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A classification of cubic bicirculants

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Abstract

The well-known Petersen graph $G(5, 2)$ admits a semi-regular automorphism α acting on the vertex set with two orbits of equal size. This makes it a *bicirculant*. It is shown that trivalent bicirculants fall into four classes. Some basic properties of trivalent bicirculants are explored and the connection to combinatorial and geometric configurations are studied. Some analogues of the polycirculant conjecture are mentioned.

Keywords: Bicirculant; Covering graph; Configuration

1. Introduction

The object of this study are trivalent (cubic) graphs admitting an action of a cyclic group having two equally sized vertex orbits. Such graphs are called *bicirculants*. For symmetry properties of more general bicirculants compare [8]. The automorphism α that generates the corresponding cyclic group is said to be *semi-regular*. Let G be a bicirculant and let α be the corresponding semi-regular automorphism. By $V(G) = V_1 \cup V_2$ we denote the decomposition of the vertex set into the two orbits where

$$V_1 = \{u_0, u_1, \ldots, u_{n-1}\},$$

$$V_2 = \{v_0, v_1, \ldots, v_{n-1}\}$$

and $\alpha(u_i) = u_{i+1}, \alpha(v_i) = v_{i+1},\ i = 0, 1, \ldots, n-1$. Note that all computations are mod n.

There are two *types* of edges with respect to α. An edge e that has both of its endpoints in the same orbit is said to be a *rim edge* (R), otherwise, if the two endpoints belong to distinct orbits, the edge is said to be a *spoke* (S). Now we distinguish four classes of bicirculants (with respect to a given semi-regular element α) and classify the vertices according to the type of incident edges. If all the edges are rim edges, we denote the corresponding vertex class by $3R$, if two of them are rim edges the class is denoted by $2R + S$, if only one edge is a rim edge, the class is denoted by $R + 2S$, while the remaining class, composed of only spoke edges is denoted by $3S$.

Recall that the *Möbius ladder* M_n on $2n$ vertices is a graph on the vertex set $\{v_0, v_1, \ldots, v_{2n-1}\}$ with two types of edges $v_k \sim v_{k+1}, v_k \sim v_{k+n},\ k = 0, 1, \ldots, n-1$.

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Also, a *generalized Petersen graph* $G(n, r)$, introduced by Mark Watkins in [25], is a graph on the vertex set

$$\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

with the adjacencies:

$$u_k \sim u_{k+1}, \quad v_k \sim v_{k+r}, u_k \sim v_k, \quad k = 0, 1, \ldots, n-1.$$

In particular, $G(5, 2)$ is the Petersen graph, $G(10, 2)$ is the dodecahedron graph, and $G(n, 1)$ is also known as the *n-prism* Π_n.

A *circulant graph* is a possibly disconnected Cayley graph of a cyclic group. It is well-known that each circulant $C(n, S)$ can be described by two parameters: an integer n and a *symbol* $S \subset \mathbb{Z}_n$, such that $0 \notin S$, $S = -S$. It follows, that for a cubic circulant we have $S = \{i, -i, n/2\}$ and therefore n has to be even. Each cubic circulant graph either consists of isomorphic copies of r-prisms Π_r or Möbius ladders.

Bicirculants have been studied in the past [19]. They form a subclass of an important class of graphs, called polycirculants; see for instance [17,19,18] and also [2,22].

We are interested only in simple graphs (graphs with no loops or parallel edges). However in describing them we use general graphs and even pre-graphs, allowing parallel edges, loops and even half-edges.

Proposition 1. *In any cubic bicirculant graph the type of vertex is constant*; *i.e. all the vertices have the same type.*

Proof. Since the type of a vertex is preserved by the automorphism α the vertices in the same orbit have the same type. As both vertex orbits have the same size, the number of spokes per vertex is constant. □

The structure of a cubic bicirculant is highly dependent on the vertex type. We shall explain the structure of each of the four classes.

2. Pre-graphs, voltage graphs, covering graphs

In order to define the four classes of graphs in a simple way we need to apply covering graph technique. First we have to slightly generalize the familiar notion of a graph to pre-graphs. Pre-graphs are allowed to have loops and parallel edges and also half-edges (or pending edges).

A pre-graph G is a quadruple $G = (V, S, i, r)$ where V is the set of vertices, S is the set of arcs (or semi-edges, darts, sides, ...), i is a mapping $i : S \to V$, specifying the origin or initial vertex for each half-edge, while r is the reversal involution: $r : S \to S, r^2 = 1$. We may also define $t : S \to V$ as $t(s) := i(r(s))$, specifying the terminal vertex for each arc. An arc s with $r(s) \neq s$ forms an edge $e = \{s, r(s)\}$, which is called proper if $|e| = 2$ and is called a half-edge if $|e| = 1$. Define $\partial(e) = \{i(s), t(s)\}$. A pre-graph without half-edges is called a general graph. Note that G is a graph if and only if the involution has no fixed points. A proper edge e with $|\partial(e)| = 1$ is called a loop and two edges e, e' are parallel if $\partial(e) = \partial(e')$. A graph without loops and parallel edges is called simple. The valence of a vertex v is defined as $val(v) = |\{s \in S | i(s) = v\}|$. A pre-graph of valence 3 is called cubic. There are two cubic pre-graphs on a single vertex and there are seven cubic pre-graphs on two vertices.

Let us consider four pre-graphs on 2 vertices. In all cases we have $V = \{u, v\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The four pre-graphs T, H, I, F can be defined by the tables.

