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Asymptotics for self-normalized random products of sums of i.i.d. random variables ☆

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Abstract

Let $\{X, X_i;\ i \geqslant 1\}$ be a sequence of independent and identically distributed positive random variables, which is in the domain of attraction of the normal law, and t_n be a positive, integer random variable. Denote $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n (X_i - \overline{X})^2$, where $\overline{X}$ denotes the sample mean. Then we show that the self-normalized random product of the partial sums, $(\prod_{k=1}^{t_n} \frac{S_k}{k\mu})^{\frac{\mu}{V_{t_n}}}$, is still asymptotically lognormal under a suitable condition about t_n.

Keywords: Self-normalized; Products; Domain of attraction of the normal law; Lognormal distribution; I.i.d. random variables

1. Introduction and main results

Throughout this paper let $\{X, X_i;\ i \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) positive random variables and define the partial sums $S_n = \sum_{j=1}^n X_j$ and $V_n^2 = \sum_{i=1}^n (X_i - \overline{X})^2$ for $n \geqslant 1$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Arnold and Villasenor [1] considered

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the limiting properties of sums of records and obtained the following version of the central limit theorem for i.i.d. exponential random variables with the mean one,

$$\frac{\sum_{k=1}^{n}\log(S_k) - n\log(n) + n}{\sqrt{2n}} \xrightarrow{d} \mathcal{N} \tag{1.1}$$

as $n \to \infty$, here and in the sequel, $\mathcal{N}$ stands for the standard normal random variable. By Stirling's formula, (1.1) can be equivalently stated as

$$\left(\prod_{k=1}^{n}\frac{S_k}{k}\right)^{\frac{1}{\sqrt{n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}.$$

Rempala and Wesolowski [12] removed the condition that the distribution of X_i is exponential and obtained the following theorem.

Theorem A. *Let $\{X, X_i;\ i \geqslant 1\}$ be a sequence of i.i.d. positive square integrable random variables. Denote $\mu = \mathsf{E}X > 0$, the coefficient of variation $\gamma = \sigma/\mu$, where $\sigma^2 = \mathsf{Var}\,X$. Then*

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.2}$$

Recently, Qi [11] and Lu and Qi [10] obtained the similar results for $\{X, X_i;\ i \geqslant 1\}$, which is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$ and $\alpha = 1$, respectively. We recall the definition of the domain of attraction of a stable law first, then state their results.

A sequence of i.i.d. random variables $\{X, X_i;\ i \geqslant 1\}$ is said to be in the domain of attraction of a stable law $\mathcal{L}_\alpha$ if there exist constants $A_n \geqslant 0$ and $B_n \in R$ $(n \geqslant 1)$ such that

$$\frac{S_n - B_n}{A_n} \xrightarrow{d} \mathcal{L}_\alpha, \tag{1.3}$$

where $\mathcal{L}_\alpha$ is one of the stable distributions with index $\alpha \in (0, 2]$.

Theorem B. *Assume that the positive random variable X has mean μ (> 0) and is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$. The constants A_n $(n \geqslant 1)$ are defined as above so that the limit $\mathcal{L}_\alpha$ in* (1.3) *has a character function as in Theorem* 2.1 *in* [11]. *Then*

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\frac{\mu}{A_n}} \xrightarrow{d} e^{(\Gamma(\alpha+1))^{1/\alpha}\mathcal{L}_\alpha}. \tag{1.4}$$

If $\alpha = 1$ and $\mathcal{L}_1$ has a character function as (iii) *in Theorem* 2.1 *in* [11] *with $\beta = 1$, then* (1.4) *holds for $\alpha = 1$.*

It is well known that the so-called self-normalized limit theorems put a totally new countenance upon classical limit theorems. We refer to Bentkus and Götze [2] for Berry–Esseen inequalities, Giné et al. [7] for the necessary and sufficient condition for the asymptotic normality, Griffin and Kuelbs [8] for the law of the iterated logarithm, Csörgő et al. [4] for studentized increments, Lin [9] for Chung-type law of the iterated logarithm, Csörgő et al. [5] for Donsker's theorem. For a survey on recent developments in this area, we refer to Shao [13] or Csörgő et al. [6]. Consequently, in this paper, we take a sequence of random variables which is in the domain of attraction of the normal law ($\mathcal{L}_\alpha$ is replaced by $\mathcal{N}$ in (1.3)) instead of a sequence of constants used in Theorem A as the power of $(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n})$, i.e., the so-called self-normalized products. Furthermore, since the investigation of the behavior of the sum of a random number of random variables is important in sequential analysis, in random walk problems, etc., we will consider the asymptotic normality of random sums for (1.1) in present paper, i.e., the self-normalized products with random index. We state our results as follows.

Theorem 1.1. *Assume that the positive random variable X has mean μ (>0) and is in the domain of attraction of the normal law and t_n be a positive integer-valued random variable, in addition, if there is a positive constant sequence $\{b_n\}$ tending to infinity as $n \to \infty$ such that $t_n/b_n \xrightarrow{p} \nu$, where ν is a positive random variable and independent of $\{X_i;\ i \geqslant 1\}$. Then we have*

$$\left(\frac{\prod_{k=1}^{t_n} S_k}{t_n!\mu^{t_n}}\right)^{\frac{\mu}{V_{t_n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.5}$$

Obviously, it follows from Theorem 1.1 we have the following consequence.

