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Cosmological solutions of time varying speed of light theories

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ABSTRACT

We consider scalar–tensor theory for describing varying speed of light in a spatially flat FRW space–time. We find some exact solutions in the metric and Palatini formalisms. Also we examine the dynamics of this theory by dynamical system method assuming a ΛCDM background and we find some exact solutions by considering the character of critical points of the theory in both formalisms. We show that for any attractor the form of non-minimal coupling coefficient is quadratic in terms of the scalar field Ψ. Also we show that only attractors of the de Sitter era satisfy the horizon criteria.

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1. Introduction

There are some ideas that suggest the constants of nature, such as gravitational constant and the speed of light, should be time–space dependent [1–3]. Although the number and the role of fundamental constants are still debated [4], there are different mechanisms leading to varying constants. Brans–Dicke scalar–tensor theory [2,5] is the first formulation of a dynamical gravitational constant as while as a theoretical explanation of Mach's principle is inherent in it. Another important physical constant which has attracted considerable attention recently [1] is the speed of light. Theories with varying speed of light (VSL) have been firstly proposed by Moffat, Albrecht, Magueijo and Barrow [6,7] as an alternative to the inflation mechanism solving some problems of Big-Bang cosmological models [6,8]. In their formulation the Lorentz invariance is broken and there is a preferred frame, called cosmological frame, in which the speed of light is only a time dependent field. In this frame there exists a pre-set function [7,9] representing the speed of light and enters in FRW equations as an input.

It is a well-known fact that it is possible to have a varying speed of light theory and *preserving* the general covariance and local Lorentz invariance [10]. The price that have to be paid for this, is to introduce a time-like coordinate x^0 which is not necessarily equal to ct. In terms of x^0 and $\vec{x}$, one has local Lorentz invariance and general covariance. The physical time t, can only be defined when dx^0/c is integrable.

The most general scalar–tensor action of gravity which allows for a dynamical speed of light is illustrated in [11]. This action is previously analyzed by many authors using the metric approach. Demianski et al. [12] present a class of cosmological models derived from Noether symmetry requirement. These models describe accelerating evolution of a FRW universe filled with dust matter and exhibit power-law dependence of coupling and potential to the scalar field. There is also some *tracking* solutions of this model [13], in which the time evolution of the scalar field tracks the expansion rate of the universe.

In our earlier paper [14] we have found the exact classical cosmological solutions assuming an exponential coupling between a scalar field, representing dynamical speed of light, and the geometry, with or without cosmological constant. Here we shall continue our previous work [14] on investigating the exact cosmological solutions with varying speed of light. In the following sections we shall use it and find it's exact cosmological solutions for the spatially flat universe. We get the results firstly for a metric theory. Then we discuss the exact solutions using the Palatini approach in which the connection and metric are independent degrees of freedom. In the last section we examine the dynamics of this theory by dynamical system method assuming a ΛCDM background. By considering the character of critical points of the theory we find some exact cosmological solutions in both formalisms.

2. The model

The Jordan–VSL action which we use here is the one presented in [11]:

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\big(F(\Psi)R-2U(\Psi)-Z(\Psi)\;g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi\big)+S_m[\phi_i,g_{\mu\nu}] \tag{1}$$

in which $F(\Psi)=(c/c_0)^4$ and $U(\Psi)$ are arbitrary regular functions of the scalar field Ψ, representing the coupling of the scalar field Ψ with geometry and it's potential energy density respectively. c_0 is a constant velocity and hereafter we shall put $8\pi G=c_0^4=1$. The first part of the above action functional is the gravitational part, including Ricci scalar R and a dynamical term for the velocity of light with arbitrary coupling function $Z(\Psi)$. The latter is the

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action for matter fields, ϕ_i, and does not involve the scalar field Ψ, so the matter is minimally coupled to gravity. As emphasized in the introduction, here we have assumed that there is a time-like coordinate x^0, which is not equal to ct and thus dx^0/c is not necessarily integrable. The dynamics of Ψ depends on the functions F and Z. But note that Z can always be set equal to unity by a redefinition of the field Ψ. Therefore only one arbitrary function remains.

