Superquadratic functions in several variables

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Abstract

The concept of superquadratic functions in several variables, as a generalization of the same concept in one variable is introduced. Analogous results to results obtained for convex functions in one and several variables are presented. These include refinements of Jensen’s inequality and its counterpart, and of Slater–Pečarić’s inequality.

Keywords: Jensen’s inequality; Convex functions; Superquadratic functions

1. Introduction

The aim of this paper is to introduce the concept of a superquadratic function in several variables. This concept is a generalization of the concept of superquadratic functions in one variable presented in [1,2].

Let \( C \) be a convex subset of a real linear space \( X \) and \( f : C \rightarrow \mathbb{R} \) a convex function. If \( \mathbf{x}_i \in C, \ i = 1, \ldots, n \ (n \geq 2) \) and \( \mathbf{p} = (p_1, \ldots, p_n) \) is any nonnegative \( n \)-tuple such that \( p_1 > 0 \), then the well-known Jensen’s inequality

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holds. When \( f \) is strictly convex the inequality in (1.1) is strict unless \( x_j = c \) for all indices \( j \) with \( p_j > 0 \), where \( c \in C \) is some fixed vector.

An integral analogue of Jensen’s inequality expressed in the language of the abstract Lebesgue integral (see for example [5, p. 61]) states that if \((\Omega, \mathcal{A}, \mu)\) is a measure space with \( 0 < \mu(\Omega) < \infty \) and if \( x \in L^1(\mu) \) is such that \( a < x(t) < b \) for all \( t \in \Omega, -\infty \leq a < b \leq \infty \), then

\[
f \left( \frac{1}{\mu(\Omega)} \int_{\Omega} x \, d\mu \right) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (f \circ x) \, d\mu \tag{1.2}
\]

holds for any convex function \( f : (a, b) \to \mathbb{R} \). In the case when \( f \) is strictly convex we have equality in (1.2) if and only if \( x \) is constant a.e. \((\mu)\) on \( \Omega \).

A function \( f : S \to \mathbb{R} \), where \( S \) is a subset of a normed real linear space \( X \), is said to have a support at a point \( x \in S \) if there exists a linear functional \( a^*(x; \cdot) \in X^* \) such that

\[
f(y) \geq f(x) + a^*(x; y - x) \tag{1.3}
\]

for all \( y \in S \).

For our purposes the following characterization of convexity is useful [4, p. 108]:

The function \( f : C \to \mathbb{R} \) defined on an open convex subset \( C \) of a real normed linear space \( X \) is convex if and only if \( f \) has a support at each point \( x \in C \). In the case when \( f \) is strictly convex the inequality in (1.3) is strict for all \( y \neq x \).

(Note that if \( f : C \to \mathbb{R} \) is convex, but if \( C \) is not open then \( f \) needs not to posses a support at every point \( x \in C \).

We are interested mostly in the case when \( X \) is the usual Euclidean space \( \mathbb{R}^m \) since then a linear functional \( a^*(x; \cdot) \in X^* \) is uniquely determined by a vector \( c(x) \in \mathbb{R}^m \) such that \( a^*(x; z) = \langle c(x), z \rangle \), \( z \in \mathbb{R}^m \), where \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( \mathbb{R}^m \). Hence in this case (1.3) can be rewritten as

\[
f(y) \geq f(x) + \langle c(x), y - x \rangle, \tag{1.4}
\]

and for \( m = 1 \) as

\[
f(s) \geq f(t) + c(t)(s - t), \tag{1.5}
\]

where \( c(t) \in \mathbb{R} \).

Moreover, for every \( x \in C \) and for the vector \( c(x) = (c_1(x), \ldots, c_m(x)) \) in (1.4) we have that \( c_j(x) \) is any value in \( [\partial_j - f(x), \partial_j + f(x)] \) where \( \partial_j - f(x) \) (\( \partial_j + f(x) \)) denotes the left (right) partial derivative of \( f \) over the \( j \)th variable at a point \( x \). In the case \( m = 1 \) \( c(t) \) is any value in \( [f'_-(t), f'_+(t)] \).

In [3] a couple of general inequalities related to Jensen’s inequality (1.1) were proved. Here we state the main result specialized to the case \( X = \mathbb{R}^m \):

**Theorem A.** [3, Theorem 2.1] Let \( f : C \to \mathbb{R} \) be a convex function defined on an open convex subset \( C \) of \( \mathbb{R}^m \). For the given vectors \( x_i \in C \), \( i = 1, \ldots, n \), and the nonnegative real numbers \( p_i, i = 1, \ldots, n \), such that \( P_n = \sum_{i=1}^n p_i > 0 \) let

\[
\bar{x} := \frac{1}{P_n} \sum_{i=1}^n p_i x_i, \quad \bar{y} := \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i).
\]
For any \( x \in C \) assume \( c(x) \in \mathbb{R}^m \) to be a vector such that (1.4) holds for all \( y \in C \). If \( a, b \in C \) are arbitrarily chosen vectors, then we have
\[
f(a) + \langle c(a), x - a \rangle \leq \bar{y} \leq f(b) + \frac{1}{P_n} \sum_{i=1}^n p_i \langle c(x_i), x_i - b \rangle.
\]
(1.6)

Also, when \( f \) is strictly convex we have equality in the left inequality in (1.6) if and only if \( x_j = a \) for all indices \( j \) with \( p_j > 0 \), while equality holds in the right inequality in (1.6) if and only if \( x_j = b \) for all indices \( j \) with \( p_j > 0 \).

We also state another result which is an integral analogue of the above theorem:

**Theorem B.** [3, Theorem 2.3] Let \((\Omega, \mathcal{A}, \mu)\) be a measure space with \( 0 < \mu(\Omega) < \infty \) and let \( f : I \to \mathbb{R} \) be a convex function defined on an open interval \( I \subseteq \mathbb{R} \). For any \( t \in I \) assume \( c(t) \in \mathbb{R} \) to be a value such that (1.5) holds for all \( s \in I \). If \( x : \Omega \to I \) is such that \( f \circ x, c(x) \) and \( x \cdot c(x) \) are all in \( L^1(\mu) \), then for arbitrarily chosen \( a, b \in I \) we have
\[
f(a) + c(a)(\bar{x} - a) \leq \bar{y} \leq f(b) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f \circ x)c(x) \, d\mu,
\]
(1.7)

where
\[
\bar{x} := \frac{1}{\mu(\Omega)} \int_{\Omega} x \, d\mu, \quad \bar{y} := \frac{1}{\mu(\Omega)} \int_{\Omega} (f \circ x) \, d\mu.
\]

Furthermore, when \( f \) is strictly convex, we have equality in the left inequality in (1.7) if and only if \( x = a \) a.e. \((\mu)\) on \( \Omega \), while the equality holds in the right inequality in (1.7) if and only if \( x = b \) a.e. \((\mu)\) on \( \Omega \).