Corollary 1.1. *Assume that the positive random variable X has mean μ (>0) and is in the domain of attraction of the normal law. Then*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^{n}}\right)^{\frac{\mu}{V_{n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.6}$$

2. Proof

Put $l(x) = \mathsf{E}(X-\mu)^2 I\{|X-\mu| \leqslant x\}$, $b = \inf\{x \geqslant 1\colon\ l(x) > 0\}$, and

$$\eta_j = \inf\left\{s\colon\ s \geqslant b+1,\ \frac{l(s)}{s^2} \leqslant \frac{1}{j}\right\}, \quad j = 1, 2, 3, \ldots.$$

Furthermore, let $B_k^2(j) = \sum_{i=1}^{k} \mathsf{E}(X_i-\mu)^2 I\{|X_i-\mu| \leqslant \eta_j\} = kl(\eta_j)$. It is easy to see that $B_n^2(n) = nl(\eta_n) \sim \eta_n^2$ as $n \to \infty$. We state some lemmas before showing the proof of Theorem 1.1.

Lemmas 2.1 and 2.2 are due to Csörgő et al. [5] and Griffin and Kuelbs [8], respectively.

Lemma 2.1. *If $\mathsf{E}X = 0$, then the following statements are equivalent*:

(a) *X is in the domain of attraction of the normal law*;
(b) $x\mathsf{E}|X|I\{|X| > x\} = o(l(x))$;
(c) $\mathsf{E}|X|^{\alpha} I\{|X| \leqslant x\} = o(x^{\alpha-2} l(x))$ *for* $\alpha > 2$.

Lemma 2.2. *Let $W_1, W_2, \ldots, W_l$ be i.i.d. random variables. Then for any* $0 \leqslant r \leqslant l-1$,

$$\mathsf{P}\left(\sum_{j=1}^{l} I\{W_j < W_1\} \geqslant l-r\right) \leqslant r/l.$$

Lemma 2.3. *Assume that the positive random variable X has mean μ (>0) and is in the domain of attraction of the normal law and t_n be a positive integer-valued random variable, in addition, if there is a positive constant sequence $\{b_n\}$ tending to infinity as $n \to \infty$ such that*

$t_n/b_n \xrightarrow{p} \lambda$, where λ is a positive random variable having a discrete distribution and independent of $\{X_i;\ i \geqslant 1\}$. Then

$$\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{t_n} \left(\frac{S_k}{k\mu} - 1 \right) \xrightarrow{d} \mathcal{N}. \tag{2.1}$$

Proof. Since λ is a positive random variable having a discrete distribution and independent of $\{X_i;\ i \geqslant 1\}$, it is easily seen that we only need to prove (2.1) under the condition $t_n/b_n \xrightarrow{p} c$, where c is a positive constant. Denote $k_n = [cb_n]$, here and in the sequel, $[x]$ stands for the integer part of x, $\sum_i^j = \sum_{[i]}^{[j]}$ and C denotes a constant whose value can differ from line to line. It is obvious that $t_n/k_n \xrightarrow{p} 1$. Put $X_j^*(k_n) = (X_j - \mu) I\{|X_j - \mu| \leqslant \eta_{k_n}\}$ and $S_n^*(k_n) = \sum_{i=1}^n X_j^*(k_n)$. We show $V_{t_n}^2 / V_{k_n}^2 \xrightarrow{p} 1$ first. To see this, we need the following fact,

$$\frac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sum_{j=1}^n (X_j - \mu)^2} \to 1 \quad \text{a.s. } (n \to \infty). \tag{2.2}$$

Indeed,

$$\begin{aligned} \frac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sum_{j=1}^n (X_j - \mu)^2} &= \frac{\sum_{j=1}^n (X_j - \mu)^2 - n(\mu - \overline{X})^2}{\sum_{j=1}^n (X_j - \mu)^2} \\ &= 1 - \frac{(\mu - \overline{X})^2}{(\sum_{j=1}^n (X_j - \mu)^2)/n}. \end{aligned} \tag{2.3}$$

We can choose two constants $M > 0$ and $0 < \delta < 1$ such that $\mathsf{P}(|X - \mu| > M) > \delta > 0$, hence, in view of the strong law of large numbers, we have for large n,

$$\begin{aligned} \frac{(\mu - \overline{X})^2}{(\sum_{j=1}^n (X_j - \mu)^2)/n} &\leqslant \frac{(\mu - \overline{X})^2}{(\sum_{j=1}^n (X_j - \mu)^2 I\{|X_j - \mu| > M\})/n} \\ &\leqslant \frac{(\mu - \overline{X})^2}{M^2(\sum_{j=1}^n I\{|X_j - \mu| > M\})/n} \\ &= \frac{o(1)}{M^2[\mathsf{P}(|X - \mu| > M) + o(1)]} \\ &= o(1) \quad \text{a.s.} \end{aligned} \tag{2.4}$$

which together with (2.3) imply (2.2). For any fixed $0 < \varepsilon < 1$, we choose a ε' small enough such that $[1/\varepsilon'] - 2/\varepsilon^2 > 0$. Moreover, we denote the event

$$A = \left\{ \frac{1}{1+\varepsilon} \frac{\sum_{j=1}^{t_n} (X_j - \mu)^2}{\sum_{j=1}^{k_n} (X_j - \mu)^2} \leqslant \frac{V_{t_n}^2}{V_{k_n}^2} \leqslant \frac{1}{1-\varepsilon} \frac{\sum_{i=1}^{t_n} (X_j - \mu)^2}{\sum_{i=1}^{k_n} (X_j - \mu)^2} \right\}.$$

