



Konishi anomaly and $N = 1$ effective superpotentials from the matrix models

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Abstract

We discuss the restrictions imposed by the Konishi anomaly on the matrix model approach to the calculation of the effective superpotentials in $N = 1$ SUSY gauge theories with different matter content. It is shown that they correspond to the anomaly deformed Virasoro L_0 constraints.

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1. Recently [1] it was suggested that the effective Veneziano–Yankielowicz type superpotentials for the composite glueball chiral superfield $S = \frac{1}{32\pi^2} \text{Tr } W_\alpha W^\alpha$ in $N = 1$ SUSY theories can be evaluated using the planar limit of the matrix models. Such superpotentials provide the tensions of the BPS domain walls which saturate the central charges in SUSY algebra and therefore enjoy the known properties under the RG flows. This approach based on the earlier analysis in [2] has been considered for the $N = 1$ theories with several matter fields in adjoint [3]. Recently it was argued [4] that planar limit in the matrix model corresponds to the Intriligator–Leigh–Seiberg nonrenormalization conjecture [5] for the generic superpotentials. The matrix model approach has been also analyzed in [6–8]. To clarify the validity of the approach it is interesting to derive the full set of the field theoretic constraints imposed by the symmetries

of the problem and check their consistency with the matrix model realization.

In this Letter we formulate the constraint on the superpotential imposed by the Konishi anomaly. Such Konishi anomaly relation has been proven very useful in the derivation of the gluino condensate since it allows to perform the transition from the weak to strong coupling regime in the controllable way [9]. Recently it was also demonstrated that it provides the effective bridge between $N = 2$ and $N = 1$ SUSY theories with some matter content [10].

We shall argue that Konishi relation is fulfilled in the theory with the single field in adjoint and in the elliptic models with the proper normalizations of the condensates. From the matrix model point of view these relations can be interpreted as a version of L_0 constraint.

2. We would like to discuss the following chiral matter superfield transformation

$$\Phi_f(x_L, \theta) \rightarrow e^{-i\alpha_f} \Phi_f(x_L, \theta). \quad (1)$$

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Contrary to the R-symmetry transformation the transformation (1) does not touch the θ variable. The corresponding Konishi current

$$J_f = \bar{\Phi}_f e^V \Phi_f \tag{2}$$

suffers from the anomaly and the following equation [11] takes place

$$\bar{D}^2 \bar{\Phi}_f e^V \Phi_f = \frac{T(f)}{2\pi^2} \text{Tr} W^2 + 4 \text{Tr} \Phi_f W'(\Phi_f), \tag{3}$$

where $W(\Phi)$ is the tree superpotential for the matter field Φ and the value of the dual Coxeter number in the matter representation $T(f)$ enters the anomaly term. For $SU(N)$ it takes values $T(\text{adj}) = N$ and $T(\text{fund}) = 1/2$. Note that in contrast to the anomaly in R-current there is no higher order correction to (3).

The key point amounting to the importance of the Konishi anomaly relation follows from the vanishing of the l.h.s. upon the averaging over the supersymmetric vacuum state. Hence one arrives at the following relation between condensates in $N = 1$ theory involving fields Φ_i in representations R_i

$$0 = 4T(R_i)\langle S \rangle + \langle \text{Tr} \Phi W'(\Phi_i) \rangle. \tag{4}$$

Note that we have such equation for each matter field. Let us emphasize that composite glueball field is uncharged with respect to the Konishi transformation while it is natural to assign charge k to the coupling t_k in the superpotential $W = \sum_k t_k \text{Tr} \Phi^k$.

We shall be interested in the low energy effective superpotentials hence let us formulate the equation on the W_{eff} following from the Konishi relation in the theory with the single adjoint field. To this aim remind that

$$\frac{dW_{\text{eff}}}{dt_k} = \langle \text{Tr} \Phi^k \rangle \tag{5}$$

and

$$1/2 \frac{dW_{\text{eff}}}{d \text{Log} \Lambda^{2N}} = -\langle S \rangle \tag{6}$$

therefore one arrives at

$$\left(\sum_k k t_k \frac{d}{dt_k} - \frac{d}{d \log \Lambda} \right) \langle W_{\text{eff}}(t_k, \Lambda) \rangle = 0, \tag{7}$$

where the superpotential is taken at some vacuum state.