The proof of Theorem A essentially depends only on the fact that (1.4) is valid for all \( x, y \in C \). Therefore, the assertions in Theorem A remain true for any convex subset \( C \) of \( \mathbb{R}^m \) (not necessarily open) provided \( f : C \to \mathbb{R} \) is a convex function for which (1.4) is valid for all \( x, y \in C \). Similarly, the assertions in Theorem B are true for any interval \( I \subseteq \mathbb{R} \) and any convex function \( f : I \to \mathbb{R} \) such that (1.5) is valid for all \( t, s \in I \).

In a recent paper [1] the concept of a superquadratic function in one variable was introduced. A function \( f : [0, \infty) \to \mathbb{R} \) is said to be superquadratic if for any \( t \geq 0 \) there exists a constant \( c(t) \) such that
\[
f(s) \geq f(t) + c(t)(s - t) + f(|s - t|)
\]
(1.8)
holds for all \( s \geq 0 \).

Condition (1.8) seems to be much stronger than the convexity condition. In [1] it was proved that this observation is true for nonnegative \( f \). Here we state some basic properties of superquadratic functions in one variable established in [1]. (Some further results on such functions are given in [2].)

**Lemma A.** [1, Lemma 2.1] Let \( f \) be a superquadratic function with \( c(t) \) as in (1.8). Then:

(i) \( f(0) \leq 0 \).
(ii) If \( f(0) = f'(0) = 0 \), then \( c(t) = f'(t) \) whenever \( f \) is differentiable at \( t > 0 \).
(iii) If \( f \geq 0 \), then \( f \) is convex and \( f(0) = f'(0) = 0 \).
A function \( h : [0, \infty) \to \mathbb{R} \) is superadditive provided \( h(t + s) \geq h(t) + h(s) \) for all \( t, s \geq 0 \). (If the reversed inequality holds, then \( f \) is said to be subadditive.)

**Lemma B.** [1, Lemma 3.1] Suppose \( f : [0, \infty) \to \mathbb{R} \) is continuously differentiable and \( f(0) \leq 0 \). If \( f' \) is superadditive or \( f'(t)/t \) is nondecreasing then \( f \) is superquadratic.

**Lemma C.** [1, Lemma 3.2] Suppose \( f \) is differentiable and \( f(0) = f'(0) = 0 \). If \( f \) is superquadratic then \( f(t)/t^2 \) is nondecreasing on \((0, \infty)\).

In Section 2 we show that all the basic properties of superquadratic functions are extendable to the case of several variables.

In Section 3 we prove the analogues of Theorems A and B for superquadratic functions. In the case when we deal with nonnegative superquadratic functions our results refine the inequalities stated in Theorems A and B as well as some inequalities which are obtained as corollaries of these two theorems.

Finally in Section 4 we give relevant examples.

### 2. Superquadratic functions in \( m \) variables

We consider the Euclidean space \( \mathbb{R}^m \) with the usual inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \) defined by

\[
\langle x, y \rangle = \sum_{i=1}^{m} x_i y_i, \quad \| x \| = \left( \sum_{i=1}^{m} x_i^2 \right)^{1/2}
\]

for \( x = (x_1, x_2, \ldots, x_m) \) and \( y = (y_1, y_2, \ldots, y_m) \). We also use the following notations:

\[
|x| = (|x_1|, |x_2|, \ldots, |x_m|), \quad x \cdot y = (x_1 y_1, x_2 y_2, \ldots, x_m y_m),
\]

\[
\text{sgn} x = (\text{sgn} x_1, \text{sgn} x_2, \ldots, \text{sgn} x_m).
\]

(2.1)

with \( \text{sgn} a = a/|a| \) for \( a \neq 0 \) and \( \text{sgn} 0 = 0 \). Also, \( x \leq y \) (\( x < y \)) means that \( x_j \leq y_j \) (\( x_j < y_j \)) for all \( j \in \{1, 2, \ldots, m\} \). The nullvector \((0, 0, \ldots, 0)\) is denoted by \( 0 \).

We define the subsets \( K_m \) and \( K_m^+ \) in \( \mathbb{R}^m \) as

\[
K_m = [0, \infty)^m = \{ x \in \mathbb{R}^m : 0 \leq x \}, \quad K_m^+ = (0, \infty)^m = \{ x \in \mathbb{R}^m : 0 < x \}.
\]

For a given function \( f : \mathcal{X} \to \mathbb{R} \), \( \mathcal{X} \subset \mathbb{R}^m \), we use the notation \( \nabla f(x) = (\partial_1 f(x), \partial_2 f(x), \ldots, \partial_m f(x)) \) for the gradient of \( f \) at a point \( x \in \mathcal{X} \) (\( \partial_j f(x) \) denotes the partial derivative of \( f \) over the \( j \)th variable at a point \( x \)).

**Definition 1.** A function \( f : K_m \to \mathbb{R} \) is said to be superquadratic if for every \( x \in K_m \) there exists a vector \( c(x) \in \mathbb{R}^m \) such that

\[
f(y) \geq f(x) + \langle c(x), y - x \rangle + f(|y - x|)
\]

holds for all \( y \in K_m \). \( f \) is said to be strictly superquadratic if (2.2) is strict for all \( x \neq y \).

From this definition we immediately obtain some simple facts:
(a) If \( f(x) = ||x||^2 = \sum_{i=1}^{m} x_i^2 \), then \( f \) is differentiable and \( \nabla f(x) = 2x \). Also it is easy to check the identity
\[
f(y) - f(x) = (\nabla f(x), y - x) + f(|y - x|).
\]

Hence \( f \) is superquadratic and we have equality in (2.2) with \( c(x) = \nabla f(x) \).

(b) Any function \( f : K_m \to \mathbb{R} \) satisfying \(-2 \leq f \leq -1\) is superquadratic. Namely, (2.2) is satisfied with \( c(x) = 0 \).