2.1. Metric approach

In the metric approach the metric and the scalar field Ψ are dynamical variables. The variation of the action (1) with respect to them gives:

$$F(\Psi)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right) = T_{\mu\nu}+\partial_\mu\Psi\partial_\nu\Psi-\frac{1}{2}g_{\mu\nu}(\partial_\alpha\Psi)^2 + \nabla_\mu\partial_\nu F(\Psi)-g_{\mu\nu}\nabla_\mu\nabla^\mu F(\Psi)-g_{\mu\nu}U(\Psi), \tag{2}$$

$$\nabla_\alpha\nabla^\alpha\Psi=-\frac{1}{2}\frac{dF}{d\Psi}R+\frac{dU}{d\Psi}. \tag{3}$$

Also the weak equivalence principle holds because the matter fields are minimally coupled to the metric. This implies:

$$\nabla_\mu T^\mu_\nu=0, \tag{4}$$

where the energy–momentum tensor of matter is defined as

$$T^{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}. \tag{5}$$

In a cosmological context, applying the field equations (2)–(4) to FRW universe in which the metric has the following form:

$$ds^2=-dx^{0^2}+a(t)^2\left(\frac{dr^2}{1-kr^2}+r^2\,d\Omega^2\right), \tag{6}$$

and leads to the following cosmological equations:

$$3F\left(H^2+\frac{k}{a^2}\right)=\rho+\frac{1}{2}\dot{\Psi}^2-3H\dot{F}+U, \tag{7}$$

$$-2F\left(\dot{H}-\frac{k}{a^2}\right)=(\rho+p)+\dot{\Psi}^2+\ddot{F}-H\dot{F}, \tag{8}$$

$$(\ddot{\Psi}+3H\dot{\Psi})=3\frac{dF}{d\Psi}\left(\dot{H}+2H^2+\frac{k}{a^2}\right)-\frac{dU}{d\Psi}, \tag{9}$$

$$\dot{\rho}+3H(\rho+p)=0. \tag{10}$$

These are c-variable FRW equations, the wave equation of Ψ field and the conservation law respectively. $H(x^0)=\frac{1}{a}\frac{da}{dx^0}$ is the Hubble parameter, ρ and p are the energy and pressure densities of a perfect fluid considered as matter field and dot denotes derivative with respect to the time-like coordinate x^0. These equations form a coupled set of nonlinear differential equations for $H(x^0)$ and $\Psi(x^0)$. The time-like coordinate x^0 is related to cosmic time by the relation:

$$dt=\frac{dx^0}{c}=\frac{dx^0}{F^{1/4}}. \tag{11}$$

In cosmological application dx^0/c is integrable and gives the physical time. Therefore, the physical Hubble parameter $H_p(t)=\frac{1}{a}\frac{da}{dt}$ can be evaluated as $H_p=H(x^0)\frac{dx^0}{dt}$.

Substituting $\frac{1}{H(x^0)}\frac{d}{dx^0}$ by $\frac{1}{H_p(t)}\frac{d}{dt}$ in (10) gives:

$$\frac{d\rho}{dt}+3H_p(\rho+p)=0. \tag{12}$$

This shows that in this model the conservation equation (10) is valid even in terms of cosmic time. However it is possible to have the non-conservation of energy–momentum if we change our model. The way that the conservation relation changes, highly depends on the model. For example, in the preferred frame approach [6], violation of the energy–momentum conservation occurs because of appearing a source term proportional to the gradient of c in the conservation equation.

The conditions which one should impose on VSL models are usually inspired by the cosmological puzzles. In order to solve the horizon problem of the standard cosmology, one should set [1] $\ddot{a}/\dot{a}-\dot{c}/c>0$ for the early universe and also one has $\dot{a}>0$. These lead to some constraints on the range of possible values of integration constants, appeared in the solutions.

The cosmological solutions of the greatest interest are those for which the time evolution of the Hubble parameter is proportional to the inverse of the cosmic time (corresponding to power-law expansion) or a constant (corresponding to de Sitter expansion). Considering a spatially flat FRW universe, we shall distinguish two cases. A c-dominated universe by which we mean $S_m=U=0$. A $(c\text{–}\Lambda)$-dominated universe means that $S_m=0$ but Λ is not zero and is of gravitational type.

2.1.1. c-dominated universe

Putting $S_m=U=0$ in the equations of motion, we get two independent equations:

$$3FH^2=\frac{1}{2}\dot{\Psi}^2-3H\dot{F}, \tag{13}$$

$$-2F\dot{H}=\dot{\Psi}^2+\ddot{F}-H\dot{F}. \tag{14}$$

Assuming a power-law dependence for the coupling coefficient F, the above equations have the following solutions:

$$H\sim\frac{1}{x^0}, \tag{15}$$

$$\Psi\sim x^{0^\alpha}. \tag{16}$$

And also the coupling function $F(\Psi)\sim\Psi^2$ which is a particular case emerged by requiring the existence of Noether symmetry [12]. The cosmic time is defined as (assuming $\alpha\neq 2$):

$$t\sim\left(x^0\right)^{1-\alpha/2}. \tag{17}$$

Thus the physical Hubble parameter and the speed of light can be written as:

$$H_p\sim\frac{1}{t}\ \Rightarrow\ a\sim t^\nu, \tag{18}$$

$$c\sim t^{\frac{\alpha}{2-\alpha}}. \tag{19}$$

Requesting an expanding universe together with horizon criteria lead to the following constraints on the range of possible values of constants:

