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Group Ramsey Theory

ANNE PENFOLD STREET

Department of Mathematics, University of Queensland, St. Lucia, Queensland, 4067, Australia

AND

EARL GLEN WHITEHEAD, JR.*

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

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A subset S of a group G is said to be a *sum-free set* if $S \cap (S + S) = \varnothing$. Such a set is *maximal* if for every sum-free set $T \subseteq G$, we have $|T| \leqslant |S|$. Here, we generalize this concept, defining a sum-free set S to be *locally maximal* if for every sum free set T such that $S \subseteq T \subseteq G$, we have $S = T$. Properties of locally maximal sum-free sets are studied and the sets are determined (up to isomorphism) for groups of small order.

1. INTRODUCTION AND STATEMENT OF RESULTS

Given an additive group G and nonempty subsets S, T of G, let $S + T$ denote the set $\{s + t \mid s \in S, t \in T\}$, $-S$ the set $\{-s \mid s \in S\}$, $\bar{S}$ the complement of S in G, and $|S|$ the cardinality of S. We call S a *sum-free set* in G if $(S + S) \subseteq \bar{S}$.

A sum-free set $S \subseteq G$ is said to be *maximal* if for every sum-free set $T \subseteq G$, we have $|T| \leqslant |S|$. We denote by $\lambda(G)$ the cardinality of a maximal sum-free set in G. There is already a considerable literature on maximal sum-free sets in finite groups; see the bibliography of [2, Part 3].

More generally, a sum-free set $S \subseteq G$ is said to be *locally maximal* if for every sum-free set T such that $S \subseteq T \subseteq G$, we have $S = T$. Clearly,

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S is a locally maximal sum-free set if and only if $S \cup \{g\}$ is not sum-free for any $g \in \bar{S}$. Hence for every $g \in \bar{S}$, we have

$$g = s_1 + s_2 \tag{1}$$

or

$$g + s_1 = s_2 \tag{2}$$

or

$$2g = s, \tag{3}$$

for some $s_1, s_2, s \in S$. We denote by $\Lambda(G)$ the set of cardinalities of all locally maximal sum-free sets in G, so that $\lambda(G) = \max \Lambda(G)$. We are also interested in the cardinality of the smallest possible locally maximal sum-free set and denote it by $\mu(G) = \min \Lambda(G)$.

A *sum-free covering* of G is a collection of sum-free sets $\mathscr{S} = \{S_1, ..., S_n\}$ such that $G^* = G \backslash \{0\} = \bigcup_{i=1}^{n} S_i$. If $\mathscr{S}$ and $\mathscr{T} = \{T_1, ..., T_n\}$ are two sum-free coverings of G such that $S_i \subseteq T_i$ for $i = 1, ..., n$, we say that $\mathscr{S}$ is *embedded* in $\mathscr{T}$. The special case of a sum-free partition of G (or strictly, G^*) where the sets S_i are pairwise disjoint, has been studied [1, 2, 4, 5, 6] in connection with finding lower bounds on the Ramsey number $N_k = N_k(3, 2)$ i.e., the smallest positive integer such that coloring the edges of the complete graph on N_k vertices in k colors forces the appearance of a monochromatic triangle.

Ramsey theory for groups involves finding the smallest number of sum-free sets needed to partition G^*. Our first result is the following:

THEOREM 1. *Every sum-free partition of G^* can be embedded into at least one covering of G^* by locally maximal sum-free sets.*

It has been shown [1, 5] that if G^* can be partitioned into k sum-free sets, then each of these k sets has cardinality less than N_{k-1}. An adaption of this argument leads to the following result.

COROLLARY 1. *Each of the locally maximal sum-free sets of Theorem 1 has cardinality less than N_{k-1}.*

Theorem 1 implies that we need only consider covering of G by locally maximal sum-free sets, so we are interested in characterizing all locally maximal sum-free sets in as many groups as possible.

