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The Weierstrass-Laguerre Transform

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1. PRELIMINARIES

In recent papers [1], [2], [3], [4], the authors studied the Hankel, the Weierstrass-Hankel, and the Poisson-Laguerre transforms. An analogous development is undertaken here for the Weierstrass-Laguerre transform, extending in part a similar investigation of the Laguerre transform by I. Hirschman in [5].

Let $\alpha \geqslant 0$ and let $L_n{}^\alpha(x)$ denote the Laguerre polynomial of degree n defined by

$$L_n{}^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\left(\frac{d}{dx}\right)^n (x^{n+\alpha}e^{-x}), \qquad n = 0, 1, \ldots . \tag{1.1}$$

For $f(n)$ a real function defined for $n = 0, 1, \ldots$, the Laguerre transform $f\hat{}(x)$ is given by

$$f\hat{}(x) = \sum_{n=0}^{\infty} L_n{}^\alpha(x) f(n)\, \rho(n), \tag{1.2}$$

where

$$\rho(n) = \frac{n!}{\Gamma(n + \alpha + 1)}. \tag{1.3}$$

By the inversion formula, we have

$$f(n) = \int_0^\infty L_n{}^\alpha(x) f\hat{}(x)\, d\Omega(x), \tag{1.4}$$

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with

$$d\Omega(x) = e^{-x}x^{\alpha}\,dx. \tag{1.5}$$

If we define the Laguerre difference operator ∇_n by

$$\nabla_n h(n) = (n+1)\,h(n+1) - (2n+\alpha+1)\,h(n) + (n+\alpha)\,h(n-1), \tag{1.6}$$

then, from the fact that

$$\nabla_n L_n^{\alpha}(x) = -\,xL_n^{\alpha}(x), \tag{1.7}$$

we have, for an arbitrary polynomial $p(x)$,

$$[p(\nabla_n)f]^{\wedge}(x) = p(-x)f^{\wedge}(x) \tag{1.8}$$

or, by inversion,

$$[p(\nabla_n)f](n) = \int_0^{\infty} f^{\wedge}(x)\,p(-x)\,L_n^{\alpha}(x)\,d\Omega(x). \tag{1.9}$$

We consider the Laguerre difference heat equation

$$\nabla_n u(n,t) = \frac{\partial}{\partial t}\,u(n,t) \tag{1.10}$$

whose fundamental solution is the function $g(n;t)$ given by

$$\begin{aligned} g(n;t) &= \int_0^{\infty} e^{-tx}L_n^{\alpha}(x)\,d\Omega(x), \qquad t > -1, \\ &= \frac{1}{\rho(n)}\,\frac{t^n}{(1+t)^{n+\alpha+1}}. \end{aligned} \tag{1.11}$$

Corresponding to $g(n;t)$ we define its conjugate $g(n^*;t)$ by

$$\begin{aligned} g(n^*;t) &= \int_0^{\infty} e^{-tx}L_n^{\alpha}(-x)\,d\Omega(x) \\ &= \frac{1}{\rho(n)}\,\frac{(2+t)^n}{(1+t)^{n+\alpha+1}}. \end{aligned} \tag{1.12}$$

We need to introduce associated functions. To this end, let

$$d(k,m,n) = \int_0^{\infty} L_k^{\alpha}(x)\,L_m^{\alpha}(x)\,L_n^{\alpha}(x)\,d\Omega(x) \tag{1.13}$$

and correspondingly

$$d(k^*,m,n) = \int_0^{\infty} L_k^{\alpha}(-x)\,L_m^{\alpha}(x)\,L_n^{\alpha}(x)\,d\Omega(x). \tag{1.14}$$