Let us identify (7) in the matrix model approach. According to [1] the effective superpotential can

be obtained from the tree one via the following procedure. For example, the starting point in the theory with the single adjoint is the integral over the Hermitian matrices

$$Z = \frac{1}{\text{Vol}[\Phi]} \int d\Phi e^{-\frac{1}{g_s} W(\Phi)} \tag{8}$$

with the tree superpotential. At large N limit one can derive the density of the eigenvalues in the saddle point approximation and calculate the partition sum in the planar limit. Given the matrix model partition function the effective superpotential can be derived from the following formula

$$W_{\text{eff}}(S, t_k) = NS \log S / \Lambda^3 - 2\pi i \tau S + N \frac{\partial \mathcal{F}_0(S, t_k)}{\partial S}, \tag{9}$$

where \mathcal{F}_0 is the free energy of the matrix model in the planar limit. Minimization of W_{eff} with respect to the composite field S which is identified with $S = g_s N$ in the matrix model amounts to the vacuum values of the superpotential yielding the holomorphic physical observables—domain wall tensions.

Note that there are essentially two different contributions to the effective superpotential. One yielding the entropy contribution to the free energy amounts from the volume of the group while the second comes from the normalized matrix integral. In what follows we shall argue that the regularized volume has to respect the condition follows from the Konishi anomaly.

In terms of the matrix model the Konishi relation can be reformulated as a anomaly modified Virasoro L_0 constraint. Namely

$$L_0 W_{\text{eff}}(\Lambda, t_k) = 4T(R)\langle S \rangle. \tag{10}$$

The Virasoro operator L_0 term is well known in the matrix models and reflect the invariance of the matrix integral under the change of variables. Let us note that it is essential to consider the complex matrix to identify properly then multiplication by the complex phase corresponding to the Konishi transformations. The second term actually reflects noninvariance of the entropy term in the effective action. It can be also expressed in terms of the derivative of W_{eff} with respect to $\log \Lambda$ for the asymptotically free theory or with respect to the bare coupling τ in the perturbed finite theory.

3. Let us discuss a few explicit examples starting with the theory with the single matter field in adjoint. Consider first the simplest superpotential $W = \mu \text{Tr} \Phi^2$. The vacuum value of the superpotential reads as

$$W_{\min} = \mu \Lambda^2 \tag{11}$$

which evidently obeys Eq. (7). The next example involves the cubic tree superpotential $W = \mu \text{Tr} \Phi^2 + g \text{Tr} \Phi^3$. The corresponding effective superpotential looks as follows [2,7]

$$W_{\text{eff}} = N S \log \frac{S}{\Lambda_{\text{eff}}^3} - 2\pi i \tau S + \frac{N}{2} \sum_k \left(\frac{8g^2}{\mu^3} \right)^k \frac{S^{k+1} \Gamma(3k/2)}{(k+1)! \Gamma(k/2+1)}. \tag{12}$$

Since S is uncharged we have to check the Konishi constraint for both terms separately. The sum evidently obey the constraint as for the entropy term we have to assume that

$$\Lambda_{\text{eff}}^3 = \mu \Lambda^2. \tag{13}$$

Such identification is natural from the matrix models indeed.

Let us turn to the softly broken $N = 4$ theory and assume that masses of adjoint scalars are different. On the field theory side we immediately derive the system of three equation on the condensates

$$-4T(\text{adj}) \langle S \rangle = \langle i \text{Tr} \Phi_1 [\Phi_2, \Phi_3] \rangle + 2M_k \langle \text{Tr} \Phi_k^2 \rangle, \tag{14}$$

where $k = 1, 2, 3$. It is evident that $M_k \langle \text{Tr} \Phi_k^2 \rangle$ does not depend on the flavor. To compare (14) with the matrix model answer recall that in this theory gluino condensate can be derived from the superpotential as follows

$$\langle S \rangle = -\frac{1}{2\pi i} \frac{dW_{\text{eff}}}{d\tau}, \tag{15}$$

where τ is the complexified bare coupling constant.