Let Z_n denote the cyclic group of order n, and $(Z_n)^m$ the direct sum of m copies of Z_n. In the known sum-free partitions of Z_5, $(Z_2)^4$, Z_{41}[1], $(Z_4)^2$ [4], $(Z_7)^2$ [6] and $Z_{13} = \{4, 6, 7, 9\} \cup \{1, 5, 8, 12\} \cup \{2, 3, 10, 11\}$, all the sets occurring in the partitions are locally maximal sum-free sets S_i with the additional property that in each case $S_i \cup (S_i + S_i) = G$, the group being partitioned. It seems likely that this property might be important so we provide the following definition.

DEFINITION. Let S be a locally maximal sum-free set in a group G. If $S \cup (S + S) \supseteq G^*$, then S is said to fill G. [Note that two cases arise here if S fills G: $S \cup (S + S) = G^*$ if and only if $S \cap (-S) = \varnothing$; otherwise $S \cup (S + S) = G$.] If every locally maximal sum-free set S in G fills G, then G is said to be a *filled group*.

Finite abelian filled groups are fully characterized.

THEOREM 2. *A finite abelian group G is a filled group if and only if it is* (1) *an elementary abelian 2-group*
or (2) Z_3
or (3) Z_5.

For nonabelian groups, we have only necessary conditions.

THEOREM 3. *Let G be a finite nonabelian filled group. Then*

(1) *For any normal subgroup $N \leqslant G$, the factor group G/N is a filled group;*

(2) $G = G'$ *or* G/G' *is an elementary abelian 2-group or* $G/G' \cong Z_5$ *and* $|G|$ *is even, where G' is the commutator subgroup of G.*

These conditions are certainly not sufficient, as we can see from the following examples. We consider first the case where the commutator factor group is an elementary abelian 2-group; in both of the following cases, S is a locally maximal sum-free set which does not fill G.

(1) $G = D_n$, the dihedral group of order $2n$, where $n = 6k + 1$, denoted by $G = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle$. Choose

$$S = \{a^{2k+1}, \dots, a^{4k}, a^{2k+1}b, \dots, a^{4k}b\}.$$

(2) $G = Q$, the quaternion group of order 8. Let S be the set consisting of the only element of order 2.

However, we find by direct computation that D_3, D_4, D_5 and D_6 are filled groups.

For the case in which $G = G'$, we know that A_5, the alternating group of degree five, is not filled by the locally maximal sum-free set

$$S = \{(14)(23), (12)(35), (13)(45), (15)(24), (25)(34), (12)(34), (15)(23), (14)(25), (24)(35), (14)(35), (123), (245)\}.$$

So by (1) of Theorem 3, $SL(2, 5)$ is also not a filled group. We have no other results for this case.

2. PROOFS OF THEOREMS

Proof of Theorem 1. Let $\mathscr{S} = \{S_1, ..., S_k\}$ be a sum-free partition of G^*. For each i, adjoin elements of G^* to S_i maintaining sum-freeness, until a locally maximal sum-free set, say T_i, is obtained. (T_i may not be unique.) Now $\mathscr{T} = \{T_1, ..., T_k\}$ is a covering of G^* by locally maximal sum-free sets, and by construction, $\mathscr{S}$ is embedded in $\mathscr{T}$.

Proof of Corollary 1.

$$\text{Let } \mathscr{L} = \{T \mid T \text{ is a locally maximal sum-free set; } T \supseteq S_i \text{ for some } i.\}$$

Let T_0 be a maximal element of $\mathscr{L}$, i.e., for every $T \in \mathscr{L}$, $|T| \leqslant |T_0|$. Suppose, without loss of generality, that $T_0 \supseteq S_1$. Consider $\mathscr{R} = \{T_0, S_2 \backslash T_0, ..., S_k \backslash T_0\}$. Now $\mathscr{R}$ is a sum-free partition of G^*, for $T_0 \supseteq S_1$ and $T_0 \cap (S_i \backslash T_0) = \varnothing$ for all $i = 2, ..., k$. By the results of [1, 5], we have $|T_0| \leqslant N_{k-1}$ and the corollary follows.

We need one preliminary result in order to prove Theorems 2 and 3.