T	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_1	s_3	s_2	s_4	s_6	s_5

H	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_5	s_6	s_1	s_2	s_3

I	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_3	s_2	s_1	s_6	s_5

F	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_5	s_3	s_1	s_2	s_6

Now we shall briefly introduce voltage graphs. Voltage graphs are obtained from pre-graphs by assigning group elements to arcs. More precisely, a *voltage graph* X is a 6-tuple $X=(V,S,i,r,\Gamma,a)$ where (V,S,i,r) is the underlying pre-graph, Γ is a group and a is a mapping $a:S\to\Gamma$ satisfying the following axiom:

For each $s\in S$ we have $a(r(s))=a^{-1}(s)$.

Any voltage graph X defines the so-called *derived graph* or *regular covering graph* Y as follows:

$$V(Y):=V\times\Gamma,\quad S(Y):=S\times\Gamma,$$

$$i(s,\gamma):=(i(s),\gamma)\quad\text{for any } (s,\gamma)\in S(Y),$$

$$r(s,\gamma):=(r(s),a(s)\circ\gamma)\quad\text{for any } (s,\gamma)\in S(Y).$$

If $s\in S$ is a half-edge, i.e. if $r(s)=s$ its voltage $a(s)$ is of order at most two: $a(s)=a^{-1}(s)$. Hence $a=a^{-1}$ or equivalently, $a^2=1$.

In this paper we restrict our attention to the case when the group Γ is cyclic $\Gamma=\mathbb{Z}_n$. In such a case the only voltages that can be assigned to a half-edge are 0 and $n/2$. A voltage 0 on a half-edge implies that the derived graph remains a pre-graph having half-edges. Since we are interested in simple covering graphs it means that the only admissible voltage for a half-edge remains $n/2$; furthermore n must be even.

The above tables for T,H,I,F can be augmented to describe voltage graphs.

T	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_1	s_3	s_2	s_4	s_6	s_5
a	$n/2$	i	$-i$	$n/2$	j	$-j$

H	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_5	s_6	s_1	s_2	s_3
a	i	j	0	$-i$	$-j$	0

I	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_3	s_2	s_1	s_6	s_5
a	0	i	$-i$	0	j	$-j$

F	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_5	s_3	s_1	s_2	s_6
a	0	i	$n/2$	0	i	$n/2$

The corresponding covering graphs are denoted by $T(n,i,j)$, $H(n,i,j)$, $I(n,i,j)$, and $F(n,i)$. As mentioned earlier, we are only interested in simple graphs. Since $T(n,i,j)$, $T(n,j,i)$, $T(n,n-i,n-j)$ and $T(n,n-j,n-i)$ are isomorphic we may assume without loss of generality that in $T(n,i,j)$ we have $0<i\leqslant j\leqslant n-1$, $i+j\leqslant n$. In the similar way we may restrict parameters for $H(n,i,j)$ to $0<i<j\leqslant n-1$, $i+j\leqslant n$, for $I(n,i,j)$ to $0<i\leqslant j\leqslant n-1$, $i\neq n/2$, $j\neq n/2$ and for $F(n,i)$ to n even and $0<i<n/2$. Note that the family of graphs $F(n,i)$ differs from the other three families since it can be described by only two parameters. We could have used three parameters n,i,j, but that would not give rise to any new (non-isomorphic) graphs. This follows from the basic theory of voltage graphs. Note that voltage graphs enable us to develop a combinatorial analog of the well-known theory of covering spaces in algebraic topology. The reader may find more on the theory of voltage graphs and covering graphs in [2,13,23].

We note in passing that there is a convention in drawing voltage graphs. Usually the voltage on only one of the pair of reversing arcs is specified. The choice of the arc is denoted by placing an appropriate arrow. When the voltage is an involution (in our case a voltage from $\{0,n/2\}$) no arrow is drawn. Furthermore, only four out of seven cubic pre-graphs were considered. The other three contain a vertex incident with at least two half-edges. This in turn, implies no simple covering graph exists in any of the three remaining cases.

3. Classification theorem

Class $3R$: Since there are no edges between V_1 and V_2 the graph $G|V_i$ induced on V_i, $i=1,2$, is a union of connected components. $G|V_i$ is therefore a cubic circulant graph. This means that $n=|V_1|=|V_2|$ has to be an even number. The only connected cubic circulants are Möbius ladders and odd prisms. This completely reveals the structure of graphs in $3R$. None of them is connected. Each graph from $3R$ has a number of vertices divisible by 4. Girth is

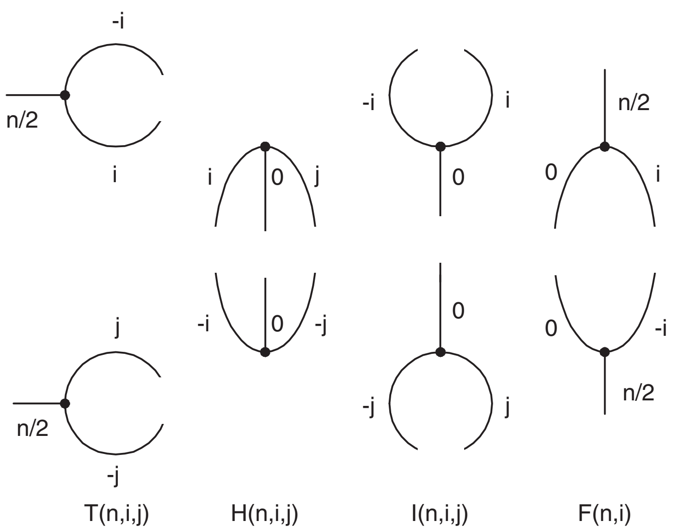

Fig. 1. Each cubic bicirculant on $2n$ vertices is a cyclic cover over one of the four cubic pre-graphs on two vertices. The voltage group in each case is $\mathbb{Z}_n$.