(c) Let \( f \) be superquadratic, \( a \in K_m \) and \( b \geq 0 \). If \( g : K_m \to \mathbb{R} \) is defined by
\[
g(x) = f(x) - \langle a, x \rangle - b, \quad x \in K_m,
\]
then \( g \) is superquadratic. Namely, for any two \( x, y \in K_m \) we have \(|y - x| - (y - x) \in K_m\) and \( \langle a, |y - x| - (y - x) \rangle \geq 0 \). Hence
\[
g(y) - g(x) - g(|y - x|) = f(y) - f(x) - f(|y - x|) + \langle a, |y - x| - (y - x) \rangle + b
\geq \langle c(x), y - x \rangle.
\]

(d) Assume that \( f, g : K_m \to \mathbb{R} \) are two superquadratic functions, and \( \alpha, \beta \geq 0 \). Define \( h : K_m \to \mathbb{R} \) as
\[
h(x) = \alpha f + \beta g.
\]
Then, \( h \) is also a superquadratic function. Namely, the function \( f \) is satisfying (2.2) with \( c(x) = c_f(x) \), and \( g \) is satisfying this inequality with \( c(x) = c_g(x) \) and it is obvious that \( h \) is satisfying (2.2) with \( c(x) = \alpha c_f(x) + \beta c_g(x) \).

**Example 1.** The function \( f : [0, \infty) \to \mathbb{R}, f(t) = t^p \) is superquadratic for \( p \geq 2 \), as it was shown in [1]. Consider the function \( g : K_m \to \mathbb{R} \) defined by
\[
g(x) = \sum_{i=1}^{m} x_i^p = (\|x\|_p)^p, \quad x \in K_m.
\]
From the fact that \( g \) is a sum of superquadratic functions, by statement (d) above, we conclude that \( g \) is a superquadratic function too.

At first glance condition (2.2) appears to be stronger than the convexity condition on \( f \), but if \( f \) takes negative values than it may be considerable weaker. On the other hand, nonnegative superquadratic functions are indeed convex as we shall see from the following:

**Lemma 1.** Let \( f : K_m \to \mathbb{R} \) be superquadratic and \( c(x) = (c_1(x), c_2(x), \ldots, c_m(x)) \) be as in Definition 1. Then:

(i) \( f(0) \leq 0 \) and \( c_j(0) \leq 0 \) for all \( j \in \{1, 2, \ldots, m\} \).

(ii) If \( f(0) = 0 \) and \( \nabla f(0) = 0 \), then \( c_j(x) = \partial_j f(x) \) whenever \( \partial_j f(x) \) exists for some index \( j \in \{1, 2, \ldots, m\} \) at \( x \in K_m \).

(iii) If \( f \geq 0 \), then \( f \) is convex and \( f(0) = 0 \) and \( \nabla f(0) = 0 \).

**Proof.** (i) Setting \( x = y \) in (2.2) we get \( f(x) \geq f(x) + \langle c(x), 0 \rangle + f(0) \) that is \( f(0) \leq 0 \). Also setting \( x = 0 \) in (2.2) we obtain \( f(y) \geq f(0) + \langle c(0), y \rangle + f(y) \) which means that for all \( y \in K_m \)
we have \( \langle c(0), y \rangle \leq -f(0) \) or \( \sum_{i=1}^{m} c_i(0)y_i \leq -f(0) \) which is possible only when \( c_i(0) \leq 0 \) for all \( i \in \{1, 2, \ldots, m\} \).

(ii) Suppose \( f(0) = 0 \) and \( \nabla f(0) = 0 \). Also suppose that \( \partial_i f(x) \) exists for fixed \( x \in K_m \). Let \( c(x) = (c_1(x), c_2(x), \ldots, c_m(x)) \) be as in Definition 1. Define \( g : [0, \infty) \to \mathbb{R} \) by \( g(t) = f(t, x_2, \ldots, x_m) \) and apply (2.2) with fixed points \( x \) and \( y = (t, x_2, \ldots, x_m) \) to obtain

\[
g(t) \geq g(x_1) + c_1(x)(t - x_1) + f(|t - x_1|, 0, \ldots, 0).
\]

For \( t < x_1 \) this is equivalent to

\[
\frac{g(t) - g(x_1)}{t - x_1} + \frac{f(x_1 - t, 0, \ldots, 0)}{x_1 - t} \leq c_1(x), \tag{2.3}
\]

and for \( t > x_1 \) it is equivalent to

\[
c_1(x) \leq \frac{g(t) - g(x_1)}{t - x_1} + \frac{f(t - x_1, 0, \ldots, 0)}{t - x_1}. \tag{2.4}
\]

By the assumption \( f(0) = 0 \) and \( \nabla f(0) = 0 \), so that

\[
\lim_{t \to x_1^-} \frac{f(x_1 - t, 0, \ldots, 0)}{x_1 - t} = \lim_{t \to x_1^+} \frac{f(t - x_1, 0, \ldots, 0)}{t - x_1} = \partial_1 f(0) = 0
\]

holds. Also, \( \partial_1 f(x) \) exists by the assumption so that

\[
\lim_{t \to x_1^-} \frac{g(t) - g(x_1)}{t - x_1} = \partial_1 f(x) = \lim_{t \to x_1^+} \frac{g(t) - g(x_1)}{t - x_1}
\]

holds. Hence, from (2.3) and (2.4) we get \( \partial_1 f(x) \leq c_1(x) \leq \partial_1 f(x) \) which means that \( c_1(x) = \partial_1 f(x) \). It is obvious that the index \( j = 1 \) can be replaced by an arbitrary \( j \in \{1, 2, \ldots, m\} \).

(iii) If \( f \geq 0 \), then we have \( f(0) \geq 0 \) and by (i) \( f(0) \leq 0 \). So \( f(0) = 0 \). Further, take any two \( x, y \in K_m \) and \( t \in [0, 1] \) and define \( \bar{x} = tx + (1-t)y \). By setting \( y = z \) and \( x = \bar{x} \) in (2.2) we get

\[
f(z) \geq f(\bar{x}) + \langle c(\bar{x}), z - \bar{x} \rangle + f(|z - \bar{x}|) \geq f(\bar{x}) + \langle c(\bar{x}), z - \bar{x} \rangle
\]

for any \( z \in K_m \). Define \( S(z) = f(\bar{x}) + \langle c(\bar{x}), z - \bar{x} \rangle \). Obviously \( f(\bar{x}) = S(\bar{x}) \) and \( f(x) \geq S(x) \), \( f(y) \geq S(y) \). It is easy to see that

\[
S(\bar{x}) = S(tx + (1-t)y) \leq tS(x) + (1-t)S(y) \leq tf(x) + (1-t)f(y).
\]

So,

\[
f(\bar{x}) = f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds, which means that \( f \) is convex on \( K_m \). Since \( f \) is convex on \( K_m^+ \) it is well known that \( f \) is differentiable almost everywhere on \( K_m^+ \). On the other side, let \( f \) be differentiable at some \( x \in K_m^+ \). We can fix that point in (2.2). Define the function \( g \) as in (ii). We can again use inequalities (2.3), (2.4) and the fact that \( \partial_1 f(x) \) exists. This implies that

\[
\lim_{t \to x_1^-} \frac{g(t) - g(x_1)}{t - x_1} = \partial_1 f(x), \quad \lim_{t \to x_1^+} \frac{g(t) - g(x_1)}{t - x_1} = \partial_1 f(x).
\]

So, from (2.3) and (2.4) we obtain
So, to prove that (2.2) is satisfied, it is sufficient to see that $0 \leq \sum_{m} c_{1}(x) - \partial_{1} f(x)$.