LEMMA 1. *Let G be a finite group and let $N \leqslant G$. Suppose that S is a locally maximal sum-free set in G/N and that S does not fill G/N. Let T be the subset of G consisting of all the elements of G belonging to the cosets of N contained in S. Then T is a locally maximal sum-free set in G which does not fill G.*

Proof. Since S is a locally maximal sum-free set in G/N, we see at once that T is a locally maximal sum-free set in G.

If T fills G, then we must have $T \cup (T + T) = G$, since T and $T + T$ consist of complete cosets of N. Hence, for every $g \in \bar{T}$, we have $g = t_1 + t_2$ for some $t_1, t_2 \in T$. So for every $g + N \in G/N \backslash S$, we have $g + N = (t_1 + N) + (t_2 + N)$ and $S \cup (S + S) = G/N$. But this is a contradiction.

Proof of Theorem 2.

(i) First we check that the groups listed in the statement of the theorem are filled groups.

In Z_3, the only locally maximal sum-free sets are $\{1\}$ and $\{2\}$; in Z_5, the only locally maximal sum-free sets are $\{1, 4\}$ and $\{2, 3\}$. Clearly these are filled groups.

Now let G be an elementary abelian 2-group and let $S \subseteq G$ be a locally maximal sum-free set. Let $g \in \bar{S}$, so that g satisfies at least one of the conditions (1)–(3). Since G has exponent 2, (3) cannot hold and (1) is equivalent to (2). So we have $g = s_1 + s_2 \in S + S$; hence $S \cup (S + S) = G$, and G is a filled group.

(ii) To show that no other finite abelian group G is filled, it is sufficient by Lemma 1 to find a quotient group H of G, such that H is not a filled group. Since G is abelian, it can be written as a direct product of cyclic groups; we choose $H = Z_n$ for some n and consider several cases.

(a) If $n = 3k + 1$, $k \geqslant 1$, choose $S = \{k + 1,..., 2k\}$.
(b) If $n = 3k - 1$, $k \geqslant 3$, choose $S = \{k - 1,..., 2k - 3\}$.
(c) If $n = 3k$, $k \geqslant 2$, choose $S = \{k,..., 2k - 1\}$.

In each of these cases, the set S is a locally maximal sum-free set which does not fill Z_n.

We have still to consider the cases where the only cyclic factor of G is Z_n, where $n = 3k - 1$ for $k = 1, 2$ and $n = 3k$ for $k = 1$, i.e., the elementary abelian groups of exponent 2, 3 or 5. We know from (i) that the elementary abelian 2-groups are filled.

(d) If G has exponent 3, but $G \neq Z_3$, then choose $H = Z_3 \times Z_3$, represented as the set of ordered pairs of integers modulo 3. Let $S = \{(1, 0), (1, 1), (1, 2)\}$.

(e) If G has exponent 5, but $G \neq Z_5$, then choose $H = Z_5 \times Z_5$, and let $S = \{(1, 0), (0, 1), (4, 4), (1, 3), (3, 2)\}$.

Again in both of these cases, the set S is a locally maximal sum-free set which does not fill H.

Proof of Theorem 3. From Lemma 1, we have at once both the first statement of the theorem and the fact that, if G' is a proper subgroup of G, then G/G' must be an elementary 2-group or Z_3 or Z_5.

If $G/G' = Z_3$, then a coset of G' (other than G' itself) is a locally maximal sum-free set which does not fill G.

If $G/G' = Z_5$, choose $a \in \bar{G}'$ so that $G = \bigcup_{i=0}^{4} (ia + G')$. If $|G|$ is odd, there exists a unique element $b \in G$, such that $2b = a$; clearly $b \in 3a + G'$. Then $S = \{a + G'\} \backslash \{a\} \cup \{b\}$ is a locally maximal sum-free set which does not fill G. If $|G|$ is even, we may not be able to choose a and b to satisfy these requirements; we have been unable to deal with this case.