Then for large n we have

$$\begin{aligned} &\mathsf{P}\big(\big|V_{t_n}^2 - V_{k_n}^2\big| > \varepsilon V_{k_n}^2\big) \\ &\quad \leqslant \mathsf{P}\left(\left| \frac{V_{t_n}^2}{V_{k_n}^2} - 1 \right| > \varepsilon, |t_n - k_n| \leqslant \varepsilon' k_n, A \right) + \mathsf{P}\big(A^c\big) + \mathsf{P}\big(|t_n - k_n| \geqslant \varepsilon' k_n\big) \end{aligned}$$

$$\begin{aligned}
&\leqslant \mathsf{P}\left(\left|\frac{\sum_{i=1}^{t_n}(X_i-\mu)^2}{\sum_{i=1}^{k_n}(X_i-\mu)^2}-1\right|>\varepsilon^2, |t_n-k_n|\leqslant \varepsilon' k_n\right)+\mathsf{P}(A^c)+\mathsf{P}(|t_n-k_n|\geqslant \varepsilon' k_n)\\
&\leqslant \mathsf{P}\left(\sum_{j=k_n+1}^{(1+\varepsilon')k_n}(X_j-\mu)^2>\varepsilon^2\sum_{j=1}^{k_n}(X_j-\mu)^2\right)\\
&\quad+\mathsf{P}\left(\sum_{j=(1-\varepsilon')k_n+1}^{k_n}(X_j-\mu)^2>\varepsilon^2\sum_{j=1}^{k_n}(X_j-\mu)^2\right)+\mathsf{P}(A^c)+\mathsf{P}(|t_n-k_n|\geqslant \varepsilon' k_n)\\
&:= P_1+P_2+P_3+P_4.
\end{aligned} \tag{2.5}$$

Obviously, $P_3 \xrightarrow{p} 0$, $P_4 \xrightarrow{p} 0$ and

$$\begin{aligned}
P_1 &\leqslant \mathsf{P}\left(2\sum_{j=k_n+1}^{(1+\varepsilon')k_n}(X_j-\mu)^2>\varepsilon^2\sum_{j=1}^{(1+\varepsilon')k_n}(X_j-\mu)^2\right)\\
&= \mathsf{P}\left(\sum_{j=1}^{(1+\varepsilon')k_n}(X_j-\mu)^2<\frac{2}{\varepsilon^2}\sum_{j=k_n+1}^{(1+\varepsilon')k_n}(X_j-\mu)^2\right).
\end{aligned}$$

Let

$$W_i=\sum_{j=(1+\varepsilon'-i\varepsilon')k_n+1}^{(1+\varepsilon'-(i-1)\varepsilon')k_n}(X_j-\mu)^2 \quad \text{for } i=1,2,\ldots,[1/\varepsilon']+1.$$

Then by Lemma 2.2 we have

$$\begin{aligned}
&\mathsf{P}\left(\sum_{j=1}^{(1+\varepsilon')k_n}(X_j-\mu)^2<\frac{2}{\varepsilon^2}\sum_{j=k_n+1}^{(1+\varepsilon')k_n}(X_j-\mu)^2\right)\\
&\quad\leqslant \mathsf{P}\left(W_1+W_2+\cdots+W_{[1/\varepsilon']+1}<\frac{2}{\varepsilon^2}W_1\right)\\
&\quad\leqslant \mathsf{P}\left(\sum_{j=1}^{[1/\varepsilon']+1}I\{W_j\geqslant W_1\}<\frac{2}{\varepsilon^2}\right)\\
&\quad= \mathsf{P}\left(\sum_{j=1}^{[1/\varepsilon']+1}I\{W_j< W_1\}\geqslant [1/\varepsilon']+1-\frac{2}{\varepsilon^2}\right)\\
&\quad\leqslant \frac{2/\varepsilon^2}{[1/\varepsilon']+1},
\end{aligned} \tag{2.6}$$

which implies $P_1 \xrightarrow{p} 0$ by letting $\varepsilon' \to 0$. Similarly, we have $P_2 \xrightarrow{p} 0$. $V_{t_n}^2/V_{k_n}^2 \xrightarrow{p} 1$ is proved. On the other hand, we have $V_{k_n}^2/B_{k_n}^2(k_n) \xrightarrow{p} 1$ from (2.2) and the formula (18) in Csörgő et al. [5]. Clearly, it suffices to prove

$$\frac{\mu}{\sqrt{2B_{k_n}^2(k_n)}}\sum_{k=1}^{t_n}\left(\frac{S_k}{k\mu}-1\right) \xrightarrow{d} \mathcal{N} \tag{2.7}$$

for showing (2.1). Note that

$$\begin{aligned}
&\frac{\mu}{\sqrt{2B_{k_n}^2(k_n)}}\sum_{k=1}^{t_n}\left(\frac{S_k}{k\mu}-1\right)\\
&=\frac{1}{\sqrt{2B_{k_n}^2(k_n)}}\sum_{k=1}^{t_n}\frac{1}{k}\Bigg[\big(S_k^*(k_n)-\mathsf{E}S_k^*(k_n)\big)+\sum_{j=1}^{k}(X_j-\mu)I\big\{|X_j-\mu|>\eta_{k_n}\big\}\\
&\quad-\mathsf{E}\sum_{j=1}^{k}(X_j-\mu)I\big\{|X_j-\mu|>\eta_{k_n}\big\}\Bigg].
\end{aligned}\tag{2.8}$$

In view of Lemma 2.1, we have

$$\begin{aligned}
&\mathsf{P}\Bigg(\Bigg|\frac{1}{\sqrt{2B_{k_n}^2(k_n)}}\sum_{k=1}^{t_n}\frac{1}{k}\Bigg[\sum_{j=1}^{k}(X_j-\mu)I\big\{|X_j-\mu|>\eta_{k_n}\big\}\\
&\quad-\mathsf{E}\sum_{j=1}^{k}(X_j-\mu)I\big\{|X_j-\mu|>\eta_{k_n}\big\}\Bigg]\Bigg|>\varepsilon\Bigg)\\
&\leqslant\frac{2(1+\varepsilon')k_n\mathsf{E}|X-\mu|I\{|X-\mu|>\eta_{k_n}\}}{\varepsilon\sqrt{2B_{k_n}^2(k_n)}}+P_4\\
&=\frac{Ck_n}{\varepsilon B_{k_n}(k_n)}\cdot o\left(\frac{l(\eta_{k_n})}{\eta_{k_n}}\right)+P_4\to 0
\end{aligned}\tag{2.9}$$

as $n\to\infty$. From Lemma 1 in [12] and the formula (16) in [5], we have

$$\frac{1}{\sqrt{2B_{k_n}^2(k_n)}}\sum_{k=1}^{k_n}\frac{S_k^*(k_n)-\mathsf{E}S_k^*(k_n)}{k}\xrightarrow{d}\mathcal{N}.\tag{2.10}$$