$$\nu>0,\qquad \nu>\frac{2}{2-\alpha}. \tag{20}$$

Taking now $\alpha=2$, the corresponding solutions are given by substituting $\alpha=2$ in the relations (15) and (16). But now the time is defined as:

$$t\sim\ln x^0, \tag{21}$$

so that:

$$H_p\sim\text{const}\ \Rightarrow\ a\sim e^{\nu t}, \tag{22}$$

$$c\sim e^{\kappa t}. \tag{23}$$

So $\alpha = 2$ leads to de Sitter expansion for cosmic scale factor. In this case it is needed that:

$$\nu > 0, \qquad \nu - \kappa > 0. \tag{24}$$

Moreover, another solution which leads to the power-law expansion is:

$$H \sim \text{const}, \qquad \psi \sim e^{\alpha x^0}, \qquad F \sim \psi^2. \tag{25}$$

This is a special choice which is used previously by many authors [10,14,15]. The cosmic time is:

$$t \sim e^{-\frac{\alpha}{2} x^0}, \tag{26}$$

so

$$H_p \sim \frac{1}{t} \;\Rightarrow\; a \sim t^{\nu}. \tag{27}$$

Considering both criteria pointed before, one gets the following constraint:

$$\nu > 0. \tag{28}$$

2.1.2. (c–Λ)-dominated universe

A mentioned before, this era corresponds to a matter free universe for which the potential U is nonzero and has the form $U = \Lambda F$ in which Λ is a constant. Demanding power-law expansion for cosmic scale factor one can easily show that the solution is:

$$H \sim \text{const}, \quad \psi \sim e^{\alpha x^0}, \qquad F \sim \psi^2, \tag{29}$$

and the cosmic time is:

$$t \sim e^{-\frac{1}{2}\alpha x^0}. \tag{30}$$

Here $\alpha > 0$, so the speed of light is decreasing in cosmic time.

2.2. Palatini approach

In the Palatini approach the metric and connections are considered as independent fields. The variation of the action (1) with respect to the metric gives:

$$F(\Psi)\left(\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\tilde{R}\right) = T_{\mu\nu} + \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu}(\partial_\alpha \Psi)^2 - g_{\mu\nu} U(\Psi), \tag{31}$$

where $\tilde{R}_{\mu\nu}$ is Ricci tensor constructed from connections and $\tilde{R} \equiv g^{\mu\nu}\tilde{R}_{\mu\nu}$. By varying the action (1) with respect to the connection, one arrives at another field equation:

$$\tilde{\nabla}_\alpha\left(\sqrt{-g}\, g^{\mu\nu} F(\Psi)\right) = 0 \tag{32}$$

in which $\tilde{\nabla}$ represents the covariant derivative with respect to the connection. This equation shows that the connections are Levi-Civita connections of a metric $h_{\mu\nu}$ related to $g_{\mu\nu}$ as:

$$h_{\mu\nu} = F(\Psi) g_{\mu\nu} \tag{33}$$

or equivalently:

$$\Gamma^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + X^\lambda_{\mu\nu} \tag{34}$$

where:

$$X^\alpha_{\beta\gamma} = \frac{1}{2}\left(\delta^\alpha_\beta \partial_\gamma \ln F(\Psi) + \delta^\alpha_\gamma \partial_\beta \ln F(\Psi) - g_{\beta\gamma} g^{\alpha\delta}\partial_\delta \ln F(\Psi)\right) \tag{35}$$

is the difference between the affine connection and the Christoffel symbols ($\left\{ {\lambda \atop \mu\nu} \right\}$). and finally the variation with respect to Ψ gives:

$$\nabla_\alpha \nabla^\alpha \Psi = \frac{dU}{d\Psi} - \frac{1}{2}\frac{dF}{d\Psi}\tilde{R}. \tag{36}$$

It is worth noting that in this case the matter energy–momentum tensor is divergence free with respect to covariant derivative defined with Levi-Civita connection of the metric, i.e. $\nabla_\mu T^\mu_\nu = 0$. This implies that the test particle shall move on the metric geodesic calculated using the Levi-Civita connection. An explicit proof of this point for a more general action can be found in [16].