Then the associated function $f(n, m)$ of a function $f(n)$ defined for $n = 0, 1, \ldots$ is given by

$$f(n, m) = \sum_{k=0}^{\infty} f(k)\, d(k, m, n)\, \rho(k) \tag{1.15}$$

and the conjugate associated function $f(n^*, m)$ of $f(n)$ by

$$f(n^*, m) = \sum_{k=0}^{\infty} f(k)\, d(k, m, n^*)\, \rho(k). \tag{1.16}$$

It readily follows that

$$f(n, m) = \int_0^{\infty} f^{\wedge}(x)\, L_n^{\alpha}(x)\, L_m^{\alpha}(x)\, d\mu(x) \tag{1.17}$$

and

$$f(n^*, m) = \int_0^{\infty} f^{\wedge}(x)\, L_n^{\alpha}(-x)\, L_m^{\alpha}(x)\, d\mu(x) \tag{1.18}$$

and the functions $f(n, m), f(n^*, m)$ satisfy the following

$$f(n, 0) = f(n) \tag{1.19}$$

$$f(n, m) = f(m, n) \tag{1.20}$$

$$\nabla_n f(n, m) = \nabla_m f(n, m) \tag{1.21}$$

$$[L_n(-\nabla_m) f]\,(m) = f(n, m) \tag{1.22}$$

$$f(n^*, m) = f(m, n^*) \tag{1.23}$$

$$\nabla_n f(n^*, m) = -\nabla_m f(n^*, m) \tag{1.24}$$

$$[L_n(\nabla_m) f]\,(m) = f(n^*, m). \tag{1.25}$$

Of central importance in our theory are the associated and conjugate associated functions of the fundamental solution $g(n; t)$. Properties and asymptotic estimates of these were developed in detail in [3] and we include here only the results needed.

The function associated with $g(n; t)$ is given by

$$\begin{aligned} g(n, m; t) &= \int_0^{\infty} e^{-tx} L_n(x)\, L_m(x)\, d\Omega(x), \qquad t > 0 \\ &= \frac{\Gamma(n + m + \alpha + 1)}{n!m!} \frac{t^{n+m}}{(1 + t)^{n+m+\alpha+1}} \\ &\quad \times {}_2F_1\left(-n, -m; -n-m-\alpha; 1 - \frac{1}{t^2}\right). \end{aligned} \tag{1.26}$$

so that

$$e^{-tx}L_n^\alpha(x) = \sum_{m=0}^{\infty} g(n, m; t)\, L_m^\alpha(x)\, \rho(m), \tag{1.27}$$

and the conjugate associated function is given by

$$\begin{aligned} g(n^*, m; t) &= \int_0^\infty e^{-tx} L_n^\alpha(-x)\, L_m^\alpha(x)\, d\Omega(x) \\ &= \frac{\Gamma(n+m+\alpha+1)}{n!m!} \frac{(2+t)^n t^m}{(1+t)^{n+m+\alpha+1}} \\ &\quad \times {}_2F_1\left(-n, -m; -n-m-\alpha; 1+\frac{1}{t^2+2t}\right), \end{aligned} \tag{1.28}$$

where upon

$$e^{-tx}L_n^\alpha(-x) = \sum_{m=0}^{\infty} g(n^*, m; t)\, L_m^\alpha(x)\, \rho(m) \tag{1.29}$$

and also

$$e^{tx}L_n^\alpha(x) = \sum_{m=0}^{\infty} g(n^*, m; t)\, L_m^\alpha(-x)\, \rho(m). \tag{1.30}$$

As may readily be established from their series representations the functions $g(n, m; t)$, $g(n^*, m; t)$ have the following asymptotic estimates.

$$g(n, m; t) \sim \frac{n^{m+\alpha}}{m!} \frac{t^{n-m}}{(1+t)^{n+m+\alpha+1}}, \qquad n \to \infty \tag{1.31}$$

$$g(n^*, m; t) \sim \frac{(-1)^m n^{m+\alpha}}{m!} \frac{(2+t)^{n-m}}{(1+t)^{n+m+\alpha+1}}, \qquad n \to \infty \tag{1.32}$$

$$g(n, m^*; t) \sim \frac{(-1)^m n^{m+\alpha}}{m!} \frac{t^{n-m}}{(1+t)^{n+m+\alpha+1}}, \qquad n \to \infty. \tag{1.33}$$

In addition, they satisfy the following Huygens-type relation.