From this we conclude that

$$\limsup_{s \to 0^{+}} \frac{f(s, 0, \ldots, 0)}{s} \leq 0$$

and since $f \geq 0$ and $f(0) = 0$ this implies that $0 = \lim_{s \to 0^{+}} \frac{f(s, 0, \ldots, 0)}{s} = \partial_{1} f(0)$. Clearly, the index $j = 1$ can be replaced by any $j \in \{1, 2, \ldots, m\}$, so that we have $\partial_{j} f(0) = 0$ for all indices $j$ i.e. $\nabla f(0) = 0$.

**Remark 1.** For $m = 1$ the assertions of the above lemma reduce to Lemma A.

Our next goal is to find a sufficient condition for $f$ to be superquadratic.

**Lemma 2.** Let $f : K_{m} \to \mathbb{R}$ be differentiable and $f(0) \leq 0$. If for all $j \in \{1, 2, \ldots, m\}$

$$\frac{\partial_{j} f(|u - v|)}{|u_{j} - v_{j}|} \leq \frac{\partial_{j} f(u) - \partial_{j} f(v)}{u_{j} - v_{j}}$$

(2.5)

for all $u, v \in K_{m}$ with $u_{j} \neq v_{j}$, then $f$ is superquadratic.

**Proof.** Take $x \in K_{m}$. If $x = 0$ then (2.2) reduces to $f(y) \geq f(0) + \langle c(0), y \rangle + f(y)$ that is $0 \geq f(0) + \langle c(0), y \rangle$ which is satisfied for all $y \in K_{m}$ with $c(0) \leq 0$, since $f(0) \leq 0$.

If $x \neq 0$, then for any fixed $y \in K_{m}$ we define a function $h : [0, 1] \to \mathbb{R}$:

$$h(s) = f(x + s(y - x)) - \langle \nabla f(x), y - x \rangle s - f(s|y - x|), \quad s \in [0, 1].$$

Since $f$ is assumed to be differentiable, the same is true for $h$ and we have

$$h'(s) = \langle \nabla f(x + s(y - x)), y - x \rangle - \langle \nabla f(x), y - x \rangle$$

$$= \langle \text{sgn}(y - x) \cdot \nabla f(s|y - x|), y - x \rangle$$

$$= \langle \nabla f(x + s(y - x)) - \nabla f(x) - \text{sgn}(y - x) \cdot \nabla f(s|y - x|), y - x \rangle.$$

Also,

$$I(x, y) = \int_{0}^{1} h'(s) \, ds = h(1) - h(0)$$

$$= f(y) - f(x) - \langle \nabla f(x), y - x \rangle - f(|y - x|) + f(0)$$

$$\leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle - f(|y - x|).$$

So, to prove that (2.2) is satisfied, it is sufficient to see that $0 \leq I(x, y)$. Now we have $I(x, y) = \sum_{j=1}^{m} I_{j}(x, y)$, where

$$I_{j}(x, y) = \int_{0}^{1} \left[ \partial_{j} f(x + s(y - x)) - \partial_{j} f(x) - \text{sgn}(y_{j} - x_{j}) \partial_{j} f(s|y - x|) \right] (y_{j} - x_{j}) \, ds.$$
If \( y_j = x_j \) then evidently \( I_j(x, y) = 0 \), while for \( y_j \neq x_j \) we must consider two cases: \( y_j > x_j \) and \( y_j < x_j \). If \( y_j > x_j \), then we have

\[
I_j(x, y) = \int_0^1 \left[ \partial_j f(x + s(y - x)) - \partial_j f(x) - \partial_j f(s|y - x|) \right] (y_j - x_j) \, ds.
\]

For \( s \in [0, 1] \) define \( u = x + s(y - x) \), \( v = x \). Then \( |u - v| = s|y - x| \) and \( u_j - v_j = s(y_j - x_j) > 0 \) for \( 0 < s \leq 1 \). Hence, condition (2.5) implies that

\[
\left[ \partial_j f(x + s(y - x)) - \partial_j f(x) - \partial_j f(s|y - x|) \right] (y_j - x_j) \geq 0
\]

for \( 0 < s \leq 1 \). Therefore, \( I_j(x, y) \geq 0 \) in case when \( y_j > x_j \). In case \( y_j < x_j \) we have

\[
I_j(x, y) = \int_0^1 \left[ \partial_j f(x + s(y - x)) - \partial_j f(x) + \partial_j f(s|y - x|) \right] (y_j - x_j) \, ds
\]

for \( 0 < s \leq 1 \). So we have again that \( I_j(x, y) \geq 0 \). Hence \( I(x, y) \geq 0 \) which implies that (2.2) is satisfied and \( f \) is superquadratic. \( \square \)

**Remark 2.** In the case that \( m = 1 \) it is easy to verify that condition (2.5) is equivalent to the requirement that \( f'(t) \) is superadditive, so that the above lemma reduces to Lemma B, and therefore as proved in [1], (2.5) is a sufficient but not necessary condition for superquadracity.

**Lemma 3.** If \( f : K_m \to \mathbb{R} \) is a differentiable superquadratic function with \( f(0) = 0 \) and \( \nabla f(0) = 0 \), then for any \( x \in K_m \)

\[
\langle \nabla f(x), x \rangle - 2f(x) \geq 0.
\]  

**Proof.** By Lemma 1(ii) we know that \( c(x) = \nabla f(x) \), and applying (2.2) with \( y = 0 \) we get

\[
f(0) \geq f(x) + \langle \nabla f(x), 0 - x \rangle + f(|0 - x|).
\]

That is

\[
0 \geq 2f(x) - \langle \nabla f(x), x \rangle.
\]  

**Remark 3.** For \( m = 1 \) the above result is equivalent to Lemma C. It was proved in [1, Example 3.4] that for \( m = 1 \) condition (2.6) is necessary, but not sufficient condition on \( f \) to be superquadratic.