Eq. (33) shows that the F-field or equivalently the c-field acts as a conformal factor of space–time metric, $g_{\mu\nu}$. Using the transformation rules of the Riemann and Ricci tensors under rescaling (33) and then inserting those in (31) and (36), we obtain:

$$\begin{aligned} F(\Psi)&\left(R^{(g)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{(g)}\right) \\ &= T_{\mu\nu} + \left(1 - \frac{3}{2F}\left(\frac{dF}{d\Psi}\right)^2\right)\left(\partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu}(\partial_\alpha \Psi)^2\right) \\ &\quad + \nabla_\mu \partial_\nu F(\Psi) - g_{\mu\nu}\nabla_\alpha \nabla^\alpha F(\Psi) - g_{\mu\nu} U(\Psi), \end{aligned} \tag{37}$$

$$\nabla_\alpha \nabla^\alpha \Psi = \frac{dU}{d\Psi} - \frac{1}{2}\frac{dF}{d\Psi}\left(R^{(g)} - \frac{3}{F}\nabla_\alpha \nabla^\alpha F(\Psi) + \frac{3}{2F^2}(\partial_\alpha F)^2\right), \tag{38}$$

where $R^{(g)} = g^{\mu\nu} R^{(g)}_{\mu\nu}$ and $R^{(g)}_{\mu\nu}$ is the Ricci tensor constructed from Christoffel symbols. In the spatially flat FRW space–time one can easily shown that the field equations (37) and (38) together the conservation law lead to:

$$3F\left(H^2 + \frac{k}{a^2}\right) = \rho + \frac{\dot{\Psi}}{2} - 3H\dot{F} - \frac{3\dot{F}^2}{4F} + U, \tag{39}$$

$$-2F\left(\dot{H} + \frac{k}{a^2}\right) = (\rho + p) + \dot{\Psi}^2 - H\dot{F} + \ddot{F} - \frac{3\dot{F}^2}{2F}, \tag{40}$$

$$\begin{aligned} \ddot{\Psi} &+ 3H\dot{\Psi} \\ &= 3\frac{dF}{d\Psi}\left(\dot{H} + 2H^2 + \frac{k}{a^2} + \frac{3}{2}\frac{H\dot{F}}{F} + \frac{\ddot{F}}{2F} - \frac{\dot{F}^2}{4F^2}\right) - \frac{dU}{d\Psi}, \end{aligned} \tag{41}$$

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{42}$$

As we did for the metric approach, one can find some exact solutions for the spatially flat universe in two cases: c-dominated and (c–Λ)-dominated. The interesting point which can be easily shown is that demanding power-law expansion for cosmic scale factor in the Palatini framework leads to the solutions which are the same as the results found using the metric formalism.

3. Dynamics of VSL theories

Another way to find out some exact solutions for cosmological models is the dynamical system method [17]. In this method by choosing some appropriate variables, one can convert the field equations of the desired theory to a set of autonomous differential equations. Then the critical points of the autonomous system describe interesting exact solutions. Also this method helps us to check the stability of the solutions. Here we consider a class of VSL theories which are described by action (1). It is worth noting that this theory is the same as to a general scalar tensor theory, but we should take into account that the volume element is defined as $dx^0 d^3x$ which is different form the canonical volume element. So as we have explained before, since in the field equations H is not the physical Hubble parameter and derivatives are with respect to x^0 coordinate, the dynamics of VSL theories should be different from the scalar tensor theories. Dynamics of a general scalar tensor theory in the Jordan frame using metric approach has been considered in [18] by demanding a background ΛCDM spatially flat cosmology.

In Section 2, to find the exact solutions, we chose $F(x^0)$ such that the corresponding solution for cosmic scale factor was physically interested. However, here we impose a general form for phys-

Table 1
The critical points of the system (57), (58), (59) and (60).

Era	Cp (x_1, x_2, x_3, x_4)	Eigenvalues
Radiation $\frac{H_p'}{H_p} = -2$	$R_1(0,0,0,1)$	$(1, -1/2, -2, 4)$
	$R_2(2,0,-1,0)$	$(1, 2, 1/2, 5-2n)$
	$R_3(4/3, 0, -1/3, 0)$	$(5/3, -1/2, 2/3, \frac{14}{3} - \frac{4n}{3})$
	$R_4(-2, 0, -1/3, 0)$	$(-1, \frac{\sqrt{41}-11}{4}, -\frac{\sqrt{41}+11}{4}, 3+2n)$
	$R_5(\frac{8}{2n-1}, \frac{(4n^2-24n+35)}{3(2n-1)^2}, \frac{4(2n^2-9n-2)}{3(2n-1)^2}, 0)$	$(\frac{4}{2n-1}, \frac{2n+3}{2n-1}, \frac{10n-29-A}{4-8n}, \frac{10n-29+A}{4-8n})$
Matter $\frac{H_p'}{H_p} = -\frac{3}{2}$	$M_1(0,0,0,0)$	$(3, -1, \frac{\sqrt{3}-3}{2}, -\frac{\sqrt{3}+3}{2})$
	$M_2(2,0,-1,0)$	$(0, 0, 1, 4-2n)$
	$M_3(\frac{6}{2n-1}, \frac{2(4+n^2-4n)}{(2n-1)^2}, \frac{2n^2-8n-1}{(2n-1)^2}, 0)$	$(-1, \frac{3}{2n-1}, \frac{(2-n)(-3+C)}{2n-1}, \frac{(2-n)(3+C)}{2n-1})$
De Sitter $\frac{H_p'}{H_p} = 0$	$\Lambda_1(0,1,0,0)$	$(-4, -3, \frac{-9+\sqrt{33+96n}}{4}, \frac{-9-\sqrt{33+96n}}{4})$
	$\Lambda_2(6,0,-7,0)$	$(-1, \frac{9+\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}, 3-6n)$
	$\Lambda_3(2,0,-1,0)$	$(-3/2, -3, -2, 1-2n)$
	$\Lambda_4(4,0,-3,0)$	$(3/2, -2, -1, 2-4n)$