$$\sum_{k=0}^{\infty} g(k, m; t_1)\, g(n, k; t_2)\, \rho(k) = g(n, m; t_1 + t_2), \qquad t_1, t_2 > 1, \tag{1.34}$$

and

$$\sum_{k=0}^{\infty} g(k, n^*; t_1)\, g(k^*, m; t_2)\, \rho(k) = g(n, m; t_2 - t_1), \qquad t_2 > t_1. \tag{1.35}$$

We define the Poisson-Laguerre transform of a function φ defined for $n = 0, 1, \ldots$ by

$$u(n, t) = \sum_{m=0}^{\infty} g(n, m; t)\, \varphi(m)\, \rho(m) \tag{1.36}$$

whenever the series converges. In [3], we established the fact that if $\sum_{m=0}^{\infty} g(n_0, m; t_0)\, \varphi(m)\, \rho(m)$ converges conditionally for some non-negative integer n_0, then the series defining the Poisson-Laguerre transform converges for all $n = 0, 1, \ldots$ and $0 < t \leqslant t_0$.

The Weierstrass-Laguerre transform of a function φ defined for $n = 0, 1, \ldots$ is given by

$$f(n) = \sum_{n=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m) \tag{1.37}$$

whenever the series converges. If it does, we know from [3] that the series defining $f(n^*)$,

$$f(n^*) = \sum_{m=0}^{\infty} g(n^*, m; 1)\, \varphi(m)\, \rho(m) \tag{1.38}$$

also converges for finite n.

2. L-Harmonic Functions

In terms of the Laguerre difference operator ∇_n, we have a theory analogous to that of harmonic functions.

Let

$$\mathcal{N} = \{-1, 0, 1, \ldots\} \tag{2.1}$$

and let $f(n, m)$ be a function defined on $\mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$ and satisfying the conditions

$$f(-1, m) = f(n, -1) = f(n, 0) = 0. \tag{2.2}$$

Definition 2.1. A point (n, m) in $\mathcal{U}$, a subset of $\mathcal{N}^2$, is an inner point of $\mathcal{U}$ iff

$$\{(n+1, m), (n-1, m), (n, m+1), (n, m-1)\} \subset \mathcal{U}. \tag{2.3}$$

Otherwise, (n, m) is a boundary point.

We denote by $\mathcal{U}^0$ the set of inner points of $\mathcal{U}$, and by $\partial\mathcal{U}$, the set of boundary points of $\mathcal{U}$.

DEFINITION 2.2. A real-valued function f defined on $\mathscr{U}$ is called L-harmonic iff

$$\Box f(n, m) = (\nabla_n + \nabla_m) f(n, m) = 0. \tag{2.4}$$

at each inner point (n, m) of $\mathscr{U}$.

It is clear that every conjugate associated function

$$f(n, m^*) = L_m^{\alpha}(\nabla_n) f(n)$$

is L-harmonic.

DEFINITION 2.3. A neighborhood of a point $P : (n, m)$ in $\mathscr{N}^2$ is the set

$$N_P = \{(n, m), (n + 1, m), (n - 1, m), (n, m + 1), (n, m - 1)\}. \tag{2.5}$$

DEFINITION 2.4. A subset $\mathscr{U}$ of $\mathscr{N}^2$ is a domain iff, given any two inner points P and Q, we can find a sequence of points $P = P_0, P_1, ..., P_n = Q$ from P to Q, where P_{k+1} is obtained from P_k by changing one coordinate of P_k by either $+1$ or -1, and where $\{P_1, P_2, ..., P_{n-1}\} \subset \mathscr{U}^0$.

DEFINITION 2.5. A domain $\mathscr{U}$ is called a simple domain iff

$$\mathscr{U} = \mathscr{U}^0 = \mathscr{U}^0 \cup \{P \in \partial\mathscr{U} \mid P \in N_Q \text{ for some } Q \in \mathscr{U}^0\}. \tag{2.6}$$

THEOREM 2.6 [maximum (minimum) principle]. *If f is L-harmonic on a finite subset $\mathscr{U}$ of $\mathscr{N}^2$, then f attains its maximum (minimum) values on $\partial\mathscr{U}$.*

PROOF. It is clearly enough to prove the theorem for the maximum.