Now, we only need to prove

$$\frac{1}{\sqrt{2B_{k_n}^2(k_n)}}\left[\sum_{k=1}^{t_n}\frac{S_k^*(k_n)-\mathsf{E}S_k^*(k_n)}{k}-\sum_{k=1}^{k_n}\frac{S_k^*(k_n)-\mathsf{E}S_k^*(k_n)}{k}\right]\xrightarrow{p}0\tag{2.11}$$

by (2.8) to (2.10). Write

$$\begin{aligned}
&\mathsf{P}\left(\frac{1}{\sqrt{2B_{k_n}^2(k_n)}}\left|\sum_{k=1}^{t_n}\frac{S_k^*(k_n)-\mathsf{E}S_k^*(k_n)}{k}-\sum_{k=1}^{k_n}\frac{S_k^*(k_n)-\mathsf{E}S_k^*(k_n)}{k}\right|>\varepsilon\right)\\
&\leqslant\mathsf{P}\left(\max_{k_n<j\leqslant(1+\varepsilon')k_n}\left(\sum_{k=1}^{j}-\sum_{k=1}^{k_n}\right)\frac{|S_k^*(k_n)-\mathsf{E}S_k^*(k_n)|}{k}>\sqrt{2}\varepsilon B_{k_n}(k_n)\right)\\
&\quad+\mathsf{P}\left(\max_{(1-\varepsilon')k_n<j\leqslant k_n}\left(\sum_{k=1}^{k_n}-\sum_{k=1}^{j}\right)\frac{|S_k^*(k_n)-\mathsf{E}S_k^*(k_n)|}{k}>\sqrt{2}\varepsilon B_{k_n}(k_n)\right)+P_4\\
&=:P_5+P_6+P_4
\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}
P_5 &= \mathsf{P}\left(\max_{1<j\leqslant\varepsilon' k_n}\sum_{k=1}^{j}\frac{|S^*_{k_n+k}(k_n)-\mathsf{E}S^*_{k_n+k}(k_n)|}{k_n+k} > \sqrt{2}\varepsilon B_{k_n}(k_n)\right)\\
&\leqslant \mathsf{P}\left(\left|S^*_{k_n}(k_n)-\mathsf{E}S^*_{k_n}(k_n)\right|\sum_{k=1}^{\varepsilon' k_n}\frac{1}{k_n+k} > \sqrt{2}\varepsilon B_{k_n}(k_n)/2\right)\\
&\quad + \mathsf{P}\left(\max_{1<j\leqslant\varepsilon' k_n}\left|\sum_{k=1}^{j}\frac{1}{k_n+k}\sum_{i=1}^{k}\left(X^*_i(k_n)-\mathsf{E}X^*_i(k_n)\right)\right| > \sqrt{2}\varepsilon B_{k_n}(k_n)/2\right)\\
&=: P_{51}+P_{52}.
\end{aligned} \tag{2.13}$$

By the Markov inequality and the formula (16) in [5] again, we have for large n,

$$\begin{aligned}
P_{51} &\leqslant \frac{C}{\varepsilon^2 B^2_{k_n}(k_n)}\left[\log(1+\varepsilon')\right]^2 \mathsf{Var}\, S^*_{k_n}(k_n)\\
&\leqslant \frac{C}{\varepsilon^2 B^2_{k_n}(k_n)}\left[\log(1+\varepsilon')\right]^2 k_n l(\eta_{k_n})\\
&\leqslant \frac{C}{\varepsilon^2}\left[\log(1+\varepsilon')\right]^2 \to 0
\end{aligned} \tag{2.14}$$

as $\varepsilon' \to 0$, and

$$\begin{aligned}
P_{52} &\leqslant \mathsf{P}\left(\max_{1<j\leqslant\varepsilon' k_n}\left|\sum_{i=1}^{j}\log(1+\varepsilon')\left(X^*_i(k_n)-\mathsf{E}X^*_i(k_n)\right)\right| > \sqrt{2}\varepsilon B_{k_n}(k_n)/4\right)\\
&\leqslant \frac{C}{\varepsilon^2 B^2_{k_n}(k_n)}\left[\log(1+\varepsilon')\right]^2 \varepsilon' k_n l(\eta_{k_n})\\
&\leqslant C\left[\log(1+\varepsilon')\right]^2 \varepsilon'/\varepsilon^2 \to 0
\end{aligned} \tag{2.15}$$

as $\varepsilon' \to 0$. $P_6 \to 0$ can be proved by the same way. The proof is completed. □

The next two lemmas are due to Blum et al. [3].

Lemma 2.4. *Let $W_n, X_{m,n}, Y^{(j)}_{m,n}$, and $Z^{(j)}_{m,n}$ be random variables for $m, n = 1, 2, \ldots$, and $j = 1, \ldots, k$. Suppose*

$$W_n = X_{m,n} + \sum_{j=1}^{k} Y^{(j)}_{m,n} Z^{(j)}_{m,n}$$

and

(A) $\lim_{m\to\infty}\limsup_{n\to\infty}\mathsf{P}(|Y^{(j)}_{m,n}| > \varepsilon) = 0$ *for every* $\varepsilon > 0$ *and* $j = 1, \ldots, k$;
(B) $\lim_{M\to\infty}\limsup_{m\to\infty}\limsup_{n\to\infty}\mathsf{P}(|Z^{(j)}_{m,n}| > M) = 0$ *for* $j = 1, \ldots, k$;
(C) *the distributions of* $\{X_{m,n}\}$ *converge to the distribution function* F *for each fixed* m.