at most 4. The graphs on $2n$ vertices $T(n, i, j)$ where n is an even integer; see Fig. 1. Furthermore we may assume $0 < i \leqslant j \leqslant n-1$, $i + j \leqslant n$. There are three types of edges:

$$u_k \sim u_{k+i}, \quad k = 0, 1, \ldots, n-1,$$

$$v_k \sim v_{k+j}, \quad k = 0, 1, \ldots, n-1,$$

$$u_k \sim u_{k+n/2}, \quad v_k \sim v_{k+n/2}, \quad k = 0, 1, \ldots, n/2-1.$$

In order to describe the structure of $T(n, i, j)$ we need new notation. let $g(n, k) = \gcd(n, k)$, if $n/\gcd(n, k)$ is even and let $g(n, k) = \gcd(n, k)/2$ if $n/\gcd(n, k)$ is odd. Furthermore, let X_t denote a Möbius ladder $M_{t/2}$ if t is even or a prism Π_t, if t is odd. Let $r = n/\gcd(n, i)$ and let $s = n/\gcd(n, j)$. Hence $T(n, i, j)$ consists of $g(n, i)$ copies of X_r and $g(n, j)$ copies of X_s. Clearly each prism or Möbius ladder lies entirely in one of the two orbits. For instance $T(6, 1, 2)$ has one component isomorphic to Π_3 and the other to M_3; see Fig. 2.

Class $3S$: All edges have one endpoint in V_1 and the other one in V_2. The graphs therefore coincide with cubic cyclic Haar graphs introduced in [16]. They can be described by three parameters n, i, j and are denoted by $H(n, i, j)$. We may assume $0 < i < j \leqslant n-1$, $i + j \leqslant n$.

$$u_k \sim v_k, \quad k = 0, 1, \ldots, n-1,$$

$$u_k \sim v_{k+i}, \quad k = 0, 1, \ldots, n-1,$$

$$u_k \sim v_{k+j}, \quad k = 0, 1, \ldots, n-1.$$

Class $2R + S$: Each vertex from V_1 is incident with two rim edges and one spoke. The same is true for each vertex from V_2. The class of graphs coincides with the so-called I-graphs, and can be described by three parameters n, i, j and are denoted by $I(n, i, j)$. These graphs that were introduced in [5] have been studied in [3]. Using the same argument as above we may assume $0 < i \leqslant j \leqslant n-1$, $i + j \leqslant n$, $i \neq n/2$, $j \neq n/2$. Note that $i = n/2$ and $j = n/2$ are forbidden as they would result in parallel edges.

$$u_k \sim v_k, \quad k = 0, 1, \ldots, n-1,$$

$$u_k \sim u_{k+i}, \quad k = 0, 1, \ldots, n-1,$$

$$v_k \sim v_{k+j}, \quad k = 0, 1, \ldots, n-1.$$

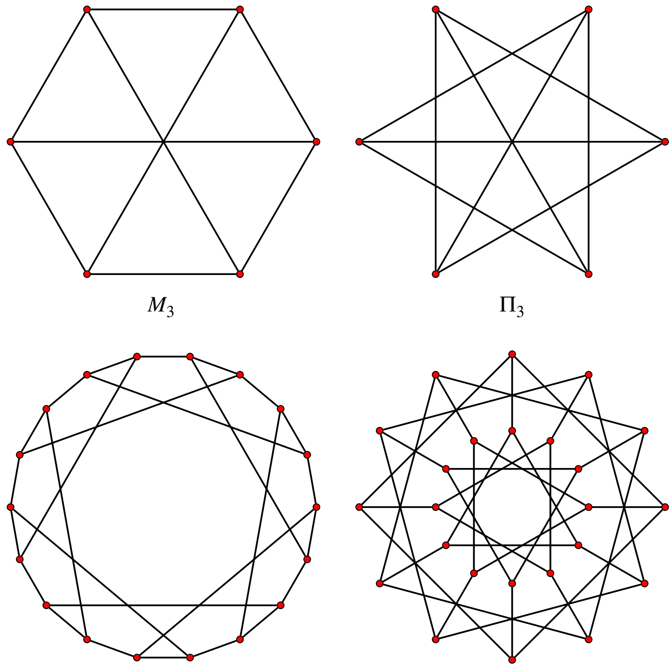

Fig. 2. $T(6, 1, 2)$ has two components M_3 and Π_3 shown above. $H(9, 1, 3)$ has many interesting properties; compare [7,16]. $I(12, 3, 4)$ is the smallest I-graph that is not isomorphic to a generalized Petersen graph; see [3]. The family of graphs $F(n, i)$ has no new members.

Class $R + 2S$: Each vertex from V_1 is incident with a single rim edge and two spokes. The same is true for each vertex from V_2. The class of graphs $F(n, i)$ can be described by two parameters n and i with $0 < i \leqslant n/2$ and n even.

$$u_k \sim v_k, \quad k = 0, 1, \ldots, n - 1,$$

$$u_k \sim v_{k+i}, \quad k = 0, 1, \ldots, n - 1,$$

$$u_k \sim u_{k+n/2}, \quad v_k \sim v_{k+n/2}, \quad k = 0, 1, \ldots, n/2 - 1.$$

It is not hard to see that $F(n, i)$ are composed of disjoint Möbius ladders or prisms. Note that a different choice of the authomorphism α would put this graph in the class $3R$.

Let us summarize the previous discussion in the form of a theorem.

Theorem 2. *Any cubic bicirculant on* $2n$ *vertices is isomorphic to one of the following graphs*: $T(n, i, j)$, $H(n, i, j)$, $I(n, i, j)$ *and* $F(n, j)$.