Let (n, m) be an inner point of $\mathscr{U}$ at which f attains a maximum value M. Then

$$\begin{aligned} M &= f(n, m) = f(n + 1, m) = f(n - 1, m) \\ &= f(n, m + 1) = f(n, m - 1). \end{aligned} \tag{2.7}$$

For, if any one of $f(n + 1, m)$, $f(n - 1, m)$, $f(n, m + 1)$, $f(n, m - 1)$ is strictly less than $M = f(n, m)$, it follows that

$$\begin{aligned} (n + 1) f(n + 1, m) + (n + \alpha) f(n - 1, m) + (m + 1) f(n, m + 1) \\ + (m + \alpha) f(n, m - 1) < 2(n + m + \alpha + 1) f(n, m) \end{aligned} \tag{2.8}$$

or

$$\Box f(n, m) < 0, \tag{2.9}$$

contrary to the assumption that f is L-harmonic. Repeating the argument a finite number of times, we find a boundary point at which f takes on the maximum value M.

COROLLARY 2.7. *If f is L-harmonic on a finite domain $\mathscr{U}$ of $\mathscr{N}^2$, then f does not attain a local maximum (minimum) on $\mathscr{U}^0$ unless f is constant on $\mathscr{U}^0$.*

THEOREM 2.8. *Let f be defined on the boundary ∂ of a finite simple domain $\mathscr{U}$. Then there exists a unique L-harmonic function g on $\mathscr{U}$ which coincides with f on $\partial\mathscr{U}$.*

PROOF. We note, first, that the uniqueness follows from the preceding theorem. For, if g_1 and g_2 are two such functions, then $g = g_1 - g_2$ is L-harmonic on $\mathscr{U}$ and is identically zero on $\partial\mathscr{U}$. But then $g \equiv 0$ on $\mathscr{U}$ so that $g_1 \equiv g_2$ on μ.

Now, let $P_1, P_2, \ldots, P_r$ be all the inner points of μ, and $Q_1, Q_2, \ldots, Q_s$ all its boundary points. Consider the r equations

$$\Box g(P_k) = 0, \qquad k = 1, 2, \ldots, r, \tag{2.10}$$

in the r unknowns $g(P_1), \ldots, g(P_r)$. The system (2.10) can be written in the form

$$\sum_{j=1}^{r} a_{kj} g(P_j) = b_k \qquad k = 1, 2, \ldots, r, \tag{2.11}$$

where

$$b_k = \sum_{j=0}^{s} c_{kj} f(Q_j). \tag{2.12}$$

We note that if $f \equiv 0$ on $\partial\mathscr{U}$, then $g \equiv 0$ on $\mathscr{U}$ and (2.11) reduces to the homogenous system

$$\sum_{j=1}^{r} a_{kj} g(P_j) = 0, \tag{2.13}$$

which admits only the trivial solution. It follows, therefore, by Cramer's rule, that the system (2.13), and equivalently (2.11), must have a unique solution g, which is the function sought.

3. FORMAL APPROACH

Before proceeding to a rigorous development of the inversion theory we seek, let us illustrate its essence by deriving it formally. We need a suitable definition for the difference operator $e^{-\nabla_n}$ so that the inversion formula

$$e^{-\nabla_n} \sum_{n=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m) = \varphi(n) \tag{3.1}$$

will hold. To this end, we formally set

$$e^{-t\nabla_n} = \sum_{k=0}^{\infty} \frac{(-t\nabla_n)^k}{k!}, \tag{3.2}$$

where ∇_n is given by (1.2). Then we have, using (2.3),

$$\begin{aligned} e^{-t\nabla_n} L_n^{\alpha}(x) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} (-\nabla_n)^k L_n^{\alpha}(x) \\ &= \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} L_n^{\alpha}(x) \\ &= e^{tx} L_n^{\alpha}(x). \end{aligned} \tag{3.3}$$