Then the distribution functions of $\{W_n\}$ *converge to* F.

Lemma 2.5. *Let $\{k_n\}$ and $\{m_n\}$ be sequences tending to infinity with $k_n < m_n$, and A_n be an event depending only on $\xi_{k_n}, \dots, \xi_{m_n}$, which is a sequence of independent and identically distributed random variables. If A is any event, then*

$$\limsup_{n\to\infty} P(A_n|A) = \limsup_{n\to\infty} P(A_n),$$

where we set $P(A_n|A) = P(A_n)$ if $P(A) = 0$.

By Lemmas 2.4 and 2.5, we will show (2.1) is still valid under the conditions of Theorem 1.1.

Lemma 2.6. *Under the conditions of Theorem* 1.1*, we have*

$$\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{t_n} \left(\frac{S_k}{k\mu} - 1 \right) \xrightarrow{d} \mathcal{N}. \tag{2.16}$$

Proof. Let m, k be positive integers, define $\mu_m = k/2^m$ when $(k-1)/2^m \leqslant \nu < k/2^m$ and

$$\mu_{m,n} = t_n + \left[b_n(\mu_m - \nu) \right].$$

Note that μ_m is discrete for each m, $0 < \mu_m - \nu \leqslant 1/2^m$ and

$$\frac{\mu_{m,n}}{b_n} = \frac{t_n}{b_n} + \frac{[b_n(\mu_m - \nu)]}{b_n} \xrightarrow{p} \mu_m > \nu \tag{2.17}$$

as $n \to \infty$. Put $S'_j = \sum_{k=1}^{j} \frac{S_k - k\mu}{k}$, $X_j^*(\mu_m b_n) = (X_j - \mu) I\{|X_j - \mu| \leqslant \eta_{\mu_m b_n}\}$ and $S_k^*(\mu_m b_n) = \sum_{i=1}^{k} X_j^*(\mu_m b_n)$. Then

$$\begin{aligned} \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{t_n} \left(\frac{S_k}{k\mu} - 1 \right) &= \frac{S'_{\mu_{m,n}}}{\sqrt{2V_{\mu_{m,n}}^2}} + \frac{S'_{t_n} - S'_{\mu_{m,n}}}{\sqrt{2B_{\mu_m b_n}^2(\mu_m b_n)}} \sqrt{\frac{2B_{\mu_m b_n}^2(\mu_m b_n)}{2V_{t_n}^2}} \\ &\quad + \frac{\sqrt{2V_{\mu_{m,n}}^2} - \sqrt{2V_{t_n}^2}}{\sqrt{2V_{t_n}^2}} \frac{S'_{\mu_{m,n}}}{\sqrt{2V_{\mu_{m,n}}^2}} \\ &=: X_{m,n} + Y_{m,n}^{(1)} Z_{m,n}^{(1)} + Y_{m,n}^{(2)} Z_{m,n}^{(2)}. \end{aligned} \tag{2.18}$$

It follows from Lemma 2.3 that for each fixed m, $X_{m,n} = Z_{m,n}^{(2)} \xrightarrow{d} \mathcal{N}$ as $n \to \infty$ by noting (2.17), so it is easy to see that $Z_{m,n}^{(2)}$ satisfies condition (B) of Lemma 2.4. Moreover, $\mathsf{P}(\nu < m/2^m) \to 0$ as $m \to \infty$ and for $m/2^m \leqslant \nu$, we have

$$\lim_{n\to\infty} \left| \frac{\mu_{m,n} - t_n}{t_n} \right| \stackrel{p}{=} \frac{\mu_m}{\nu} - 1 \leqslant (1 + 1/m)^{-1} - 1 \to 0 \tag{2.19}$$

as $m \to \infty$, which together with (2.17) imply that

$$\frac{t_n}{\mu_m b_n} \xrightarrow{p} 1 \tag{2.20}$$

as $n \to \infty$ and $m \to \infty$. By the same way which is used in Lemma 2.3 we have

$$\frac{V^2_{\mu_{m,n}}}{V^2_{\mu_m b_n}} \xrightarrow{p} 1, \qquad \frac{V^2_{t_n}}{V^2_{\mu_m b_n}} \xrightarrow{p} 1 \tag{2.21}$$

and

$$\frac{V^2_{t_n}}{B^2_{\mu_m b_n}(\mu_m b_n)} \xrightarrow{p} 1 \tag{2.22}$$

as $n \to \infty$ and $m \to \infty$. (2.21) and (2.22) imply that $Y^{(2)}_{m,n}$ and $Z^{(1)}_{m,n}$ satisfy the conditions (A) and (B) of Lemma 2.4, respectively. Next, we only need to show $Y^{(1)}_{m,n} \xrightarrow{p} 0$ as $n \to \infty$ and $m \to \infty$ for showing (2.16). For any $\varepsilon > 0$,