ical Hubble parameter which is related to ΛCDM cosmology, that is:

$$H_p(z)^2 = H_0^2\left[\Omega_{0m}(1+z)^3 + \Omega_{0r}(1+z)^4 + \Omega_\Lambda\right] \tag{43}$$

where $\Omega_{0r} = \frac{\rho_r}{\rho_{cr}} \simeq 10^{-4}$, $\Omega_{0m} = \frac{\rho_m}{\rho_{cr}} \simeq 0.3$ and $\Omega_\Lambda = 1 - \Omega_{0m} - \Omega_{0r}$, then we seek for the corresponding solutions for scalar field. We also assume $U(\Psi) \sim F(\Psi)^n$ where n is a constant [19]. In order to express Eqs. (39), (40) and (41) as a dynamical system of the first order differential equations, we first write them in dimensionless form as:

$$1 = \Omega_m + \frac{\rho_r}{3FH^2} + \frac{\Psi'^2}{6F} + \frac{U}{3FH^2} - \frac{F'^2}{F^2} - \frac{F'}{F}, \tag{44}$$

$$-2\frac{H'}{H} = \frac{\rho_m}{FH^2} + \frac{4\rho_r}{3FH^2} + \frac{\Psi'^2}{F} - \frac{F'}{F} + \frac{H'F'}{HF} + \frac{F''}{F} - \frac{3F'^2}{F^2}, \tag{45}$$

$$\frac{\Psi''}{F} = -\frac{H'\Psi'}{HF} + \frac{1}{F}\frac{dF}{d\Psi}\left[\frac{3H'}{H} + 6 + \frac{9F'}{2F} + \frac{3H'F'}{2HF} + \frac{3F''}{2F} - \frac{3F'^2}{4F^2}\right] - \frac{1}{FH^2}\frac{dU}{d\Psi} - \frac{3\Psi'}{F}, \tag{46}$$

where $' = \frac{d}{d\ln a} = \frac{1}{H(x^0)}\frac{d}{dx^0} = \frac{1}{H_p(t)}\frac{d}{dt}$. Now we use a set of dimensionless phase-space variables $x_1, \ldots, x_4$ similar to those introduced in [18], that is:

$$x_1 = -\frac{F'}{F}, \qquad x_2 = \frac{U}{3FH^2}, \qquad x_3 = \frac{\Psi'^2}{6F}, \qquad x_4 = \frac{\rho_r}{3FH^2}. \tag{47}$$

Now defining $\Omega_m = \frac{\rho_m}{3FH^2}$ we write Eqs. (44) and (45) as:

$$\Omega_m = 1 - x_4 - x_3 - x_2 - x_1 + \frac{1}{4}x_1^2, \tag{48}$$

$$x_1' = 3 + 2\frac{H'}{H} + x_4 + 3x_3 - 3x_2 - \left(2 + \frac{H'}{H}\right)x_1 + \frac{1}{4}x_1^2. \tag{49}$$

Differentiating x_4 and x_2 with respect to $\ln a$ gives:

$$x_4' = x_4\left[x_1 - 4 - 2\frac{H'}{H}\right], \tag{50}$$

$$x_2' = x_2\left[x_1(1-n) - 2\frac{H'}{H}\right], \tag{51}$$

where $n = \frac{F}{U}\frac{dU}{dF}$. Finally, differentiating x_3 with respect to $\ln a$ and using (46), we have:

$$x_3' = -2\left(\frac{H'}{H} + 3\right)x_3 + \frac{1}{2}x_1\left(x_1 + x_4 + 5x_3 + (2n-3)x_2 - 1\right) - \frac{1}{8}x_1^3. \tag{52}$$