Proof. By definition, a cubic bicirculant is a cyclic covering graph over a cubic pre-graph on two vertices. Since there are exactly four cubic pre-graphs on two vertices, the bicirculants naturally fall into four classes as shown in Fig. 1. In each case we have, in addition to n, the number of layers of each fiber, one or two parameters i, j that arise by voltage assignment to the edges; compare [23]. □

4. Some basic properties

Fig. 2 depicts some graphs described in Theorem 2. In the remainder of the paper we use the results that were obtained during the study of graphs $H(n, i, j)$; [16] and $I(n, i, j)$; [3]. According to [7] a *zero-symmetric graph* is a vertex-transitive, trivalent graph whose edges are partitioned into three orbits by its automorphism group. The smallest zero-symmetric graph is $H(9, 1, 3)$; see [7].

Here we will explore three basic properties of bicirculants.

Proposition 3. *A cubic bicirculant graph G is connected if and only if*:

1. $G = H(n, i, j)$ *and* $\gcd(n, i, j) = 1$.
2. $G = I(n, i, j)$ *and* $\gcd(n, i, j) = 1$.
3. $G = F(n, i)$ *and* $\gcd(n, i) = 1$ *or* $\gcd(n, i) = 2$ *and* $n/2$ *is odd*.

Proof. Since none of the graphs $T(n, i, j)$ is connected we have to consider only the remaining three classes. Case 1 follows from [16], Proposition 3.1. Case 2 is described in [3]. The remaining case $F(n, i)$ is clearly disconnected if $\gcd(n, i) > 2$ and is connected if $\gcd(n, i) = 1$. If $\gcd(n, i) = 2$ the spokes alone form two cycles that are connected by the rim edges if and only if both cycles are odd. □

Since the semi-regular automorphism α produces only two orbits, there are only two cases concerning vertex-transitivity.

Proposition 4. *A cubic bicirculant graph G is vertex-transitive if and only if*

1. $G = T(n, i, j)$ *and* $\gcd(n, i) = \gcd(n, j)$.
2. $G = H(n, i, j)$.
3. *Let* $k = \gcd(n, i, j)$ *and let* $n_0 = n/k$, $i_0 = n/i$, $j_0 = n/j$. $G = I(n, i, j)$ *and* $\gcd(n_0, i_0) = \gcd(n_0, j_0) = 1$, $I(n_0, i_0, j_0)$ *is isomorphic to the dodecahedron graph* $G(10, 2)$ *or else there exists and integer r such that* $j_0 = r.i_0 (\mathrm{mod}\ n_0)$ *and* $i_0 = \pm r.j_0\ (\mathrm{mod}\ n_0)$.
4. $F(n, i)$.

Proof. The graphs in the first case are vertex-transitive if and only if they have all connected components isomorphic. The second case is covered in [16]. Each cyclic Haar graph is a Cayley graph. The third case follows from [3] where it is shown that the only transitive I-graphs are certain generalized Petersen graphs. The class of vertex-transitive generalized Petersen graphs was established in [9]. The last case has all connected components isomorphic and each of them is vertex-transitive. □

Theorem 5. *Each connected cubic bicirculant graph G is* 3-*connected*.

Proof. According to Proposition 3 there are three possible cases: H, I, and F. By Proposition 4 two out of three cases, namely H and F are always vertex-transitive. For cubic, vertex-transitive graphs the result follows from [26]. The only remaining case is therefore a connected graph $I(n, i, j)$. Case by case analysis shows that removal of any pair of vertices keeps the graph connected. In general there are $\gcd(n, i)$ disjoint cycles of length $n/\gcd(n, i)$ in the outer rim and $\gcd(n, j)$ disjoint cycles of length $n/\gcd(n, j)$ in the inner rim. We may assume that $\gcd(n, i) \leqslant \gcd(n, j)$. If $\gcd(n, i) = 1$, the graph $I(n, i, j)$ is a generalized Petersen graph with at least one rim cycle C of length n. Let x and y be the vertices that are removed and let G be the graph obtained from $I(n, i, j)$ after removing x and y. If neither x nor y belong to C then G is connected. Namely C is connected to the remaining vertices of G by the spokes. If x belongs to C and y does not belong to C, G is connected by a similar argument. Let x' denote the vertex in the inner rim connected to x by a spoke. Clearly Cx is a path in G By spokes it is connected to the vertices of the other rim except x'. If $x' = y$ the result follows. If not, x' has two neighbors in the outer rim and at most one of them is y. Finally, let us assume that both x and y belong to C. The second case $\gcd(n, i) = 2$ has only one subcase that has to be treated separately from the last case $\gcd(n, i) > 2$. Let x and y belong to two different cycles, say X and Y of the outer rim (and there are no other cycles in that rim). The connectedness of G is again established by the spokes and the fact that there exists a cycle in the other rim attached to X and Y by the spokes in G. In the last case, $\gcd(n, i) > 2$, we use the fact that we may remove not only x and y but also the rim cycle (or cycles) that contain x and y and the resulting graph is connected.

An alternative proof would be obtained by adapting to bicirculants the method that is used in [10], Theorem 3.4.2. □

Since the girth of any Möbius ladder or prism graph is at most 4, it follows that girth$(T(n, i, j)) \leqslant 4$ and girth$(F(n, i)) \leqslant 4$. In [16] it was shown that girth$(H(n, i, j)) \leqslant 6$ and in [3] it was shown that girth$(I(n, i, j)) \leqslant 8$. In these references one can find recipes for computing girth of these graphs.

