \leq t - 1$ and $g_j + 1 \leq i \leq i_{j+1} - 1$, let $I(b_i) = (i, i + 1]$,
- for $i_t \leq i \leq s - 1$, let $I(b_i) = [i + 1, i + 2)$.

From the block structure of G and the gap property, it follows easily that I is an integral $\mathcal{U}$-intersection representation of G.

(2) ⇒ (3): This implication is trivial.

(3) ⇒ (1): This implication follows immediately from Lemma 5(a) and (b). □

We remark that the proof of Theorem 9 implicitly contains a linear-time algorithm that recognizes whether a given diamond-free graph is a $\mathcal{U}$-graph, respectively, has an integral $\mathcal{U}$-intersection representation, and if so, constructs an (integral) $\mathcal{U}$-intersection representation.

Theorems 8 and 9 immediately imply the following special case of Conjecture 6.

Corollary 10. *Conjecture 6 is true for diamond-free graphs.*

Acknowledgment

We acknowledge partial support by a CAPES/DAAD Probral project "Cycles, Convexity, and Searching in Graphs".

References

[1] K.P. Bogart, D.B. West, A short proof that "proper = unit", Discrete Math. 201 (1999) 21–23.
[2] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
[3] D.G. Corneil, A simple 3-sweep LBFS algorithm for the recognition of unit interval graphs, Discrete Appl. Math. 138 (2004) 371–379.
[4] D.G. Corneil, H. Kim, S. Natarajan, S. Olariu, A.P. Sprague, Simple linear time recognition of unit interval graphs, Inform. Process. Lett. 55 (1995) 99–104.
[5] D.G. Corneil, S. Olariu, L. Stewart, The ultimate interval graph recognition algorithm? (Extended abstract), in: Proceedings of the 9th Annual ACM–SIAM Symposium on Discrete Algorithms, SODA, 1998, pp. 175–180.
[6] P.C. Fishburn, Interval Orders and Interval graphs. A Study Of Partially Ordered Sets, John Wiley & Sons, 1985.
[7] P. Frankl, H. Maehara, Open-interval graphs versus closed-interval graphs, Discrete Math. 63 (1987) 97–100.
[8] P.W. Goldberg, M.C. Golumbic, H. Kaplan, R. Shamir, Four strikes against physical mapping of DNA, J. Comput. Biol. 2 (1995) 139–152.
[9] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, vol. 57. Annals of Discrete Mathematics, Amsterdam, The Netherlands, 2004.
[10] P. Hell, R. Shamir, R. Sharan, A fully dynamic algorithm for recognizing and representing proper interval graphs, SIAM J. Comput. 31 (2001) 289–305.
[11] C.M. Herrera de Figueiredo, J. Meidanis, C. Picinin de Mello, A linear-time algorithm for proper interval graph recognition, Inform. Process. Lett. 56 (1995) 179–184.
[12] H. Kaplan, R. Shamir, Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques, SIAM J. Comput. 25 (1996) 540–561.
[13] D.G. Kendall, Incidence matrices, interval graphs, and seriation in archaeology, Pacific J. Math. 28 (1969) 565–570.
[14] C.H. Papadimitriou, M. Yannakakis, Scheduling interval-ordered tasks, SIAM J. Comput. 8 (1979) 405–409.
[15] D. Rautenbach, J.L. Szwarcfiter, Unit interval graphs of open and closed intervals, J. Graph Theory (in press).
[16] F.S. Roberts, Indifference graphs, in: F. Harary (Ed.), Proof Techniques in Graph Theory, Academic Press, 1969, pp. 139–146.
[17] S.B. Sadjad, H.Z. Zadeh, Unit interval graphs, properties and algorithms, School of Computer Science, University of Waterloo, February 2004.
[18] A. Tucker, Structure theorems for some classes of circular-arc graphs, Discrete Math. 7 (1974) 167–195.