3. Summary of Calculations

We give here a summary of preliminary calculations of locally maximal sum-free sets in groups of small order. In carrying out these calculations we used isomorph rejection [3], several results on maximal sum-free sets

[2, Part 3, Theorems 6.15, 6.16, 7.9 and Corollary 7.3], Theorem 2 and the following observations.

LEMMA 2. *Let S be a sum-free set in a group G. If H is a subgroup of G, and* $x \in \bar{H}$, *let* $S_0 = S \cap H$, $S_x = S \cap (H + x)$. *If* $|S_0| \geqslant 1$, *then* $|S_x| \leqslant (1/2)|H|$ *for every coset of H.*

Proof. $(S_0 + S_x) \cup (S_x + S_0) = R \subseteq H + x$. Since S is sum-free, $S \cap R = \varnothing$. But $|R| \geqslant |S_x|$, so $2|S_x| \leqslant |H|$ and Lemma 2 follows.

LEMMA 3. *Let H be a subgroup of index 2 in a group G, and let S be a locally maximal sum-free set in G. If* $|S| < |H|$, *then* $|S \cap H| \geqslant 1$.

Proof. By Corollary 7.3 of [2, Part 3], a maximal sum-free set in G must be a coset of a subgroup of index 2. Hence a locally maximal sum-free set S, with $|S| < |H|$, cannot be contained in the coset complementary to H. But this implies that $|S \cap H| \geqslant 1$.

TABLE I

$\|G\|$	G	$\Lambda(G)$	Locally maximal sum-free sets
2	Z_2	{1}	{1}
3	Z_3	{1}	{1}
4	Z_4	{1, 2}	{2}, {1, 3}
	$(Z_2)^2$	{2}	{01, 11}
5	Z_5	{2}	{1, 4}
6	Z_6	{2, 3}	{2, 3}, {2, 5}, {1, 3, 5}
	S_3	{2, 3}	{(123), (12)}, {(12), (13), (23)}
7	Z_7	{2}	{2, 3}, {3, 4}
8	Z_8	{2, 3, 4}	{1, 6}, {2, 6}, {3, 4, 5}, {1, 3, 5, 7}
	$Z_4 \times Z_2$	{2, 4}	{20, 01}, {01, 11, 21, 31}, {10, 11, 30, 31}
	$(Z_2)^3$	{4}	{001, 011, 101, 111}
	D_4	{3, 4}	$\{a^2, b, ab\}, \{b, ab, a^2b, a^3b\}, \{a, a^3, b, a^2b\}$
	Q	{1, 4}	$\{a^2\}, \{b, ab, a^2b, a^3b\}$
9	Z_9	{3}	{3, 4, 5}, {1, 4, 7}
	$(Z_3)^2$	{3}	{01, 11, 21}
10	Z_{10}	{3, 4, 5}	{3, 4, 5}, {4, 5, 6}, {1, 4, 6, 9}, {1, 3, 5, 7, 9}
	D_5	{4, 5}	$\{a^2, a^3, a^2b, a^3b\}, \{b, ab, a^2b, a^3b, a^4b\}$
11	Z_{11}	{3, 4}	{3, 4, 5}, {4, 5, 6, 7}
13	Z_{13}	{3, 4}	{3, 4, 5}, {1, 3, 9}, {4, 6, 7, 9}, {4, 5, 6, 7}, {5, 6, 7, 8}
14	Z_{14}	{4, 5, 7}	{2, 5, 9, 13}, {4, 5, 6, 7}, {5, 6, 7, 8, 9}, {1, 3, 5, 7, 9, 11, 13}
	D_7	{4, 5, 7}	$\{a, ab, a^3b, a^6b\}, \{a^2, a^3, b, ab\},$ $\{a^3, a^4, b, ab, a^2b\}, \{b, ab, a^2b, a^3b, a^4b, a^5b, a^6b\}$
16	$(Z_2)^4$	{5, 8}	{1111, 1000, 0100, 0010, 0001}, {1111, 1000, 1001, 1010, 1100, 1011, 1101, 1110}

More generally, we use the fact that no proper subset of a locally maximal sum-free set is locally maximal.