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{n\to\infty} \mathsf{P}\left(\left|\frac{S'_{t_n} - S'_{\mu_{m,n}}}{\sqrt{2B^2_{\mu_m b_n}(\mu_m b_n)}}\right| > \varepsilon\right) \\
&\quad \leqslant \limsup_{m\to\infty}\limsup_{n\to\infty}\left\{\mathsf{P}\left(\left|\frac{S'_{t_n} - S'_{\mu_{m,n}}}{\sqrt{2B^2_{\mu_m b_n}(\mu_m b_n)}}\right| > \varepsilon, \left|\frac{t_n}{b_n} - \nu\right| \leqslant 2^{-m}, \left|\frac{\mu_{m,n}}{b_n} - \nu\right| \leqslant 2^{-m+1}\right)\right. \\
&\qquad \left. + \mathsf{P}\left(\left|\frac{t_n}{b_n} - \nu\right| > 2^{-m}\right) + \mathsf{P}\left(\left|\frac{\mu_{m,n}}{b_n} - \mu_m\right| > 2^{-m}\right) + \mathsf{P}\big(|\nu - \mu_m| > 2^{-m}\big)\right\} \\
&\quad \leqslant \limsup_{m\to\infty}\limsup_{n\to\infty} \mathsf{P}\left(\max_{\substack{|\frac{i}{b_n}-\nu|\leqslant 2^{-m}\\ |\frac{j}{b_n}-\nu|\leqslant 2^{-m+1}}}\left|\frac{S'_i - S'_j}{\sqrt{2B^2_{\mu_m b_n}(\mu_m b_n)}}\right| > \varepsilon\right) \\
&\quad \leqslant \limsup_{m\to\infty}\limsup_{n\to\infty} \sum_{k=m}^{m\cdot 2^m} \mathsf{P}\left(\frac{k-1}{2^m} \leqslant \nu < \frac{k}{2^m}, \max_{\substack{|\frac{i}{b_n}-\nu|\leqslant 2^{-m}\\ |\frac{j}{b_n}-\nu|\leqslant 2^{-m+1}}}\left|\frac{S'_i - S'_j}{\sqrt{2B^2_{\mu_m b_n}(\mu_m b_n)}}\right| > \varepsilon\right) \\
&\qquad + \limsup_{m\to\infty}\limsup_{n\to\infty} \mathsf{P}\left(\nu < \frac{m-1}{2^m} \text{ or } m \leqslant \nu\right) \\
&\quad \leqslant \limsup_{m\to\infty}\limsup_{n\to\infty} \sum_{k=m}^{m\cdot 2^m} \mathsf{P}\left(\frac{k-1}{2^m} \leqslant \nu < \frac{k}{2^m}, \max_{\substack{|\frac{i}{b_n}-\nu|\leqslant 2^{-m}\\ |\frac{j}{b_n}-\nu|\leqslant 2^{-m+1}}}\frac{|S'_i - S'_t| + |S'_j - S'_t|}{\sqrt{2B^2_{\mu_m b_n}(\mu_m b_n)}} > \varepsilon\right) \\
&\qquad \big(\text{where } t = \big[b_n(k-3)2^{-m}\big]\big) \\
&\quad \leqslant \limsup_{m\to\infty} \sum_{k=m}^{m\cdot 2^m} \limsup_{n\to\infty} 2\mathsf{P}\left(\max_{b_n(k-3)2^{-m} < r < b_n(k+3)2^{-m}}\left|S'_r - S'_t\right| > \frac{\varepsilon}{2}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\,\right| \\
&\qquad \left.\frac{k-1}{2^m} \leqslant \nu < \frac{k}{2^m}\right) \cdot \mathsf{P}\left(\frac{k-1}{2^m} \leqslant \nu < \frac{k}{2^m}\right).
\end{aligned} \tag{2.23}$$

Denote $s = [6b_n 2^{-m}]$. When $(k-1)/2^m \leqslant \nu < k/2^m$, we have

$$\begin{aligned}
&\mathsf{P}\left(\max_{b_n(k-3)2^{-m} < r < b_n(k+3)2^{-m}}\left|S'_r - S'_t\right| > \frac{\varepsilon}{2}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right) \\
&\quad \leqslant \mathsf{P}\left(\max_{t<r<t+s}\left|S'_r - S'_t\right| > \frac{\varepsilon}{2}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \mathsf{P}\left(\max_{1\leqslant r\leqslant s}\left|\sum_{i=1}^{r}\frac{S_{t+i}-\mathsf{E}S_{t+i}}{t+i}\right| > \frac{\varepsilon}{2}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right) \\
&\leqslant \mathsf{P}\left(|S_t-\mathsf{E}S_t|\sum_{i=1}^{s}\frac{1}{t+i} > \frac{\varepsilon}{4}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right) \\
&\quad + \mathsf{P}\left(\max_{1\leqslant r\leqslant s}\left|\sum_{i=1}^{r}\frac{1}{t+i}\sum_{j=1}^{i}(X_{t+j}-\mu)\right| > \frac{\varepsilon}{4}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right) \\
&=: P_7+P_8. \qquad (2.24)
\end{aligned}$$

Write

$$\begin{aligned}
P_7 &\leqslant \mathsf{P}\left(\left|S_t^*(\mu_m b_n)-\mathsf{E}S_t^*(\mu_m b_n)\right|\sum_{i=1}^{s}\frac{1}{t+i} > \frac{\varepsilon}{8}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\right) \\
&\quad + \mathsf{P}\Bigg(\Bigg|\sum_{j=1}^{t}(X_j-\mu)I\big\{\left|(X_j-\mu)\right| > \eta_{\mu_m b_n}\big\} \\
&\quad - \sum_{j=1}^{t}\mathsf{E}(X_j-\mu)I\big\{\left|(X_j-\mu)\right| > \eta_{\mu_m b_n}\big\}\Bigg|\sum_{i=1}^{s}\frac{1}{t+i} > \frac{\varepsilon}{8}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\Bigg) \\
&=: P_{71}+P_{72}. \qquad (2.25)
\end{aligned}$$