On the other hand one can easily verify that $\frac{H'}{H} = \frac{H_p'}{H_p} + \frac{1}{4}x_1$, so by substituting this relation to Eqs. (49), (50), (51) and (52) we have:

$$x_1' = 3 - \left(\frac{3}{2} + \frac{H_p'}{H_p}\right)x_1 - 3x_2 + 3x_3 + 2\frac{H_p'}{H_p} + x_4, \tag{53}$$

$$x_2' = x_2\left[x_1\left(\frac{1}{2} - n\right) - 2\frac{H_p'}{H_p}\right], \tag{54}$$

$$x_3' = -2\left(\frac{H_p'}{H_p} + 3\right)x_3 + \frac{1}{2}x_1\left(x_1 + x_4 + 4x_3 + (2n-3)x_2 - 1\right) - \frac{1}{8}x_1^3, \tag{55}$$

$$x_4' = x_4\left[\frac{1}{2}x_1 - 4 - 2\frac{H_p'}{H_p}\right]. \tag{56}$$

These equations describe the cosmological dynamics of the VSL theory in Palatini formalism. By a completely similar procedure one can write the field equations of the metric formalism as follows:

$$x_1' = -3 - x_1\left(\frac{3}{2} + \frac{H_p'}{H_p}\right) - 3x_2 + x_4 + 2x_3 + \frac{3}{4}x_1^2 + 2\frac{H_p'}{H_p}, \tag{57}$$

$$x_2' = x_2\left[x_1\left(\frac{1}{2} - n\right) - 2\frac{H_p'}{H_p}\right], \tag{58}$$

$$x_3' = x_1\left[\frac{1}{2}x_3 - 2 - \frac{H_p'}{H_p} + nx_2\right] - 6x_3 - 2\frac{H_p'}{H_p}x_3 - \frac{1}{4}x_1^2, \tag{59}$$

$$x_4' = x_4\left[\frac{1}{2}x_1 - 4 - 2\frac{H_p'}{H_p}\right]. \tag{60}$$

It is important to note that H_p'/H_p is not always constant so the dynamical equations of the metric and Palatini formalisms are not autonomous and we cannot find the critical points in any regime. On the other hand we know that H_p'/H_p for ΛCDM background is approximately constant in the matter, radiation and de Sitter eras and so we can use the dynamical system method for these eras [18]. The critical points and their corresponding eigenvalues are shown in Table 1 for the metric formalism and in Table 2 for the Palatini formalism.

Considering Table 2, we see that the only physically interested coordinates are R_1, R_5, M_1, M_3 and Λ_1, for other points $x_3 < 0$ for any n. For $n < \frac{1}{2}$, Λ_1, M_3 and R_5 are attractors, but demanding positive values for x_3 imposes another range to n, say $-0.93 < n < -0.13$. Note that $A = \sqrt{281 + 128n^3 + 924n - 732n^2}$ and $C = \sqrt{3 + 12n}$. It should be noted that R_1, M_1 and Λ_1 correspond to general relativity with constant speed of light. Also it is

Table 2
The critical points of the system (53), (54), (55) and (56).

Era	Cp (x_1, x_2, x_3, x_4)	Eigenvalues
Radiation $\frac{H_p'}{H_p} = -2$	$R_1'(\frac{8}{2n-1}, \frac{4n^2-24n+35}{3(2n-1)^2}, \frac{4(2n^2-9n+10)}{3(2n-1)^2}, 0)$	$(0, \frac{2n+3}{2n-1}, \frac{10n-29-A}{4-8n}, \frac{10n-29+A}{4-8n})$
	$R_2'(0,0,0,1)$	$(-2, 4, \frac{1+\sqrt{17}}{4}, \frac{1-\sqrt{17}}{4})$
	$R_3'(2,0,0,0)$	$(0, 1/2, 2, 5-2n)$
	$R_4'(4/3, 0, 1/9, 0)$	$(0, -1/2, 5/3, \frac{14}{3}(1-n))$
	$R_5'(-2, 0, 2/3, 0)$	$(0, \frac{\sqrt{41}-11}{4}, -\frac{\sqrt{41}+11}{4}, 3+2n)$
Matter $\frac{H_p'}{H_p} = -\frac{3}{2}$	$M_1'(2,0,0,0)$	$(0, 1, -1, 4-2n)$
	$M_2'(0,0,0,0)$	$(3, -1, \frac{\sqrt{3}-3}{2}, -\frac{\sqrt{3}+3}{2})$
	$M_3'(\frac{6}{2n-1}, \frac{2(4+n^2-4n)}{(2n-1)^2}, \frac{2(4+n^2-4n)}{(2n-1)^2}, 0)$	$(-1, \frac{3}{2n-1}, \frac{(2-n)(-3+C)}{2n-1}, \frac{(2-n)(3+C)}{2n-1})$
De Sitter $\frac{H_p'}{H_p} = 0$	$\Lambda_1'(0,1,0,0)$	$(-4, -3, \frac{-9+\sqrt{33+96n}}{4}, \frac{-9-\sqrt{33+96n}}{4})$
	$\Lambda_2'(6,0,2,0)$	$(-4, \frac{9+\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}, 3-6n)$
	$\Lambda_3'(2,0,0,0)$	$(-3/2, -4, -2, 1-2n)$
	$\Lambda_4'(4,0,1,0)$	$(3/2, -4, -1, 2-4n)$