In Table I, we list all nonisomorphic locally maximal sum-free sets in groups of orders 2,..., 11, 13 and 14. The cyclic group Z_n is denoted by integers modulo n; the direct product $Z_{n_1} \times \cdots \times Z_{n_k}$ is denoted by ordered k-tuples of integers modulo $n_1, \ldots, n_k$, respectively. Nonabelian groups are written multiplicatively: the quaternion group as $Q = \langle a, b \mid a^4 = 1, a^2 = b^2, ab = ba^3 \rangle$; the nonabelian group of order 6 as S_3, the symmetric group of degree 3; the dihedral group as $D_n = \langle a, b; a^n = b^2 = 1, ab = ba^{-1} \rangle$ for $n \geqslant 4$. The sets listed are unique up to isomorphisms.

Further results and a discussion of the techniques used in their computation will appear in a forthcoming paper.

4. Unsolved Problems

Theorem 1 tells us that every group can be covered by locally maximal sum-free sets. However, the partitions of $(Z_4)^2$ and of the additive groups of the finite fields, quoted earlier, are all examples of locally maximal sum-free partitions. This raises the following problem: which groups can be partitioned into locally maximal sum-free sets?

If we consider Z_{13} again, we find at least three distinct ways in which it can be partitioned into locally maximal sum-free sets. Firstly, by considering the cubic residues {1, 5, 8, 12} in $GF\{13\}$, and their multiplicative cosets, we obtain the partition quoted earlier,

$$Z_{13} = \{1, 5, 8, 12\} \cup \{2, 3, 10, 11\} \cup \{4, 6, 7, 9\}.$$

Secondly, by considering the quartic residues {1, 3, 9} in $GF\{13\}$, and their multiplicative cosets, we obtain the partition

$$Z_{13} = \{1, 3, 9\} \cup \{2, 5, 6\} \cup \{4, 10, 12\} \cup \{7, 8, 11\}.$$

Finally, by considering the difference set {0, 1, 3, 9} in Z_{13} and its shifts containing 0, we obtain the partition

$$Z_{13} = \{1, 3, 9\} \cup \{2, 8, 12\} \cup \{4, 5, 7\} \cup \{6, 10, 11\}.$$

Each of the sum-free sets in these partitions is locally maximal, so this raises a further problem: In how many ways can a group be partitioned into locally maximal sum-free sets?

We notice the variation in size of the locally maximal sum-free sets, for example from $\mu(G) = 1$ to $\lambda(G) = 4$ in the quaternion group of order 8. It would be interesting to have bounds for $\mu(G)$, the cardinality of the smallest sum-free set. The only information we have in the following. Let $G = Z_n$, let h denote any element of G^*, and consider the sum-free set $R = \{h, ..., 2h - 1\}$. Now

$$R + R = \{2h, ..., 4h - 2\} \quad \text{or} \quad \{2h, ..., 0, ..., 4h - n - 2\}$$

and

$$R - R = \{0, \pm 1, ..., \pm(h - 1)\} = \{n - h + 1, ..., 0, ..., h - 1\}.$$

If $4h - 2 \geqslant n - h$, then R is certainly locally maximal. So we know that $n + 2 \leqslant 5h$, where $\mu(G) \leqslant h$.

Finally, we raise the question of whether any pattern can be found relating $|G|$ and the elements of $\Lambda(G)$. The values in Table I suggest that an element of $\Lambda(G)$ must have prime divisors in common with $|G|$, $|G| + 1$ or $|G| - 1$. But in Z_{17}, we see that $\{5, 6, 7, 8, 9\}$ is a locally maximal sum-free set, so that $5 \in \Lambda(Z_{17})$, contradicting this idea. Again from Lemma 1, by taking unions of appropriate cosets it follows that

$$\Lambda(G) \supseteq |N| \cdot \Lambda(G/N)$$

for every normal subgroup N of G. Since $5 \in \Lambda((Z_2)^4)$, this shows that $10 \in \Lambda((Z_2)^5)$ and in general $5.2^{n-4} \in \Lambda((Z_2)^n)$.

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