We estimate P_{71} first,

$$\begin{aligned}
P_{71} &\leqslant \frac{32}{\varepsilon^2 B^2_{kb_n/2^m}(\mu_m b_n)}\mathsf{Var}\left(\sum_{j=1}^{t}(X_j-\mu)I\big\{\left|(X_j-\mu)\right|\leqslant \eta_{\mu_m b_n}\big\}\right)\cdot\log(1+s/t) \\
&\leqslant \frac{Ctl(\eta_{\mu_m b_n})}{\varepsilon^2 B^2_{kb_n/2^m}(\mu_m b_n)}\cdot\frac{s}{t} \\
&\leqslant \frac{C}{\varepsilon^2 k}. \qquad (2.26)
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
P_{72} &\leqslant \frac{Ct\mathsf{E}|X-\mu|I\{|X-\mu| > \eta_{\mu_m b_n}\}}{\varepsilon\sqrt{B^2_{kb_n/2^m}(\mu_m b_n)}}\cdot\log(1+s/t) \\
&\leqslant \frac{Cb_n 2^{-m}}{\varepsilon\sqrt{B^2_{kb_n/2^m}(\mu_m b_n)}}\cdot o\left(\frac{l(\eta_{\mu_m b_n})}{\eta_{\mu_m b_n}}\right) \\
&\leqslant \frac{Cb_n 2^{-m}}{\varepsilon\mu_m b_n}\cdot o(1) \\
&= \frac{C}{\varepsilon k}\cdot o(1). \qquad (2.27)
\end{aligned}$$

In view of Lemma 2.5 and (2.25)–(2.27) we have

$$\limsup_{m\to\infty}\sum_{k=m}^{m\cdot 2^m}\limsup_{n\to\infty}\mathsf{P}\left(|S_t-\mathsf{E}S_t|\sum_{i=1}^{s}\frac{1}{t+i}>\frac{\varepsilon}{4}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\;\Big|\;\frac{k-1}{2^m}\leqslant\nu<\frac{k}{2^m}\right)$$
$$\cdot\,\mathsf{P}\left(\frac{k-1}{2^m}\leqslant\nu<\frac{k}{2^m}\right)=0. \tag{2.28}$$

Similarly, we have

$$\limsup_{m\to\infty}\sum_{k=m}^{m\cdot 2^m}\limsup_{n\to\infty}\mathsf{P}\left(\max_{1\leqslant r\leqslant s}\left|\sum_{i=1}^{r}\frac{1}{t+i}\sum_{j=1}^{i}(X_{t+j}-\mu)\right|>\frac{\varepsilon}{4}\sqrt{2B^2_{kb_n/2^m}(\mu_m b_n)}\;\Big|\right.$$
$$\left.\frac{k-1}{2^m}\leqslant\nu<\frac{k}{2^m}\right)\cdot\mathsf{P}\left(\frac{k-1}{2^m}\leqslant\nu<\frac{k}{2^m}\right)=0. \tag{2.29}$$

So $Y^{(1)}_{m,n}\xrightarrow{p}0$ is proved by (2.23), (2.24) (2.28) and (2.29). The proof is completed. □

Proof of Theorem 1.1. Denote $T_k=\frac{S_k}{k\mu}$, $k=1,2,\ldots.$ By the strong law of large numbers, it follows that for any $\delta>0$ there exists a positive integer R such that

$$\mathsf{P}\Big(\sup_{k\geqslant R}|T_k-1|>\delta\Big)<\delta.$$

Consequently, there exist two sequences $\{\delta_m\}\downarrow 0$ $(\delta_1=1/2)$, $\{R^{(1)}_m\}\uparrow\infty$ $(m=1,2,\ldots)$ such that

$$\mathsf{P}\Big(\sup_{k\geqslant R^{(1)}_m}|T_k-1|>\delta_m\Big)<\delta_m.$$

At the same time, the strong law of large numbers also guarantees that there exists a sequence $\{R^{(2)}_m\}\uparrow\infty$ $(m=1,2,\ldots)$ such that

$$\sup_{k\geqslant R^{(2)}_m}|T_k-1|\leqslant 1/m\quad\text{a.s.}$$

Let $R_m=\max\{R^{(1)}_m,R^{(2)}_m\}$, we have

$$\mathsf{P}\Big(\sup_{k\geqslant R_m}|T_k-1|>\delta_m\Big)<\delta_m\quad\text{and}\quad\sup_{k\geqslant R_m}|T_k-1|\leqslant 1/m\quad\text{a.s.} \tag{2.30}$$

For any real x, write

$$\begin{aligned}
&\mathsf{P}\left(\frac{\mu}{\sqrt{2V^2_{t_n}}}\sum_{k=1}^{t_n}\log(T_k)\leqslant x\right)\\
&\quad=\mathsf{P}\left(\frac{\mu}{\sqrt{2V^2_{t_n}}}\sum_{k=1}^{t_n}\log(T_k)\leqslant x,\ \sup_{k\geqslant R_m}|T_k-1|>\delta_m\right)\\
&\qquad+\mathsf{P}\left(\frac{\mu}{\sqrt{2V^2_{t_n}}}\sum_{k=1}^{t_n}\log(T_k)\leqslant x,\ \sup_{k\geqslant R_m}|T_k-1|\leqslant\delta_m\right)\\
&\quad=:P_9+P_{10}.
\end{aligned}\tag{2.31}$$