The following is an example of a function which is superquadratic in case \( m = 1 \) (as proved in [1]), but not for \( m = 2 \).
Example 2. Define $f : K_m \to \mathbb{R}$ as

$$f(x) = \frac{1}{2} \| x \|^2 \ln \| x \|^2, \quad x \neq 0; \quad f(0) = 0.$$  

Then we have

$$\partial_j f(x) = x_j (\ln \| x \|^2 + 1), \quad \nabla f(x) = (\ln \| x \|^2 + 1) x.$$  

Now,

$$\langle \nabla f(x), x \rangle = \langle \ln \| x \|^2 + 1 \rangle (x, x) = (\ln \| x \|^2 + 1) \| x \|^2 = 2 f(x) + \| x \|^2,$$

which may be rearranged to yield

$$\langle \nabla f(x), x \rangle - 2 f(x) = \| x \|^2 \geq 0.$$  

But, for $m \geq 2$, $f$ is not superquadratic. Namely, for $x = (t, 0)$ and $y = (0, T)$, for $t, T > 0$. The inequality

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + f(|y - x|)$$

becomes

$$\frac{1}{2} T^2 \ln T^2 \geq t^2 \ln t - t^2 (\ln t^2 + 1) + \frac{1}{2} (t^2 + T^2) \ln (t^2 + T^2),$$

which may be rearranged to yield

$$t^2 (\ln t + 1) \geq \frac{1}{2} (t^2 + T^2) \ln (t^2 + T^2) - \frac{1}{2} T^2 \ln T^2.$$  

The function $\varphi(s) = s^2 \ln s$ is strictly increasing for $s \geq (1/\sqrt{e})$, while $\ln s + 1 < 0$ for $s < (1/e)$. Therefore the above inequality cannot be fulfilled when $t$ and $T$ are such that $t < (1/e)$ and $T \geq (1/\sqrt{e})$.

Example 3. For $p \geq 1$

$$f(x) = \| x \|_p = \left( \sum_{i=1}^{m} x_i^p \right)^{\frac{1}{p}}, \quad x \in K_m,$$

is not superquadratic. Namely, for $x \neq 0$ we have:

$$\partial_j f(x) = \frac{1}{p} \left( \sum_{i=1}^{m} x_i^p \right)^{\frac{1}{p}-1} \cdot p x_j^{p-1} = \| x \|_p^{1-p} x_j^{p-1} = \left( \frac{x_j}{\| x \|_p} \right)^{p-1}.$$  

Hence, using the notation $x^{p-1} = (x_1^{p-1}, x_2^{p-1}, \ldots, x_m^{p-1})$ we get:

$$\nabla f(x) = \frac{x^{p-1}}{\| x \|_p^{p-1}}, \quad \langle \nabla f(x), x \rangle = \| x \|_p < 2 \| x \|_p = 2 f(x).$$  

Therefore, $\langle \nabla f(x), x \rangle - 2 f(x) < 0$ which by Lemma 3 allows us to conclude that $f$ is not superquadratic. On the contrary, the function $g(x) = -f(x) = -\| x \|_p$ is superquadratic. Namely by the triangle inequality for norm we have that

$$-\| y \|_p \geq -\| x \|_p - \| y - x \|_p = -\| x \|_p + \langle 0, y - x \rangle - \| y - x \|_p$$

holds for all $x, y \in K_m$, which is inequality (2.2) for the function $g$ with $c(x) = 0$. So the function $g$ is superquadratic.
By a calculation similar to the one used in Example 3 we can consider \( \|x\|_p = \left( \sum_{i=1}^{m} |x_i|^p \right)^{1/p} \) for any \( p > 0 \) and prove the following simple result which in the case \( m = 1 \) turns out to be equivalent to Lemma C.

**Lemma 4.** Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a differentiable superquadratic function with \( f(0) = 0 \) and \( \nabla f(0) = 0 \). For any \( p > 0 \) denote

\[
g(x) = f(x)/\|x\|_p^2, \quad x \in \mathbb{R}^m \setminus \{0\}.
\]

Then for every \( x \in \mathbb{R}^+ \) we have

\[
\langle \nabla g(x), x \rangle \geq 0.
\]

### 3. Companion inequalities to refined Jensen’s inequality

We start with an analogue of Theorem A for superquadratic functions.

**Theorem 1.** Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a superquadratic function with \( f(0) = 0 \). Let \( x_i \in \mathbb{R}^m, i = 1, \ldots, n \), be given vectors and \( p_i, i = 1, \ldots, n \), nonnegative numbers with \( \sum_{i=1}^{n} p_i > 0 \). Define

\[
\bar{x} := \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \quad \bar{y} := \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i).
\]

For any \( x \in \mathbb{R}^m \) assume \( c(x) \in \mathbb{R}^m \) is a vector such that (2.2) holds for all \( y \in \mathbb{R}^m \). If \( a, b \in \mathbb{R}^m \) are arbitrarily chosen vectors, then

\[
f(a) + \langle c(a), x - a \rangle + \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - a|) \\
\leq \bar{y} \leq f(b) + \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - b|) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - b|).
\]

(3.1)

If \( f \) is strictly superquadratic, then we have equality in the left inequality in (3.1) if and only if \( x_j = a \) for all \( j \) with \( p_j > 0 \), while the equality holds in the right inequality in (3.1) if and only if \( x_j = b \) for all \( j \) with \( p_j > 0 \).

In particular, for \( a = b = \bar{x} \) we get a Jensen type inequality and its counterpart, for superquadratic functions:

\[
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - \bar{x}|) \leq \bar{y} - f(\bar{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \langle c(x_i), x_i - \bar{x} \rangle - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - \bar{x}|).
\]

(3.2)

If \( f \) is strictly superquadratic, then we have equalities throughout (3.2) if and only if there exist some \( z \in \mathbb{R}^m \) such that \( x_j = z \) for all \( j \) with \( p_j > 0 \).