Proposition 6. *A cubic bicriculant graph G is bipartite if and only if*

1. $G = T(n, i, j)$, *n even, and* $n/\gcd(n, i)$ *and* $n/\gcd(n, j)$ *are odd numbers.*
2. $G = H(n, i, j)$.
3. *Let* $k = \gcd(n, i, j)$ *and let* $n_0 = n/k$, $i_0 = n/i$, $j_0 = n/j$. $G = I(n, i, j)$ *and* n_0 *is even and* i_0 *and* j_0 *are odd.*
4. $G = F(n, i)$, *n is even and i is odd.*

Proof. In the first and the last case we only have to distinguish even Möbius ladders from odd prisms. The second case follows from [16] and the third one from [3]. □

Proposition 7. *A cubic bicirculant graph G is a Cayley graph if and only if*

1. $G = T(n, i, j)$ *and* $\gcd(n, i) = \gcd(n, j)$.
2. $G = H(n, i, j)$.
3. *Let* $k = \gcd(n, i, j)$ *and let* $n_0 = n/k$, $i_0 = n/i$, $j_0 = n/j$. $G = I(n, i, j)$ *and* $\gcd(n_0, i_0) = \gcd(n_0, j_0) = 1$ *and there exists and integer r such that* $j_0 = r.i_0 \pmod{n_0}$ *and* $i_0 = r.j_0 \pmod{n_0}$.
4. $G = F(n, i)$.

Proof. In the first case we have to make sure that all connected components are isomorphic. The second case follows from [16] and the third one from [3]. The last case is trivial. □

5. Bicirculants and configurations

Finally, let us touch on a problem of configurations. A combinatorial (v_3) configuration is an incidence structure composed of v points and v lines and there are exactly three points on each line and there are exactly three lines passing through each point. Furthermore, any pair of distinct points determines at most one line. For the basic facts on configurations, the reader may consult for instance [2]. These configurations were counted in [1] for $v \leqslant 18$. There is an important connection between configurations and graphs. A *Levi graph* (see [6]) of a (v_3) configuration is a bipartite cubic graph where black vertices represent points and white vertices represent lines of the configuration. The incidence of point and line is represented as an edge of the corresponding Levi graph. The girth of any Levi graph is at least 6. A cycle of length 6 in the Levi graph corresponds to a triangle in the combinatorial configuration. Similarly, a cycle of length 8 corresponds to a quadrangle in the configuration. The importance of connections between combinatorial aspects of configurations and graphs was repeatedly emphasized by Harald Gropp [11,12]. In [1] triangle-free configurations have been counted as well. This means that cubic bipartite graphs of girth at least 8 would have been counted. Recent paper [4] studies small triangle-free configurations (Fig. 3). The paper [20] applies deep results from graph theory to the study of certain symmetries of configurations.

If we restrict our attention to the configurations whose Levi graphs are bicirculants, then each configuration contains either a triangle or a quadrangle. There is another approach to this connection. In [2] polycyclic configurations have been studied. Levi graphs of polycyclic configurations are polycirculants. A bicirculant is a polycirculant of order 2. The polycirculants arising from polycyclic configurations have an additional property that each orbit is an independent set of vertices. Let us call such polycirculants *independent*. The only independent bicirculants are cyclic Haar graphs. They correspond to cyclic configurations.

It is possible to characterize Levi graphs among the bicirculants.

Theorem 8. *A cubic bicirculant is a Levi graph of a combinatorial configuration if and only if*

1. $G = H(n, i, j)$ *and* $i \neq j, 2i \neq j, 2j \neq i, i + j \neq 0, n \neq 2i, n \neq 2j$.
2. $G = I(n, i, j)$ *is bipartite and* $n \neq 4i, n \neq 4j, i \neq j, i + j \neq 0$.

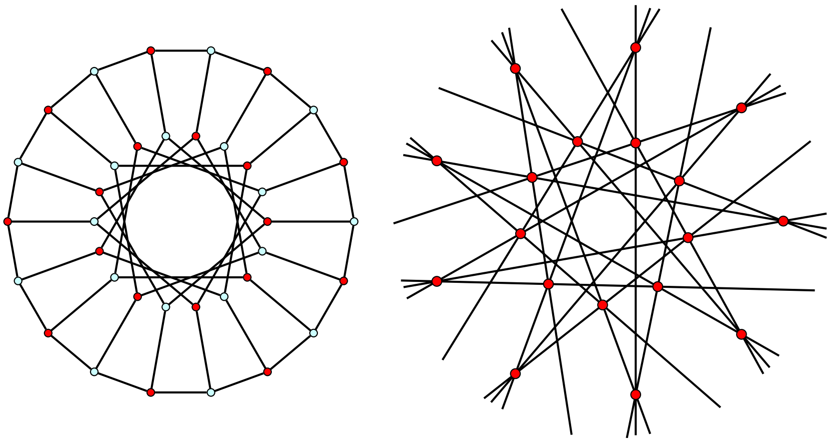

Fig. 3. The graph $G(18, 5) = I(18, 1, 5)$ is the smallest bicirculant graph that is a Levi graph of a triangle-free configuration; compare [4].