Now, if

$$f(n) = \int_0^{\infty} L_n^{\alpha}(x) f^{\wedge}(x)\, d\Omega(x), \tag{3.4}$$

it follows, formally, that

$$\begin{aligned} e^{-t\nabla_n} f(n) &= \int_0^{\infty} [e^{-t\nabla_n} L_n^{\alpha}(x)] f^{\wedge}(x)\, d\Omega(x) \\ &= \int_0^{\infty} e^{tx} L_n^{\alpha}(x) f^{\wedge}(x)\, d\Omega(x). \end{aligned} \tag{3.5}$$

But, by (1.30),

$$e^{tx} L_n^{\alpha}(x) = \sum_{m=0}^{\infty} g(n^*, m; t) L_m^{\alpha}(-x)\, \rho(m), \tag{3.6}$$

and so

$$\begin{aligned} e^{-t\nabla_n} f(n) &= \sum_{m=0}^{\infty} g(n^*, m; t)\, \rho(m) \int_0^{\infty} f^{\wedge}(x) L_m^{\alpha}(-x)\, d\Omega(x) \\ &= \sum_{m=0}^{\infty} g(n^*, m; t) f(m^*)\, \rho(m). \end{aligned} \tag{3.7}$$

We are thus led to the following definition

$$e^{-\nabla_n} f(n) = \lim_{t\to 1^-} \sum_{m=0}^{\infty} g(n^*, m; t) f(m^*)\, \rho(m), \tag{3.8}$$

which yields the inversion formula. For, let

$$f(n) = \sum_{m=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m), \qquad n = 0, 1, 2,..., \tag{3.9}$$

with $\varphi(m)$ defined for $m = 0, 1, 2,...$. Then

$$e^{-\nabla_n} f(n) = \lim_{t\to 1^-} \sum_{m=0}^{\infty} g(n^*, m; t) f(m^*)\, \rho(m)$$

$$= \lim_{t\to 1^-} \sum_{m=0}^{\infty} g(n^*, m; t)\, \rho(m) \sum_{k=0}^{\infty} g(m^*, k; 1)\, \varphi(k)\, \rho(k)$$

$$= \lim_{t\to 1^-} \sum_{k=0}^{\infty} \varphi(k)\, \rho(k) \sum_{m=0}^{\infty} g(n^*, m; t)\, g(m^*, k; 1)\, \rho(m)$$

$$= \lim_{t\to 1^-} \sum_{k=0}^{\infty} g(n, k; 1 - t)\, \varphi(k)\, \rho(k),$$

where the last equality follows from (1.35). Hence

$$e^{-\nabla_n} f(n) = \lim_{t\to 1^-} \sum_{k=0}^{\infty} \varphi(k)\, \rho(k) \int_0^{\infty} e^{-(1-t)x} L_n^{\alpha}(x)\, L_k^{\alpha}(x)\, d\Omega(x)$$

$$= \lim_{t\to 1^-} \int_0^{\infty} e^{-(1-t)x} L_n^{\alpha}(x)\, \varphi^{\wedge}(x)\, d\Omega(x)$$

$$= \int_0^{\infty} \varphi^{\wedge}(x)\, L_n^{\alpha}(x)\, d\Omega(x)$$

$$= \varphi(n),$$

and the inversion is established.

4. Inversion

Our principal inversion theorem depends on the following fundamental inversion formula derived in [3].

THEOREM 4.1. *If $\varphi(m)$ is a real-valued function defined for $m = 0, 1, \ldots$, and the Weierstrass-Laguerre transform*

$$\sum_{m=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m) \tag{4.1}$$

converges for some nonnegative integer n_0, then

$$\lim_{t \to 0^+} \sum_{m=0}^{\infty} g(n, m; t)\, \varphi(m)\, \rho(m) = \varphi(n). \tag{4.2}$$

We need, in addition, the following readily established estimate. See [3].

LEMMA 4.2. *For α a positive number and $| r | < 1$,*

$$\sum_{k=0}^{\infty} k^{\alpha} r^{k} = O\left\{\frac{[\alpha] + 1}{(1 - r)^{[\alpha]+2}}\right\}, \qquad n \to \infty, \tag{4.3}$$

where $[\alpha]$ denotes the greatest integer in α.