**Proof.** Since \( f \) is superquadratic, inequality (2.2) holds for all \( x, y \in \mathbb{R}^m \). Setting \( x = a \) and \( y = x_i \) we get

\[
f(a) + \langle c(a), x_i - a \rangle + f(|x_i - a|) \leq f(x_i)
\]

(3.3)
for \( i = 1, \ldots, n \). Multiplying by \( p_i \geq 0 \) and summing over \( i = 1, \ldots, n \), we obtain
\[
P_n f(a) + \langle c(a), P_n \boldsymbol{x} - P_n a \rangle + \sum_{i=1}^{n} p_i f(|x_i - a|) \leq P_n \tilde{y}.
\]
Finally, we divide this by \( P_n > 0 \) to obtain the left inequality in (3.1). Inequality (2.2) with \( x = x_i \) and \( y = b \) can be rewritten in the form
\[
f(x_i) \leq f(b) + \langle c(x_i), x_i - b \rangle - f(|b - x_i|).
\]
From this, after multiplying by \( p_i \geq 0 \) and summing over \( i = 1, \ldots, n \), we get
\[
P_n \tilde{y} \leq P_n f(b) + \sum_{i=1}^{n} p_i \langle c(x_i), x_i - b \rangle - \sum_{i=1}^{n} p_i f(|x_i - b|).
\]
We again divide this by \( P_n > 0 \) to obtain the right inequality in (3.1). The assertions on the equality cases are obviously true, since for a strictly superquadratic \( f \) we have strict inequality in (3.3) for \( x_i \neq a \) and strict inequality in (3.4) for \( x_i \neq b \).

**Remark 4.** Let \( x_i = (\xi_{i1}, \ldots, \xi_{im}) \in K_m \) and \( c(x_i) = (c_1(x_i), \ldots, c_m(x_i)) \in \mathbb{R}^m \) \((i = 1, \ldots, n)\) be the vectors as in Theorem 1. If \( \sum_{i=1}^{n} p_i c_j(x_i) \neq 0 \) for all \( j = 1, \ldots, m \), we can define a vector \( \tilde{x} \) by
\[
\tilde{x} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_m), \quad \tilde{\xi}_j := \frac{\sum_{i=1}^{n} p_i c_j(x_i) \xi_{ij}}{\sum_{i=1}^{n} p_i c_j(x_i)}, \quad j = 1, \ldots, m.
\]
If \( \tilde{x} \in K_m \), then choosing \( b = \tilde{x} \) in Theorem 1, and using the fact that
\[
\sum_{i=1}^{n} p_i \langle c(x_i), x_i - \tilde{x} \rangle = 0,
\]
we get the following Slater type inequality for superquadratic functions
\[
\tilde{y} \leq f(\tilde{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - \tilde{x}|).
\]
If \( f \) is strictly superquadratic, then equality in (3.6) holds if and only if \( x_j = \tilde{x} \) for all \( j \) with \( p_j > 0 \).

**Remark 5.** Considering the case when \( f \) is a nonnegative superquadratic function, by Lemma 1(iii) \( f \) is also convex, so the assertions of Theorem A are also true for such function. Since the terms
\[
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - a|), \quad \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - b|)
\]
are both nonnegative in this case, we conclude that inequalities (3.1) refine inequalities (1.6). In the special case when \( X = \mathbb{R}^m \), by [3, Corollary 2.2], we have for convex \( f \)
\[
0 \leq \tilde{y} - f(\tilde{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \langle c(x_i), x_i - \tilde{x} \rangle,
\]
while for a nonnegative and superquadratic (and therefore convex) function \( f \), from (3.2) we get a refinement of (3.7)

\[
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - \bar{x}|) \leq \bar{y} - f(\bar{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i [c(x_i), x_i - \bar{x}] - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(|x_i - \bar{x}|).
\]

Similarly, if \( f \) is nonnegative and superquadratic, then inequality (3.6) refines Pečarić–Slater’s inequality

\[
\bar{y} \leq f(\bar{x}), \tag{3.8}
\]

where \( \bar{x} \) is defined by (3.5), which was also considered in [3].

As expected the above results have an integral analogue:

**Theorem 2.** Let \((\Omega, A, \mu)\) be a measure space with \( 0 < \mu(\Omega) < \infty \) and let \( f : [0, \infty) \to \mathbb{R} \) be a superquadratic function with \( f(0) = 0 \). For any \( t \geq 0 \) let \( c(t) \in \mathbb{R} \) be such that (1.8) holds for all \( s \geq 0 \). If \( x : \Omega \to [0, \infty) \) is such that \( x, f \circ x, c(x), x \cdot c(x) \) are all in \( L^1(\mu) \), then for arbitrarily chosen \( a, b \geq 0 \) we have

\[
f(a) + c(a)(\bar{x} - a) + \frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - a|) \, d\mu \\
\leq \bar{y} \leq f(b) + \frac{1}{\mu(\Omega)} \int_{\Omega} c(x)(x - b) \, d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - b|) \, d\mu, \tag{3.9}
\]

where

\[
\bar{x} := \frac{1}{\mu(\Omega)} \int_{\Omega} x \, d\mu, \quad \bar{y} := \frac{1}{\mu(\Omega)} \int_{\Omega} (f \circ x) \, d\mu.
\]

Furthermore, when \( f \) is strictly superquadratic, we have equality in the left inequality in (3.9) if and only if \( x = a \) a.e. \((\mu)\) on \( \Omega \), while the equality holds in the right inequality in (3.9) if and only if \( x = b \) a.e. \((\mu)\) on \( \Omega \).

In particular, for \( a = b = \bar{x} \) we get a Jensen type inequality and its counterpart, for superquadratic functions:

\[
\frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - \bar{x}|) \, d\mu \\
\leq \bar{y} - f(\bar{x}) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} c(x)(x - \bar{x}) \, d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - \bar{x}|) \, d\mu. \tag{3.10}
\]

If \( f \) is strictly superquadratic, then we have equalities throughout in (3.10) if and only if there exists some real number \( z \geq 0 \) such that \( x = z \) a.e. \((\mu)\) on \( \Omega \).

**Proof.** We argue quite analogously to the proof of Theorem 1 but using (1.8) instead of (2.2) and integration over \( \Omega \) with respect to the measure \( \mu \) instead of summation over \( i = 1, \ldots, n \). The details are left to the reader. \( \square \)
**Remark 6.** Under the assumptions of the theorem above, similar observations as in the discrete case can be made.

(i) When \( \int_{\Omega} c(x) \, d\mu \neq 0 \), we can define \( \bar{x} \) by

\[
\bar{x} := \int_{\Omega} xc(x) \, d\mu / \int_{\Omega} c(x) \, d\mu.
\]

If \( \bar{x} \geq 0 \), then we have \( \int_{\Omega} c(x) (x - \bar{x}) \, d\mu = 0 \) and the second inequality in (3.9) with \( b = \bar{x} \) reduces to a Slater type inequality for superquadratic functions:

\[
\bar{y} \leq f(\bar{x}) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - \bar{x}|) \, d\mu. \tag{3.12}
\]

Furthermore, when \( f \) is strictly superquadratic, then equality in (3.12) holds if and only if \( x = \bar{x} \) a.e. \((\mu)\) on \( \Omega \).