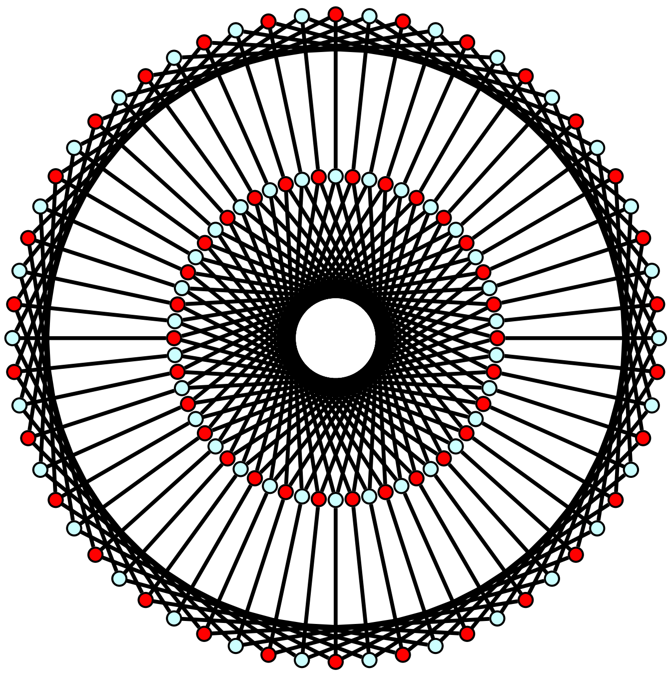

Fig. 4. In Table 4 only one graph from each equivalence class is presented. For instance there are eight graphs isomorphic to the smallest one $I(60, 5, 3)$. One of the isomorphs is $I(60, 25, 9)$ and is easier to visualize. The paper [3] addresses the problem of isomorphism of I-graphs in detail.

Proof. Only Haar graphs and I-graphs apply for girth to be at least 6. The claim follows from [3], Theorems 3 and 7. For bipartite $I(n, i, j)$ the conditions for the girth can be simplified since no odd cycles are possible. □

Theorem 9. *A cubic bicirculant is a Levi graph of a combinatorial triangle free configuration if and only if* $G = I(n, i, j)$ *is bipartite and* $0 \neq 2i, 0 \neq 2j, 0 \neq 4i, 0 \neq 4j, i \neq j, i + j \neq 0, 3i \neq j, 3j \neq i, 3i \neq -j, 3j \neq -i, 0 \neq 6i, 0 \neq 6j, 2i \neq 2j, 2i + 2j \neq 0$ (Figs. 4–6).

Proof. Only certain I-graphs have girth greater than 6. The result follows from [3], Theorems 3 and 7. □

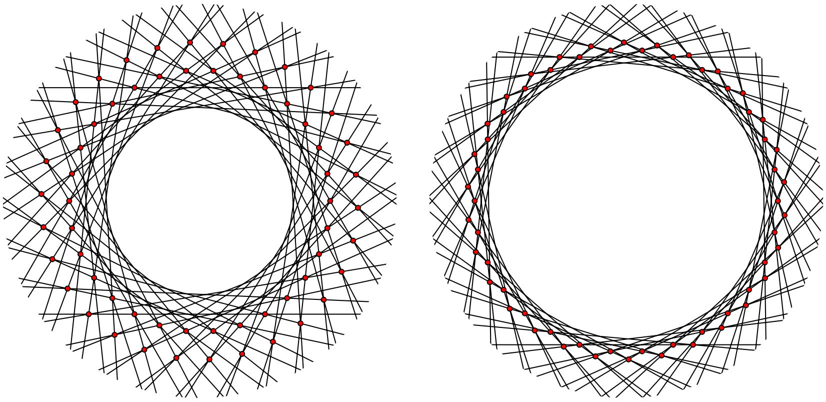

Fig. 5. Using the theory from [3] we can construct two geometrically distinct (60_3) triangle free astral configurations with the same Levi graph $I(60, 5, 3)$.

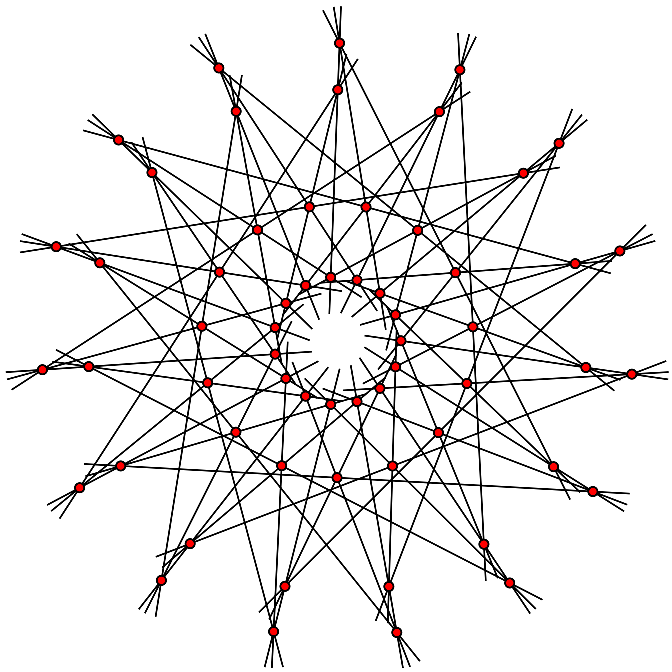

Fig. 6. The graph $I(60, 25, 9)$ from Fig. 4 is a Levi graph for a number of geometric configurations. Four astral ones are shown in Fig. 5. Some features are more clearly visible in this non-astral geometric (60_3) configuration.

Table 1
Parameters for the graphs $H(n, i, j)$ of girth 6. They constitute the class of Levi graphs of cyclic (n_3) configurations $n \leqslant 13$; see [1]

7	3	1	8	3	1	9	3	1	10	3	1	11	3	1
12	3	1	12	8	1	12	9	8	13	3	1	13	10	1

If we want the graph $I(n, i, j)$ to be different from any generalized Petersen graph, we have to add two more conditions: $\gcd(n, i) > 1$, $\gcd(n, j) > 1$.

Using the above theorems we produced our tables (Tables 1–4).