From the asymptotic relation (1.31), the following result is immediate.

LEMMA 4.3. *If the series*

$$\sum_{m=0}^{\infty} g(n, m; t)\, \varphi(m)\, \rho(m), \qquad n = 0, 1, \ldots, \tag{4.4}$$

converges, then so does the series

$$\sum_{m=0}^{\infty} m^{n} \left(\frac{t}{1 + t}\right)^{m} \varphi(m). \tag{4.5}$$

We are now ready to prove the main theorem leading to the inversion desired.

THEOREM 4.4. *Let $\varphi(m)$ be a real-valued non-negative function defined for $m = 0, 1, \ldots$ and let the series*

$$\sum_{m=0}^{\infty} g(n, m; 2)\, \varphi(m)\, \rho(m) \tag{4.6}$$

converge. Then, if

$$f(n) = \sum_{m=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m), \tag{4.7}$$

it follows that

$$e^{-t\nabla_n} f(n) = \sum_{m=0}^{\infty} g(n, m; 1 - t)\, \varphi(m)\, \rho(m), \tag{4.8}$$

where

$$e^{-t\nabla_n} f(n) = \sum_{m=0}^{\infty} g(n^*, m; t) f(m^*)\, \rho(m). \tag{4.9}$$

PROOF. Since the series (4.6) converges, it follows that the Weierstrass-Laguerre transform converges, and so, also, the series

$$f(n^*) = \sum_{k=0}^{\infty} g(n^*, k; 1)\, \varphi(k)\, \rho(k). \tag{4.10}$$

Hence we have

$$\begin{aligned} e^{-t\nabla_n} f(n) &= \sum_{m=0}^{\infty} g(n^*, m; t) f(m^*)\, \rho(m) \\ &= \sum_{m=0}^{\infty} g(n^*, m; t)\, \rho(m) \sum_{k=0}^{\infty} g(m^*, k; 1)\, \varphi(k)\, \rho(k) \\ &= \sum_{k=0}^{\infty} \varphi(k)\, \rho(k) \sum_{m=0}^{\infty} g(n^*, m; t)\, g(m^*, k; 1)\, \rho(m), \end{aligned} \tag{4.11}$$

provided the double series converges and the change in order of summation is justified. This follows from the fact that by the estimates (1.32), (1.33), the right-hand side of (4.11) is dominated by

$$\frac{1}{2^{\alpha+1} n!\, t^n (1 + t)^{n+\alpha+1}} \sum_{k=0}^{\infty} \frac{\varphi(k)}{6^k k!\, k^\alpha} \sum_{m=0}^{\infty} m^{n+k+\alpha} \left[\frac{3t}{2(1 + t)}\right]^m,$$

or, by an appeal to Lemma 4.2, by

$$A_n \sum_{k=0}^{\infty} \left(\frac{1 + t}{6 - 3t}\right)^k \varphi(k) \frac{(n + k + [\alpha] + 1)!}{k!\, k^{[\alpha]}} \sim A_n \sum_{k=0}^{\infty} \left(\frac{1 + t}{6 - 3t}\right)^k k^{n+1} \varphi(k).$$

and the series converges by the hypothesis and Lemma 4.3. Applying (1.35) to the right-hand side of (4.11), we obtain (4.9) and the proof is complete.

As a consequence of this theorem and Theorem 4.1, we have our principal result.

THEOREM 4.5. *Under the conditions of Theorem 4.4*

$$\lim_{t \to 1^-} e^{-t\nabla_n} f(n) = \varphi(n).$$

5. REPRESENTATION

Our goal here is to characterize those functions which are Weierstrass-Laguerre transforms of positive functions.

We need to consider Laguerre temperatures defined as C^1 solutions of the Laguerre difference heat equation. We denote by H the class of all Laguerre temperatures. In [3] it was shown that a convergent Poisson-Laguerre transform belongs to H in its region of convergence. In addition, if we set

$$e^{-t\nabla_n} f(n) = \sum_{m=0}^{\infty} g(n^*, m; t) f(n^*) \rho(n), \tag{5.1}$$

then the function

$$u(n, t) = e^{-(1-t)\nabla_n} f(n) \tag{5.2}$$

belongs to H for $0 < t < 1$ provided that

$$f(n^*) = o(ng(n^*; 1)), \qquad n \to \infty. \tag{5.3}$$

To establish our principal result, we state a representation theorem for Laguerre temperatures proved in [3].