Obviously, $P_9 < \delta_m$. As to P_{10}, we use the expansion $\log(1+x) = x + \frac{x^2}{(1+\theta x)^2}$, where $\theta \in (0,1)$ depends on $x \in (-1,1)$. Write

$$\begin{aligned}
P_{10} &= \mathsf{P}\Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \log(T_k) + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} \log(1+T_k-1) \leqslant x, \\
&\quad \sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Bigg) \\
&= \mathsf{P}\Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \log(T_k) + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} (T_k-1) \\
&\quad + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} \frac{(T_k-1)^2}{(1+(T_k-1)\theta_k)^2} \leqslant x,\ \sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Bigg) \\
&= \mathsf{P}\Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \log(T_k) + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} (T_k-1) \\
&\quad + \Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} \frac{(T_k-1)^2}{(1+(T_k-1)\theta_k)^2}\Bigg) I\Big\{\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big\} \leqslant x\Bigg) \\
&\quad - \mathsf{P}\Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \log(T_k) + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} (T_k-1) \leqslant x, \\
&\quad \sup_{k \geqslant R_m} |T_k - 1| > \delta_m\Bigg) \\
&=: P_{10,1} - P_{10,2}, \qquad (2.32)
\end{aligned}$$

then we have $P_{10,2} < \delta_m$ and

$$\begin{aligned}
P_{10,1} = \mathsf{P}\Bigg(&\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \big(\log(T_k) - T_k + 1\big) + \frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{t_n} (T_k-1) \\
&+ \Bigg(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m\wedge(t_n-1))+1}^{t_n} \frac{(T_k-1)^2}{(1+(T_k-1)\theta_k)^2}\Bigg) I\Big\{\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big\} \leqslant x\Bigg).
\end{aligned} \qquad (2.33)$$

First, in view of Lemma 2.6, we have $\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{t_n} (T_k - 1) \xrightarrow{d} \mathcal{N}$. Secondly, observe that for any fixed m, it is easy to obtain

$$\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=1}^{R_m \wedge (t_n-1)} \big(\log(T_k) - T_k + 1\big) \xrightarrow{p} 0 \qquad (2.34)$$

as $n \to \infty$ by noting that $V_{t_n}^2 \xrightarrow{p} \infty$. At last, we deal with the third term in the large brackets of (2.33) for two cases.

(1) if $R_m \geqslant t_n - 1$, then as n large enough,

$$\begin{aligned}
&\left(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m \wedge (t_n-1))+1}^{t_n} \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2}\right) I\Big\{\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big\} \\
&\quad \leqslant \frac{\mu}{\sqrt{2V_{t_n}^2}} \frac{(T_{t_n} - 1)^2}{(1 + (T_{t_n} - 1)\theta_{t_n})^2} \\
&\quad \overset{\text{a.s.}}{\leqslant} \frac{C}{\sqrt{2V_{t_n}^2}} \xrightarrow{p} 0
\end{aligned} \tag{2.35}$$

as $n \to \infty$.

(2) if $R_m < t_n - 1$, then $R_m + 1 < t_n$. We suppose $t_n / b_n \xrightarrow{p} c$ (c is a positive constant) first. By denoting $k_n = [cb_n]$, we have $t_n / k_n \xrightarrow{p} 1$ and

$$\begin{aligned}
&\left(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m \wedge (t_n-1))+1}^{t_n} \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2}\right) I\Big(\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big) \\
&\quad = \left(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=R_m+1}^{k_n} \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2}\right) I\Big\{\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big\} \\
&\qquad + \left(\frac{\mu}{\sqrt{2V_{t_n}^2}} \left(\sum_{k=R_m+1}^{t_n} - \sum_{k=R_m+1}^{k_n}\right) \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2}\right) I\Big\{\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big\} \\
&\quad =: P_{11} + P_{12}.
\end{aligned} \tag{2.36}$$

Let n, m be large enough, by noting (2.30), we have

$$\begin{aligned}
\frac{\mu}{\sqrt{2B_{t_n}^2(k_n)}} \sum_{k=R_m+1}^{k_n} \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2} &\leqslant \frac{C}{\sqrt{B_{k_n}^2(k_n)}} \sum_{k=R_m+1}^{k_n} (T_k - 1)^2 \\
&\overset{\text{a.s.}}{\leqslant} \frac{C}{m\sqrt{B_{k_n}^2(k_n)}} \sum_{k=R_m+1}^{k_n} |T_k - 1| \xrightarrow{p} 0
\end{aligned} \tag{2.37}$$

as $m \to \infty$ by the same way used in (2.8)–(2.10), which together with the fact $V_{t_n}^2 / B_{k_n}^2(k_n) \xrightarrow{p} 1$ imply that $P_{11} \xrightarrow{p} 0$. Similarly, $P_{12} \xrightarrow{p} 0$ can be proved by the same way used in (2.11) and by noting (2.30) again. It is easily seen that $P_{11} \xrightarrow{p} 0$ and $P_{12} \xrightarrow{p} 0$ still hold if t_n / b_n convergence in probability to a positive and discrete random variable which is independent of $\{X_i;\ i \geqslant 1\}$. Then, imitate the similar way used in Lemma 2.6, we can conclude that

$$\left(\frac{\mu}{\sqrt{2V_{t_n}^2}} \sum_{k=(R_m \wedge (t_n-1))+1}^{t_n} \frac{(T_k - 1)^2}{(1 + (T_k - 1)\theta_k)^2}\right) I\Big(\sup_{k \geqslant R_m} |T_k - 1| \leqslant \delta_m\Big) \xrightarrow{p} 0 \tag{2.38}$$

when $t_n / b_n \xrightarrow{p} \nu$, the detail is omitted here for sake of avoiding the repetitions. Consequently, $P_{10,1} \to \Phi(x)$, a standard normal distribution function. Write

$$\mathsf{P}\left(\log\left(\prod_{k=1}^{t_n}\frac{S_k}{k\mu}\right)^{\frac{\mu}{V_{t_n}}}\leqslant\sqrt{2}x\right)=\mathsf{P}\left(\frac{\mu}{\sqrt{2V_{t_n}^2}}\sum_{k=1}^{t_n}\log T_k\leqslant x\right)$$
$$=P_9+P_{10,1}-P_{10,2},\tag{2.39}$$

then the proof is completed by noting $|P_9-P_{10,2}|<2\delta_m\to 0$ as $m\to\infty$ and $P_{10,1}\xrightarrow{w}\Phi(x)$ as $n\to\infty$. □

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