Table 2
Parameters for the bipartite graphs $I(n, i, j)$ of girth at least 6, $n \leqslant 20$

8	3	1	10	3	1	12	5	1	14	3	1	16	3	1
16	5	3	18	3	1	20	3	1	20	7	3			

Table 3
Parameters for the bipartite graphs $I(n, i, j)$ of girth 8, $n \leqslant 30$. For the smallest graph see Fig. 3

18	5	1	22	5	1	24	5	1	24	5	3	24	7	1
26	5	1	26	5	3	28	5	1	30	7	1	30	7	3
30	11	1												

Table 4
Parameters for the bipartite graphs $I(n, i, j)$ that are not generalized Petersen graphs of girth 8, $n \leqslant 198$

60	5	3	70	7	5	84	7	3	90	5	3	90	9	5
110	11	5	120	5	3	120	9	5	126	7	3	126	9	7
130	13	5	132	11	3	140	7	5	150	5	3	150	9	5
154	11	7	156	13	3	168	7	3	168	9	7	170	17	5
180	5	3	180	9	5	180	21	5	182	13	7	190	19	5
198	11	3	198	11	9									

In addition to combinatorial configuration we also have geometric configurations of points and lines in the Euclidean plane. The problem is which combinatorial configurations admit geometric realizations.

According to Theorem 5 all connected configurations of this paper are 3-connected. This means that the corresponding Levi graph is 3-connected. The smallest two cyclic configurations (7_3) and (8_3) are not geometrically realizable.

In [4] the following conjecture is posed:

Conjecture 1. Every 3-connected combinatorial (n_3) configuration with $v > 10$ is geometrically realizable.

If this conjecture is true, then all other configurations of this paper are geometrically realizable. Note that for $n \leqslant 12$ there are only three examples that violate this Conjecture. In addition to the Fano plane (7_3) and the Möbius–Kantor configuration (8_3) there is only one more. Among the ten (10_3) combinatorial configurations, there is one that admits no geometric realizations, however its Levi graph is not a bicirculant.

The smallest I-graph of girth 8 is the generalized Petersen graph $G(18, 5)$ that has an important role in connection with astral configurations, see [4] and Fig. 3.

When Branko Grünbaum introduced the notion of an *astral* configuration in [15] he had in mind geometric configurations. A geometric (v_3) configuration is called *astral* if its group of isometric symmetries has only two point orbits and two line orbits. The only groups of symmetry possible for an astral configuration are cyclic groups or dihedral groups, see Fig. 8.

The smallest non-generalized Petersen I-graph of girth 8 is $I(60, 5, 3)$, see Fig. 4; compare [3]. There are eight I-graphs isomorphic to $I(60, 5, 3)$. However, there is only one self-dual combinatorial configuration resulting from these graphs. As we have seen in Fig. 6 this combinatorial (60_3) configuration admits a geometric realization. Using theory developed in [2] we can construct other symmetric geometric realizations of this combinatorial configuration. For instance, exactly two of them are astral; see Fig. 5.

This rises one more question that would be interesting to address. We have seen that different I-graphs can be isomorphic. On the other hand the same combinatorial configuration may produce distinct geometric configurations. A natural question is: how many distinct geometric forms can a combinatorial configuration possess? Some geometric configurations admit a lot of freedom. Sometimes we may continuously move a point and keep the required incidences until we reach a different configuration. Obviously, there are only finite number of possibilities. It would be interesting to explore this idea and at least give some non-trivial upper bound for a number of distinct geometric realizations of given

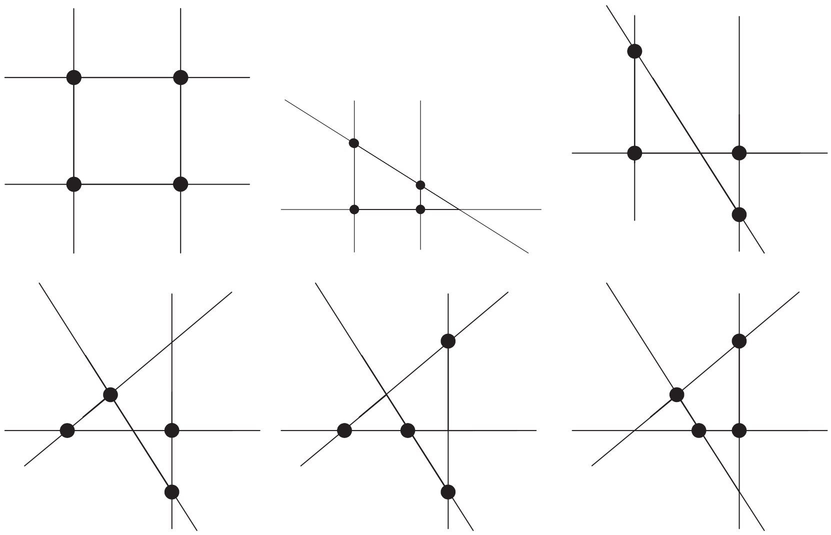

Fig. 7. Six Euclidean realizations of the combinatorial (4_2) configuration. Projectively they are all indistinguishable. Note that the 6 Euclidean realizations define only 4 distinct Euclidean line arrangements.

(v_3) configuration. Obviously, before we can answer this question we have to define when two geometric configurations are "the same". Since any geometric configuration is a line arrangement [14,24] we may have to use the idea of equality of line arrangements. But that is a subject for a separate work. We conclude this topic by exhibiting a projectively rigid configuration (4_2) that admits six Euclidean configurations and defines four Euclidean line arrangements; see Fig. 7.

6. Bicirculants and the polycirculant conjecture

The original polycirculant conjecture as stated in 1981 by Dragan Marušič [17] can be formulated as follows:

Conjecture 2. Every vertex transitive graph admits a non-trivial semi-regular automorphism.

In [21] the authors have proven the conjecture for the case of cubic graphs.

Motivated by configurations one can try to modify the conjecture to a different environment.

Conjecture 3. Every point- and line-transitive combinatorial configuration is a polycyclic configuration.