(ii) In the case when \( f \) is a nonnegative superquadratic (and therefore convex) function we can apply both Theorems B and 2. Since the terms

\[
\frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - a|) \, d\mu, \quad \frac{1}{\mu(\Omega)} \int_{\Omega} f(|x - b|) \, d\mu
\]

are both nonnegative in this case, inequalities (3.9) refine inequalities (1.7). Further, in [3] the following integral version of (3.7) for convex \( f \) was considered:

\[
0 \leq \bar{y} - f(\bar{x}) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} c(x)(x - \bar{x}) \, d\mu.
\]

Clearly, for a nonnegative and superquadratic (and therefore convex) function \( f \), the above inequalities are refined by (3.10).

(iii) If \( f \) is nonnegative and superquadratic and \( \bar{x} \) is defined by (3.11), then the inequality (3.12) refines integral Slater’s inequality

\[
\bar{y} \leq f(\bar{x}),
\]

which was also discussed in [3].

**Remark 7.** It makes sense to extend the definition of superquadratic functions in the following way. In place of the set \( K_m \) we can take \( K \subseteq \mathbb{R}^m \) defined as

\[
K = I_1 \times I_2 \times \cdots \times I_m,
\]

where for every \( j = 1, \ldots, n \), the set \( I_j \) is either: \([0, \infty)\), \([0, a_j]\) or \([0, a_j)\). Namely, for \( x = (x_1, x_2, \ldots, x_m) \in K \) and \( y = (y_1, y_2, \ldots, y_m) \in K \), we have

\[
|x - y| = (|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_m - y_m|) \in K.
\]

Indeed, the fact that \( x_j, y_j \in I_j \), implies \( |x_j - y_j| \in I_j \) for all \( j = 1, \ldots, n \).

If we change Definition 1 by replacing \( K_m \) with a set \( K \) described above it is easy to see that all results in this paper remain valid.
4. Some further results and examples

In this section we show how to produce nontrivial superquadratic functions. First we give two simple results for the one variable case.

**Proposition 1.** If \( h : (0, \infty) \to \mathbb{R} \) is a continuous and nondecreasing function with

\[
\lim_{x \to 0^+} x h(x) = 0,
\]

then \( f : [0, \infty) \to \mathbb{R} \) defined by

\[
f(x) = \int_0^x t h(t) \, dt, \quad x > 0; \quad f(0) = 0
\]

is a differentiable superquadratic function with \( f'(0) = 0 \).

**Proof.** We have \( f'(x) = xh(x) \) for \( x > 0 \), so that \( \lim_{x \to 0^+} f'(x) = 0 = f'(0) \). Hence, \( f \) is continuously differentiable on \([0, \infty)\). By our assumption \( f'(x)/x = h(x) \) is nondecreasing, so that \( f \) is a superquadratic function by Lemma B. \( \square \)

**Example 4.** (a) The function \( h(x) = \sinh x \) is nondecreasing and satisfies (4.1). Therefore the function

\[
f(x) = \int_0^x t \sinh t \, dt = x \cosh x - \sinh x, \quad x \geq 0,
\]

is superquadratic. Moreover it is convex by Lemma A(iii).

(b) It is easy to see that the function \( h(x) = (x - 2)/\sqrt{x^2 + 1}, x > 0 \), is nondecreasing on \((0, \infty)\) and \( \lim_{x \to 0^+} x h(x) = 0 \). Therefore the function

\[
f(x) = \int_0^x \frac{t(t - 2)}{\sqrt{t^2 + 1}} \, dt = \frac{1}{2} x \sqrt{x^2 + 1} - 2 \sqrt{x^2 + 1} + \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + 2, \quad x \geq 0,
\]

is superquadratic. This function \( f \) is not convex which is easily verified.

**Proposition 2.** Let \( G : (0, \infty) \to \mathbb{R} \) be a continuously differentiable and nondecreasing function with

\[
\lim_{x \to 0^+} x G(x) = 0, \quad \lim_{x \to 0^+} x^2 G'(x) = 0
\]

and such that the function \( x \mapsto x G'(x) \) is nondecreasing. Then \( f : [0, \infty) \to \mathbb{R} \) defined by

\[
f(x) = x^2 G(x), \quad x > 0; \quad f(0) = 0
\]

is a differentiable and superquadratic function with \( f'(0) = 0 \).

**Proof.** From the first equality in (4.2) it follows that \( f \) is continuous on \([0, \infty)\). We have

\[
f'(x) = 2x G(x) + x^2 G'(x), \quad x > 0,
\]
so that \( (4.2) \) implies that \( \lim_{x \to 0^+} f'(x) = 0 = f'(0) \). Hence, \( f \) is continuously differentiable on \([0, \infty)\). By the assumption \( G \) is nondecreasing and therefore
\[
G(x + y) \geq G(x) \quad \text{and} \quad G(x + y) \geq G(y)
\]
for all \( x, y \geq 0 \). Multiplying the above inequalities by \( x \) and \( y \) respectively and summing it we get
\[
(x + y)G(x + y) \geq xG(x) + yG(y),
\]
which means that the function \( x \mapsto xG(x) \) is superadditive. By a similar argument from the assumption that \( x \mapsto xG'(x) \) is nondecreasing it follows that the function \( x \mapsto x^2G'(x) \) is also superadditive. Hence we conclude that the function \( f'(x) = 2xG(x) + x^2G'(x) \) is superadditive so that by Lemma B \( f \) must be superquadratic.

**Example 5.** The function \( G(x) = \ln x, x \in (0, \infty) \), is differentiable and nondecreasing function which satisfies \( (4.2) \). Also \( xG'(x) = 1 \) is nondecreasing. Therefore
\[
f(x) = x^2G(x) = x^2\ln x, \quad x > 0; \quad f(0) = 0
\]
is a differentiable and superquadratic function with \( f(0) = f'(0) = 0 \). This function \( f \) is not convex which is easily verified.