THEOREM 5.1. *A necessary and sufficient condition that*

$$u(n, t) = \sum_{m=0}^{\infty} g(n, m; t) \varphi(m) \rho(m) \tag{5.4}$$

with $\varphi(m)$ nonnegative and the series converging for $n = -1, 0, \ldots,$ $0 < t < c$ is that $u(n, t)$ be a nonnegative Laguerre temperature there.

· We need, in addition, the following result.

LEMMA 5.2. *Let f be defined for $n = -1, 0, 1,\ldots$ and let $f(n, m^*)$ be its conjugate associated function. If*

$$f(n, m^*) = O(g(m^*; 1)), \qquad m \to \infty, \tag{5.5}$$

then for $0 \leqslant t \leqslant 1$

$$\sum_{m=0}^{\infty} g(m; t) [\nabla_m f(n, m^*)] \rho(m) = \sum_{m=0}^{\infty} [\nabla_m g(m; t)] f(n, m^*) \rho(m). \tag{5.6}$$

PROOF. We note that since $f(n, m^*)$ is L-harmonic, the series on the left of (5.6) is equal to

$$-\nabla_n \sum_{m=0}^{\infty} g(m; t) f(n, m^*) \rho(m), \tag{5.7}$$

which converges absolutely and uniformly because of (5.5). Now, defining

$$\delta^+ f(n) = f(n+1) - f(n) \tag{5.8}$$

and

$$\delta^- f(n) = f(n) - f(n-1) \tag{5.9}$$

and noting that

$$\nabla_n f(n) = (n+1)\, \delta^+ f(n) - (n+\alpha)\, \delta^- f(n), \tag{5.10}$$

we have

$$\sum_{m=0}^{\infty} g(m; t) [\nabla_m f(n, m^*)] \rho(m)$$

$$= \sum_{m=0}^{\infty} g(m; t) \{(m+1)\, \delta_m^+ f(n, m^*) - (m+\alpha)\, \delta_n^- f(n, m^*)\} \rho(m)$$

$$= -\sum_{m=0}^{\infty} f(n, m^*) [\delta_m^-(m+1)\, \rho(m)\, g(m; t) - \delta_m^+(m+\alpha)\, \rho(m)\, g(m; t)],$$

where the last equality is a result of a summation by parts, the summed part vanishing. Since the factor in brackets of the last series is $-[\nabla_m g(m; t)] \rho(m)$, the proof is established.

We now give the characterization of a function which has a Laguerre-Weierstrass transform representation.

THEOREM 5.3. *Let f be defined on $n = -1, 0, 1,...$ and let $f(n, m^*)$ be its conjugate associated function. If*

$$f(n, m^*) = O[g(m^*; 1)], \qquad m \to \infty \tag{5.11}$$

and

$$e^{-t\nabla_n} f(n) \geqslant 0, \qquad 0 \leqslant t < 1, \qquad n = 0, 1,..., \tag{5.12}$$

then there exists a nonnegative function φ, defined for $n = 0, 1,...,$ such that

$$f(n) = \sum_{m=0}^{\infty} g(n, m; 1)\, \varphi(m)\, \rho(m) \tag{5.13}$$

PROOF. Let

$$u(n, t) = e^{-(1-t)\nabla_n} f(n). \tag{5.14}$$

By (5.11), it is clear that $u(n, t)$ is well defined. Condition (5.12) implies that $u(n, t) \geqslant 0$, and we know that $u(n, t)$ is a Laguerre temperature in $0 < t < 1$. Hence by Theorem 5.1, there exists a nonnegative function φ such that

$$u(n, t) = \sum_{m=0}^{\infty} g(n, m; t)\, \varphi(m)\, \rho(m). \tag{5.15}$$

But by (5.14)