worth noting that these results are different from those obtained in [18] for similar scalar tensor theory. Considering the critical points of Table 2 we see that M_3' is an attractor if $-\frac{1}{4} < n < \frac{1}{2}$. Also in the de Sitter era Λ_1' for $n < \frac{1}{2}$ and Λ_3' for $n > \frac{1}{2}$ are attractors. The noteworthy feature of the Palatini VSL theory is that there exists no attractor critical point in the radiation dominated universe and unlike the metric formalism, for all critical points, x_3 can be a positive quantity. However, in the Palatini formalism we have three critical points R_3', M_1' and Λ_3', which are meaningless because for them $F'(\Psi) \neq 0$ while $\Psi' = 0$. It is also important to mention that again R_2', M_2' and Λ_1' correspond to general relativity with constant speed of light.

It is interesting to note that for any solutions which are extracted from the analysis of the critical points, the non-minimal coupling coefficient takes the form $F(\Psi) \sim (\Psi - \Psi_0)^2$. For any regime and critical point, the speed of light takes the form $c(t) \sim a(t)^{-x_1/4}$ in both formalisms. The scalar field takes the form $\Psi(t) = \lambda t^{-x_1/4} + \Psi_0$ and the horizon criteria is $x_1 > 4$ in the radiation dominated era. In the matter dominated era, the scalar field is $\Psi(t) = \lambda t^{-x_1/3} + \Psi_0$ and the horizon criteria takes the form $x_1 > 2$. On the other hand considering the above tables, we see that the critical points in the matter and radiation eras do not satisfy the horizon criteria. However, in the de Sitter era the horizon criteria is satisfied by any physical critical point and the solution for Ψ is $\Psi(t) = \lambda e^{-x_1 t/4} + \Psi_0$.

4. Conclusions

In this Letter, we have considered a class of VSL theories which are described by the action (1), in both metric and Palatini formalisms. In Section 2 we have shown that this model exhibits a power-law or de Sitter expansion for the cosmic scale factor. In any case applying both formalisms, we have shown that $F(\Psi)$ is a quadratic function in terms of the dynamical scalar field Ψ. This is a particular case emerged by requiring the existence of the Noether symmetry for any cosmological point-like Lagrangian of a general scalar tensor theory [12]. On the other hand, one aspect of our model is if the scalar field can be interpreted as dark energy as well as variable speed of light. Many authors studied, for example, the change of fine structure constant based on quintessence [20]. To give a look at this point here, note that in the metric formalism, using Eqs. (7) and (8), we define $\omega_{\rm eff}$ as follows:

$$\omega_{\rm eff} = \frac{P_{\rm eff}}{\rho_{\rm eff}} = \frac{\dot\Psi^2 + 2\ddot F + 4H\dot F - 2U}{\dot\Psi^2 - 6H\dot F + 2U}. \tag{61}$$

The choice of $\omega_{\rm eff} = -1$ leads to the following equation:

$$\dot\Psi^2 - H\dot F + \ddot F = 0. \tag{62}$$

In the matter–radiation free universe, by using equations (7), (8) and (62) we have:

$$6FH_0^2 + 5H_0\dot F + \ddot F - 2U = 0, \tag{63}$$

where $H = H_0$ is a constant. By using these equations one can verify that the coupling coefficient takes the form $F(\Psi) \sim \Psi^2$ in both c-dominated and Λ-dominated universe. As an example of this point, consider a Λ-dominated universe. Eq. (63) leads to the following solution for the coupling function:

$$F \sim e^{\frac{-1}{2}(5H_0 + \sqrt{H_0^2 + 8\Lambda})x^0}, \tag{64}$$

and by substituting this result in (62) we obtain:

$$\Psi \sim e^{\frac{-1}{4}(5H_0 + \sqrt{H_0^2 + 8\Lambda})x^0}. \tag{65}$$

So $F(\Psi) \sim \Psi^2$. Similarly, in the Palatini formalism, we define $\omega_{\rm eff}$ as follows:

$$\omega_{\rm eff} = \frac{P_{\rm eff}}{\rho_{\rm eff}} = \frac{\dot\Psi^2 + 2\ddot F + 4H\dot F - \frac{3\dot F^2}{2F} - 2U}{\dot\Psi^2 - 6H\dot F - \frac{3\dot F^2}{2F} + 2U}. \tag{66}$$