The effective superpotential on the field theory side which effectively sums the contributions from the fractional instantons has been suggested in [3] and in the k th confining vacuum looks as follows

$$W_{\text{eff}} = M_1 M_2 M_3 \frac{N^3}{24} \times \left[E_2(\tau) - \frac{1}{N} E_2\left(\frac{\tau+k}{N}\right) + A(\tau, N) \right], \tag{16}$$

where $A(N, \tau)$ is some unknown holomorphic function of τ which does not depend on k and $E_2(\tau)$ is the regulated second Eisenstein series

$$E_2 = \frac{3}{\pi^2} \sum_{n,m} \frac{1}{(m+n\tau)^2}. \tag{17}$$

The matrix model answer has been discussed in [3] where it was found that in the vacua corresponding to $\Phi_i = 0$ classical configuration the superpotential differs from the field theory answer by the additive constant. To check the Konishi relation we have to calculate the vacuum expectation value of the operator $\langle \text{Tr} \Phi_1 [\Phi_2, \Phi_3] \rangle$ independently. To this aim let us consider more general model involving additional couplings [12] with the tree superpotential

$$W_{\beta,\lambda} = \text{Tr}(i\lambda\Phi[\Phi_+, \Phi_-]_{\beta} + M\Phi_+\Phi_- + \mu\Phi^2), \tag{18}$$

where the Leigh–Strassler deformation [13] is considered

$$[\Phi_+, \Phi_-]_{\beta} = \Phi_+\Phi_-e^{i\beta/2} - \Phi_-\Phi_+e^{-i\beta/2}. \tag{19}$$

The matrix partition function for this model

$$Z = \int d\Phi_- d\Phi_+ d\Phi e^{-W_{\beta,\lambda}/g_s} \tag{20}$$

can be done in the saddle point approximation using [14] amounting to the effective superpotentials in the massive (p, k) vacuum states [12]

$$W_{\text{eff}} = \frac{pN\mu M^2 \theta_1'(p\beta/2|\tilde{\tau})}{2\lambda^2 \sin \beta \theta_1(p\beta/2|\tilde{\tau})} \tag{21}$$

where

$$\tilde{\tau} = \frac{p(\tau + iN \log \lambda/\pi) + k}{q}, \tag{22}$$

where $k = 0, 1, \dots, q-1$ and the following representation is known

$$\frac{\theta_1'(x|\tau)}{\theta_1(x|\tau)} = \cot x + 4 \sum_{n=1} \frac{q^{2n} \sin 2nx}{1 - q^{2n}}. \tag{23}$$

Now we can calculate relevant condensates just from derivatives of W_{eff} with respect to couplings μ, M, β and λ . This can be most easily seen by taking the simplified case $\beta = 0$. In this case we can calculate condensate

$$\langle \text{Tr} \Phi_1 [\Phi_2, \Phi_3] \rangle = -i \frac{dW_{\text{eff}}}{d\lambda}. \tag{24}$$

Collecting all derivatives of the superpotential in the vacuum state we see that Konishi relation

$$\left(-\frac{Ni}{\pi} \frac{d}{d\tau} + \lambda \frac{d}{d\lambda} + 2\mu \frac{d}{d\mu}\right) W_{\text{eff}}(\lambda, \mu, \tau) = 0 \quad (25)$$

is fulfilled.

One more example involves softly broken $N = 2$ theory with the matter fields in the fundamental. The field theory predicts system of $N_f + 1$ equations for condensates

$$4T(f)\langle S \rangle = \langle \text{Tr } \tilde{Q}_i \Phi Q_i \rangle + m_i \langle \text{Tr } \tilde{Q}_i Q_i \rangle, \quad (26)$$

$$4T(a)\langle S \rangle = \sum_i \langle \text{Tr } \tilde{Q}_i \Phi Q_i \rangle + \sum_k kt_k \langle \text{Tr } \Phi^k \rangle, \quad (27)$$

where $i = 1, \dots, N_f$. The linear combination of the equations does not contain the Yukawa terms

$$4(T(a) - N_f T(f))\langle S \rangle = -\sum_i m_i \langle \text{Tr } \tilde{Q}_i Q_i \rangle + \sum_k kt_k \langle \text{Tr } \Phi^k \rangle \quad (28)$$

and can be translated into condition for the superpotential

$$\left(-\sum_i m_i \frac{d}{dm_i} + \sum_k kt_k \frac{d}{dt_k} - \frac{d}{d \log \Lambda}\right) \times \langle W_{\text{eff}}(t_k, \Lambda, m_i) \rangle = 0. \quad (29)$$

Note that in the perturbed superconformal theory with $N_f = 2N_c$ anomaly contribution in (28) vanishes.