In Section 2 we gave only few examples of differentiable superquadratic functions \( F \) in \( m \) variables for \( m \geq 2 \). All those examples could be written as
\[
F(x) = \sum_{i=1}^{m} f_i(x_i), \quad (4.3)
\]
where the functions \( f_i : [0, \infty) \to \mathbb{R} \) are superquadratic. The following theorem enables us to produce simple examples of nonpositive differentiable superquadratic functions with \( m \geq 2 \) variables which are not of the form \( (4.3) \).

**Theorem 3.** Let \( q : [0, \infty) \to [0, \infty) \) be a continuous, nondecreasing and subadditive function with \( q(0) = 0 \). Let
\[
f(t) = -\int_{0}^{t} q(s) \, ds, \quad t \geq 0.
\]
Then \( f \) is superquadratic and \( F : K_m \to \mathbb{R} \) defined by
\[
F(x) = f(\|x\|_1) = -\int_{0}^{\|x\|_1} q(s) \, ds, \quad x \in K_m,
\]
where \( \|x\|_1 = \sum_{i=1}^{m} x_i \), is a nonpositive differentiable superquadratic function in \( m \) variables.

**Proof.** We have \( f(0) = 0 \), and by continuity of \( q \) we get \( f'(t) = -q(t) \), so that \( f'(0) = -q(0) = 0 \). Hence, by the assumptions on \( q \), the function \( f' \) is superadditive and therefore by Lemma B \( f \) is superquadratic. For \( F(x) = f(\|x\|_1) \) we have
\[
\partial_j F(x) = f'(\|x\|_1)\partial_j(\|x\|_1) = f'(\|x\|_1).
\]
Therefore, the sufficient condition on $F$ to be superquadratic from Lemma 2 is reduced to
\[ \frac{f'(\|u - v\|)}{|u_j - v_j|} \leq \frac{f'(\|u\|) - f'(\|v\|)}{u_j - v_j}, \quad j = 1, \ldots, m, \tag{4.4} \]
which in the case $u_j > v_j$ is
\[ f'(\|u - v\|) + f'(\|v\|) \leq f'(\|u\|), \tag{4.5} \]
while in the case $u_j < v_j$ is
\[ f'(\|v - u\|) + f'(\|u\|) \leq f'(\|v\|). \tag{4.6} \]
First we consider the case $u_j > v_j$. As we have just seen, the function $f'$ is superadditive and also decreasing. Hence by the norm inequality
\[ \|u\| \leq \|u - v\| + \|v\| \]
we obtain
\[ f'(\|u\|) \geq f'(\|u - v\| + \|v\|) \geq f'(\|u - v\|) + f'(\|v\|) \]
that is inequality (4.5). Similarly, in the case $u_j < v_j$, we start with the inequality
\[ \|v\| \leq \|v - u\| + \|u\| \]
and in the same way we obtain (4.6). Therefore, the function $F$ satisfies condition (4.4) and so, by Lemma 2, we conclude that $F$ is a superquadratic function.

**Remark 8.** The assumptions on $q$ in the above theorem can be replaced by the stronger assumptions that $q : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing and concave function with $q(0) = 0$. Namely it is easily verified that any concave function $q$ which is not negative at 0 is subadditive.

**Example 6.** (a) The function $q(t) = \frac{t}{t+1}$, $t \geq 0$, is continuous, nonnegative, increasing and concave with $q(0) = 0$ and therefore subadditive. By Theorem 3 the function
\[ f(t) = -\int_0^t q(s) \, ds = \ln(1 + t) - t, \quad t \geq 0, \]
is superquadratic and $F : K_m \to \mathbb{R}$ defined by
\[ F(x) = f(\|x\|) = \ln \left( 1 + \sum_{i=1}^m x_i \right) - \sum_{i=1}^m x_i \]
is a nonpositive differentiable superquadratic function.

(b) The function $q(t) = \arctan t$, $t \geq 0$, is continuous, nonnegative, increasing and concave with $q(0) = 0$. Consequently by Theorem 3 the function
\[ f(t) = -\int_0^t q(s) \, ds = \frac{1}{2} \ln(1 + t^2) - t \arctan t, \quad t \geq 0, \]
is superquadratic and $F : K_m \to \mathbb{R}$ defined by

$$F(x) = f(\|x\|_1) = \frac{1}{2} \ln \left[ 1 + \left( \sum_{i=1}^{m} x_i \right)^2 \right] - \left( \sum_{i=1}^{m} x_i \right) \arctan \left( \sum_{i=1}^{m} x_i \right)$$

is a nonpositive differentiable superquadratic function.

Up to this point the examples of superquadratic functions for $m \geq 2$ which are not of the form (4.3) follow by using Theorem 3 and all are nonpositive. To exhibit an example of a nontrivial nonnegative differentiable superquadratic function we can use fact (d) that follows Definition 1. Namely, by choosing superquadratic functions $F$ and $G$ such that $F(x) \geq 0$, $G(x) \leq 0$ and $F(x) \geq -G(x)$, for all $x \in K_m$, we see that for any constant $\alpha \geq 1$

$$H(x) = \alpha F(x) + G(x)$$

is also a nonnegative superquadratic function.

**Example 7.** Consider again the function $q(t) = \frac{t}{t + 1}$. It is increasing and concave with $q(0) = 0$ and $q'(0) = 1$ so that the tangent line of its graph at the origin is $s = t$. Therefore we have $t \geq q(t)$ for all $t \geq 0$ and

$$0 \leq \int_{0}^{t} (s - q(s)) \, ds = \frac{1}{2} t^2 + \ln(t + 1) - t, \quad t \geq 0.$$ 

Now, if we take $x = (x_1, \ldots, x_m) \in K_m$, and set $t = \|x\|_1$, from the above inequality we obtain

$$0 \leq \frac{1}{2} \|x\|_1^2 + \ln(1 + \|x\|_1) - \|x\|_1.$$ 

Moreover we easily see that $\|x\|_1^2 = (\sum_{i=1}^{m} x_i)^2 \leq m \sum_{i=1}^{m} x_i^2$ so that from the above inequality we obtain

$$0 \leq \frac{m}{2} \sum_{i=1}^{m} x_i^2 + \ln \left( 1 + \sum_{i=1}^{m} x_i \right) - \sum_{i=1}^{m} x_i.$$ 

Finally, if $m \geq 2$ and

$$F(x) = \sum_{i=1}^{m} x_i^2, \quad G(x) = \ln \left( 1 + \sum_{i=1}^{m} x_i \right) - \sum_{i=1}^{m} x_i,$$

then $H(x) = \frac{m}{2} F(x) + G(x) \geq 0$ is a nonnegative differentiable superquadratic function which is not of the form (4.3).

**References**