$$u(n, t) = \sum_{m=0}^{\infty} g(n^*, m; 1 - t) f(m^*)\, \rho(m). \tag{5.16}$$

Let

$$v(n, t) = \sum_{m=0}^{\infty} g(m; 1 - t) f(n, m^*)\, \rho(m). \tag{5.17}$$

Since (5.11) holds, the series (5.17) converges absolutely and uniformly. Applying the operator ∇_n to $v(n, t)$ we have

$$\nabla_n v(n, t) = \sum_{m=0}^{\infty} g(m; 1 - t)\, \nabla_n f(n, m^*)\, \rho(m)$$

$$= - \sum_{m=0}^{\infty} g(m; 1 - t)\, \nabla_m f(n, m^*)\, \rho(m),$$

since $f(n, m^*)$ is L-harmonic. By the preceding lemma, we then have

$$\nabla_n v(n, t) = - \sum_{m=0}^{\infty} [\nabla_m g(m; 1 - t)] f(n, m^*) \rho(m)$$

$$= \frac{\partial}{\partial t} v(n, t),$$

so that $v(n, t) \in H$. Further,

$$\nabla_n v(n, t) \mid_{n=0} = - \sum_{m=0}^{\infty} [\nabla_m g(m; 1 - t)] f(m^*) \rho(m).$$

Now, applying the operator ∇_n to $u(n, t)$, we have

$$\nabla_n u(n, t) = \sum_{m=0}^{\infty} \nabla_n g(n^*, m; 1 - t) f(m^*) \rho(m)$$

$$= - \sum_{m=0}^{\infty} [\nabla_m g(n^*, m; 1 - t)] f(m^*) \rho(m),$$

by the L-harmonic property of $g(n^*, m; 1 - t)$. Hence

$$\nabla_n u(n, t) \mid_{n=0} = - \sum_{m=0}^{\infty} [\nabla_m g(m; 1 - t)] f(m^*) \rho(m)$$

$$= \nabla_n v(n, t) \mid_{n=0}, \tag{5.18}$$

or

$$[u(1, t) - u(0, t)] - \alpha[u(0, t) - u(-1, t)] = [v(1, t) - v(0, t)] - \alpha[v(0, t) - v(-1, t)]. \tag{5.19}$$

Since $u(-1, t) = v(-1, t) = 0$, and clearly $u(0, t) = v(0, t)$, (5.19) yields

$$u(1, t) = v(1, t).$$

Inductively, we obtain

$$\nabla_n{}^k u(n, t) \mid_{n=0} = \nabla_n{}^k v(n, t) \mid_{n=0}$$

with $u(n, t) = v(n, t)$ for $n = -1, 0, \ldots, k - 1$, so that

$$u(k, t) = v(k, t).$$

Hence

$$u(n, t) = v(n, t), \qquad n = -1, 0, 1, \ldots .$$

We thus have

$$u(n, t) = v(n, t)$$

$$= \sum_{m=0}^{\infty} g(m; 1 - t) f(n, m^*) \rho(m),$$

and letting $t \to 1^-$, we obtain

$$\lim_{t \to 1^-} u(n, t) = \sum_{m=0}^{\infty} \delta(0, m) f(n, m^*)$$

$$= f(n).$$

We now must show the convergence of the series (5.15) for $t = 1$. To this end, we note that for $N > 0$, we have

$$\sum_{m=0}^{N} g(n, m; t) \varphi(m) \rho(m) \leqslant u(n, t), \qquad n = 0, 1, \ldots,$$

or letting $t \to 1^-$, we obtain

$$\sum_{m=0}^{N} g(n, m; 1) \varphi(m) \rho(m) \leqslant u(n, 1^-) = f(n).$$

Since $g(n, m; 1) \varphi(m) \rho(m) \geqslant 0$ for all n, m, we have

$$\sum_{m=0}^{\infty} g(n, m; 1) \varphi(m) \rho(m) \leqslant f(n).$$

Hence we have the required convergence and letting $t - 1^-$ in (5.15) yields the desired representation.

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