Note that in the theory with fundamentals there are also additional constraints amounting from the Ward identities [15]. In particular in the theory with large adjoint mass when it effectively decouples the following relation takes place

$$\langle S \rangle = Nm \frac{d}{dm} \langle S \rangle \quad (30)$$

which claims that $\langle S \rangle$ is the holomorphic function of m and $\langle S \rangle \propto m^{1/N}$.

4. In this Letter we discussed the constraint imposed by the Konishi anomaly on the vacuum values of the superpotentials calculated within the matrix model approach. We focused on the single cut solution in the matrix models corresponding to the simplest gauge group splitting. It was argued that the constraint can be considered in the matrix model as some version of the Virasoro constraint modified by anomaly term. We

have tested several models with respect to constraint. It appeared that it is satisfied in the $N = 1$ theory with the single adjoint field and in the elliptic models. The condition for the superpotential in the theory with the fundamental matter is presented.

A few additional comments are in order. First let us note that one could expect additional constraints imposed on the domain wall tensions or equivalently on the vacuum values of superpotentials. Actually these should be formulated as a kind of Picard–Fuchs equations for the integrals of the forms over the corresponding cycles. Indeed it is known how the domain wall tensions can be calculated in terms of such integrals of the holomorphic three forms [2]. It is natural to expect the whole multiplet of the domain wall tensions considered as a function on couplings obey some single higher order differential Picard–Fuchs equation which has singularities at Argyres–Douglas points where collisions of vacua happen and vanishing cycles emerge. On the other hand it would be interesting to realize the hypothetical symmetry meaning of the higher Virasoro constraints evident in the matrix model on the field theory side.

There is also some analogy with the low energy description of the nonsupersymmetric QCD in terms of chiral Lagrangian. The matrix integral over the flavor unitary matrices in QCD contains the important information concerning the order parameters of the low energy theory. The reason for the matrix model to work is that it captures the information about the spectrum of the Dirac operator in the complicated instanton ensemble background and such matrix models mimics the integration over the instanton moduli space. For instance, the counterpart of the expression for the gluino condensate in terms of the integrals of the spectral density in the matrix model is the Casher–Banks relation for the chiral condensate in QCD

$$\langle \bar{\Psi} \Psi \rangle = \text{const} \cdot \rho(0), \quad (31)$$

where $\rho(0)$ is the value of the spectral density of the Dirac operator at origin. Since the matrix model relevant for $N = 1$ SYM has the interpretation in terms of the ADHM construction for D-instantons [16] it would be interesting to pursue this analogy further.

One more comment concerns the interpretation in terms of the classical integrable many-body systems. The vacuum states in $N = 1$ SYM theory corresponds to the equilibrium states in the corresponding inte-

grable many body system [17] (see [18] for review). The point we would like to mention is that fermions arising in the matrix model are related to the eigenfunctions of the Lax operator in the integrable system. The physical model which provides the additional intuition concerning meaning of the corresponding spectral curve is the Peierls model of one-dimensional superconductivity. In particular one can map the generation of the scale in the $N = 1$ SYM theory into the gap formation for fermions in Peierls type models [19].

Finally it is interesting to question if the matrix model could provide information concerning another BPS objects existing in $N = 1$ theories. Actually there are two types of BPS objects in $N = 1$ SYM theory saturating central charges in $N = 1$ SUSY algebra. Apart from the domain walls saturating the central charge in $\{Q, \bar{Q}\}$ there are strings or domain wall junctions saturating central terms in $\{\bar{Q}, Q\}$. To discuss junctions one has to care on both central charges [20] and it would be interesting to understand whether matrix model could capture the information about the objects with $1/4$ instead of $1/2$ amount of SUSY.

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