If we translate this conjecture to the language of graphs and we forget about the girth 6, we can reformulate the conjecture as follows:

Conjecture 4. Every bipartite graph whose automorphism group acts transitively on black vertices and acts transitively on white vertices, admits a semi-regular color-preserving automorphism.

This gives rise to the following generalization of a semi-regular element $\alpha \in \mathrm{Aut}\, G$. Instead of semi-regular element α one can view the corresponding semi-regular cyclic group, generated by α. More generally, a subgroup $\Gamma \leqslant \mathrm{Aut}\, G$ acting on the vertex set of G is called semi-regular, if all vertex orbits are of the same size.

A natural combinatorial counterpart to astral configurations can be defined as follows. A cubic bipartite graph is called *astral* if it admits a semi-regular group of automorphism $\Gamma \leqslant \mathrm{Aut}\, G$ with two black and two white orbits. Using examples of Fig. 8 it is clear that there exist cubic bipartite graphs that are cyclically or dihedrally astral. It would be interesting to see what can be said about other groups than can appear in astral cubic bipartite graphs.

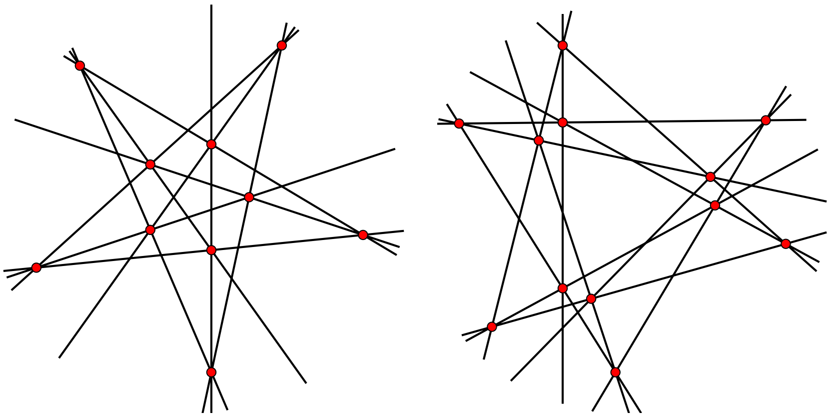

Fig. 8. Two astral configurations. The left one has cyclic while the right one has dihedral group of symmetries. Each one is the smallest in its class.

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References

[1] A. Betten, G. Brinkmann, T. Pisanski, Counting symmetric configurations v_3, Discrete Appl. Math. 99 (2000) 331–338.
[2] M. Boben, T. Pisanski, Polycyclic configurations, European J. Combin. 24 (2003) 431–457.
[3] M. Boben, T. Pisanski, A. Žitnik, *I*-graphs, their Kronecker covers and corresponding configurations, J. Combin. Des. 13 (2005) 406–424.
[4] M. Boben, B. Grünbaum, T. Pisanski, A. Žitnik, Small triangle-free configurations of points and lines, Discrete Comput. Geom. 35 (2006) 405–427.
[5] I.Z. Bouwer, W.W. Chernoff, B. Monson, Z. Star, The Foster Census, Charles Babbage Research Centre, 1988.
[6] H.S.M. Coxeter, Self-dual configurations and regular graphs, Bull. Amer. Math. Soc. 56 (1950) 413–455.
[7] H.S.M. Coxeter, R. Frucht, D. Powers, Zero-symmetric Graphs, Academic Press, New York, 1981.
[8] B. Frelih, A. Malnič, D. Marušič, P. Šparl, Symmetry structures of bicirculants, Discrete Math., this volume.
[9] R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, Proc. Camb. Phil. Soc. 70 (1971) 211–218.
[10] C. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics, vol. 207, New York, Springer, 2001.
[11] H. Gropp, Configurations and graphs, Discrete Math. 111 (1993) 269–276.
[12] H. Gropp, Configurations and graphs—II, Discrete Math. 164 (1997) 155–163.
[13] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley Intersicence, New York, 1987.
[14] B. Grünbaum, Arrangements and Spreads, CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, vol. 10, 1972.
[15] B. Grünbaum, Astral (n_k) configrations, Geombinatorics 3 (1993) 32–37.
[16] M. Hladnik, D. Marušič, T. Pisanski, Cyclic Haar Graphs, Discrete Math. 244 (2002) 137–152.
[17] D. Marušič, On vertex symmetric digraphs, Discrete Math. 36 (1981) 69–81.
[18] D. Marušič, Vertex symmetric graphs and digraphs of order p^k, Ann. Discrete Math. 27 (1985) 115–128.
[19] D. Marušič, Strongly regular bicirculants and tricirculants, Ars Combin. 25 (1988) 11–15.
[20] D. Marušič, T. Pisanski, Weakly flag-transitive configurations and half-arc-transitive graphs, European J. Combin. 20 (1999) 559–570.
[21] D. Marušič, R. Scapellato, Permutation groups, vertex-transitive graphs and semiregular automorphisms, European J. Combin. 19 (1998) 707–712.
[22] T. Pisanski, M. Boben, D. Marušič, A. Orbanić, A. Graovac, The 10-cages and derived configurations, Discrete Math. 275 (2004) 265–276.
[23] T. Pisanski, M. Randić, Bridges between Geometry and Graph Theory, in: C.A. Gorini (Ed.), Geometry at Work, MAA Notes, vol. 53, 2000, pp. 174–194.
[24] I. Streinu, Arrangements of geometric objects, in: K.H. Rosen (Editor-in-chief), Handbook of Discrete and Combinatorial Mathematics, CRC Press, Boca Raton, FL, 2000, pp. 805–824.
[25] M. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory 6 (1969) 152–164.
[26] M. Watkins, Connectivity of transitive graphs, J. Combin. Theory 8 (1970) 23–29.