Again, $\omega_{\rm eff} = -1$ leads to the following equation:

$$\dot\Psi^2 - H\dot F + \ddot F - \frac{3\dot F^2}{2F} = 0. \tag{67}$$

By considering the modified Friedman equations of this formalism, equations (39) and (40), in the matter–radiation free universe, one can see that the coupling function satisfying the condition (63). So by using the equations (63) and (67), we see that $F(\Psi) \sim \Psi^2$ in both c-dominated and Λ-dominated universe. So the consistency with Noether symmetry and having constant $\omega_{\rm eff}$, specially $\omega_{\rm eff} = -1$, are both satisfied for $F(\Psi) \sim \Psi^2$. But it should be stressed here that unlike the usual ΛCDM or quintessence models, the expression $\omega_{\rm eff} = -1$ dose not mean necessarily an accelerated universe (i.e. $\frac{d^2a}{dt^2} > 0$). To clarify this point let us combine (7) and (8) in the metric formalism (or (39) and (40) in the Palatini formalism) as follows:

$$\frac{\ddot a}{a} = -\frac{1}{6}\big[(1+3\omega)\rho + (1+3\omega_{\rm eff})\rho_{\rm eff}\big]. \tag{68}$$

Converting the derivatives in terms of x^0 to derivatives with respect to the cosmic time, for a matter–radiation free universe, we have:

$$\frac{1}{a}\frac{d^2a}{dt^2} = \frac{1}{4}\frac{1}{F}\frac{dF}{dt}H_p - \frac{(1+3\omega_{\rm eff})}{6F^{1/2}}\rho_{\rm eff}. \tag{69}$$

Note that this equation is correct in both formalisms with only different expression of ρ_{eff} so the following results are common for them. From this equation we see that the evolution of the speed of light has a crucial role in constructing an inflationary universe. If the speed of light is decreasing in cosmic time, then the necessary condition for $\frac{d^2a}{dt^2} > 0$ is $\omega_{\text{eff}} < -\frac{1}{3}$, but this is not sufficient condition. On the other hand, if the speed of light is increasing in time, then the sufficient condition for $\frac{d^2a}{dt^2} > 0$ is $\omega_{\text{eff}} < -\frac{1}{3}$, but this is not necessary condition. In the other word, by a constant ω_{eff}, the sign of $\frac{d^2a}{dt^2}$ can change with time. For a simple example, consider the solution (64), where $H(x^0) = H_0$ and $\omega_{\text{eff}} = -1$. By using Eq. (69) we have:

$$\frac{1}{a}\frac{d^2a}{dt^2} = H_p^2\left(1 - \alpha t^2\right), \tag{70}$$

where α is a positive constant.

We recall that the action (1) which is used in this Letter, allows for a dynamical gravitational constant as well as dynamical speed of light. The only difference is that we should take x^0 coordinate as a time-like coordinate in the latter case. So the equations of variable speed of light and variable gravitational constant scalar–tensor cosmology are not similar when we write them in terms of the physical time and physical Hubble parameter. To clarify better, starting action (1) as a variable gravitational constant theory and then applying it to cosmology, considering a matter–radiation free universe, it can be easily shown that the first term in Eq. (69) is absent in this case. In this theory $\omega_{\text{eff}} < -\frac{1}{3}$ leads to an inflationary universe if the matter–radiation density is negligible.

In Section 3 we have considered the dynamics of the VSL theories in order to find out other exact solutions. We have constructed the cosmological dynamical system which is constrained to obey the ΛCDM cosmic history. Also by considering the corresponding critical points, we have shown that for both formalisms, variable speed of light takes the form $c \sim a^{-\frac{x_1}{4}}$ in each era. But the time-dependence of dynamical scalar field is different in the radiation and matter eras and it has the forms $\Psi \sim t^{-\frac{x_1}{4}}$ and $\Psi \sim t^{-\frac{x_1}{3}}$, respectively.

By calculating the variables $x_1, \ldots, x_4$ for exact solutions of the c-dominated universe which have been obtained in Section 2 and comparing them with the results of the section 3, one can show that they are not critical points in metric formalism. But by choosing appropriate values for α, υ and k in the relations (18)–(27), they can correspond to the critical points R_4', R_5', Λ_2' and Λ_4' in Palatini formalism.

It is worth to note that for any solution which are extracted from the analysis of critical points, the non-minimal coupling coefficient takes the form $F(\Psi) \sim (\Psi - \Psi_0)^2$. Also unlike the metric approach, there is no attractor critical point in the radiation dominated era for the Palatini approach. It is also important to note that critical points in the matter and radiation eras do not satisfy the horizon criteria. However, in the de Sitter era the horizon criteria is satisfied by any physical critical point and the scalar field has the exponential dependence to cosmic time.

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