Normality criteria of meromorphic functions with multiple zeros

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1. Introduction and main results

Let $F$ be a meromorphic function in $\mathbb{C}$. In 1959, W.K. Hayman [9] proposed the conjecture: If $F$ is transcendental, then $F^n F'$ assumes every finite non-zero complex number infinitely often for any positive integer $n$. Hayman [9,10] himself confirmed it for $n \geq 3$ and for $n \geq 2$ in the case of an entire $F$. Further, it was proved by E. Mues [15] when $n \geq 2$; J. Clunie [6] when $n \geq 1$, $F$ is entire; W. Bergweiler and A. Eremenko [3] if $n=1$ and $F$ is of finite order, and finally by H.H. Chen and M.L. Fang [4] for the case $n=1$.

Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$. Correspondingly, a conjecture of Hayman [10] on normal family, which is related to above problem on value distribution, is as follows: If each $f \in \mathcal{F}$ satisfies $f^n f' \neq a$ for a positive integer $n$ and a finite non-zero complex number $a$, then $\mathcal{F}$ is normal. This conjecture has been shown to be true by Yang and Zhang [29] (for $n \geq 5$ and for $n \geq 2$ in case that $\mathcal{F}$ is a family of holomorphic functions), Gu [12] (for $n=3,4$), Oshkin [16] (for holomorphic functions, $n=1$; cf. [13]), and Pang [18] (for $n \geq 2$ in general; cf. [8]). As indicated by X. Pang [18] (or see [4,31]), the conjecture for $n=1$ is a consequence of Chen–Fang's theorem and his theorem which is a generalization of Zalcman's lemma (cf. [30]). Thus, the Hayman's conjecture on normal family is also verified completely.

Lately, Q.C. Zhang [33] proved that $\mathcal{F}$ is also normal when each pair $(f,g)$ of $\mathcal{F}$ is such that $f^n f' + g^n g'$ share a finite non-zero complex number $a$ IM for $n \geq 2$ (or see [32]), where, by definition, two meromorphic functions $F$ and $G$ are said to share a IM (ignoring multiplicity) if $F^{-1}(a) = G^{-1}(a)$ (see [7]). There are examples showing that this result is not true if $n=1$. For the case of high derivatives, a similar result was obtained by J.M. Qi, D.W. Meng and H.X. Yi [20].

W. Hennekemper [11] extended Clunie's result above by proving an inequality on value distribution, which means particularly that if $F$ is a transcendental entire function, then $(F^k)'(k)$ assumes every finite non-zero complex number infinitely often for any positive integer $k$. Fix positive integers $n$ and $k$. In 1998, Y.F. Wang and M.L. Fang [24] proved that if $F$ is transcendental meromorphic functions in $\mathbb{C}$, $n \geq k+1$, then $(F^n)'(k)$ assumes every non-zero complex number infinitely often. Correspondingly, W. Schwick [22] proved a theorem of normal families related to this result, that is, when $n \geq k+3$, $(F^n)'(k) \neq 1$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in $D$. Recently, Y.T. Li, Y.X. Gu [14] further extend Schwick's result as follows:

Take positive integers $n,k \geq 2$. Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k$. If, for each pair $(f,g)$ in $\mathcal{F}$, $f(f^{(k)})^p$ and $g(g^{(k)})^p$ share a non-zero complex number $a$ ignoring multiplicity, then $\mathcal{F}$ is normal in $D$. Crown Copyright © 2009 Published by Elsevier Inc. All rights reserved.

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When \( n \geq k + 2 \), and if \((f^n)_{(k)}\) and \((g^n)_{(k)}\) share a non-zero complex number \( a \) IM for each pair \((f, g)\) of \( \mathcal{F} \), then \( \mathcal{F} \) is normal in \( D \).

Related to above Hayman’s problem on value distribution, L. Yang and C.C. Yang [28] proposed the conjecture: If \( F \) is transcendental, then \( FF_{(k)} \) assumes every finite non-zero complex number infinitely often for any positive integer \( k \). C.C. Yang and P.C. Hu [26] obtained a part of an answer. To understand this problem well, many authors studied the functions of the form \( F(F_{(k)})^n \) along the researching route of Hayman’s problem. C.C. Yang, L. Yang and Y.F. Wang [27] proved that if \( n \geq 2 \) and \( F \) is a transcendental entire function, then the only possible Picard value of \( F(F_{(k)})^n \) is the value zero. In 1998, Z.F. Zhang and G.D. Song [34] announced that if \( F \) is transcendental, \( a \notin [0, \infty] \), \( n \geq 2 \), then \( F(F_{(k)})^n - a \) has infinitely many zeros. A simple proof was given by A. Alotaibi [1]. In fact, they proved a more stronger result that this fact is true if \( a \) \((\neq 0)\) is a small meromorphic function of \( F \).

Influenced from Bloch’s principle (cf. [31]), that is, there is a normality criterion corresponding to every Liouville–Picard type theorem, in this paper we investigate the problem on normal families related to above theorems of value distribution due to Zhang, Song [34] and Alotaibi [1] by proving the following result:

**Theorem 1.1.** Take positive integers \( n \) and \( k \) with \( n, k \geq 2 \) and take a non-zero complex number \( a \). Let \( \mathcal{F} \) be a family of meromorphic functions in the plane domain \( D \) such that each \( f \in \mathcal{F} \) has only zeros of multiplicity at least \( k \). For each pair \((f, g)\) in \( \mathcal{F} \), if \( f(f_{(k)})^n \) and \( g(g_{(k)})^n \) share a IM, then \( \mathcal{F} \) is normal in \( D \).

**Example 1.2.** Let \( D = \{z \in \mathbb{C} | |z| < 1\} \) and take a non-zero complex number \( a \). Fix two integers \( n \geq 2 \), \( k \geq 2 \). We consider the family
\[
\mathcal{F} = \left\{ f_n(z) = mz + a \left\lfloor \frac{m}{n} \right\rfloor z^{k-1} | m = 1, 2, \ldots, n \right\}.
\]

Obviously, for distinct positive integers \( m, l \), we have \( f_m(f_{(k)})^n \) and \( f_l(f_{(k)})^n \) share a IM. However, the family \( \mathcal{F} \) is not normal at \( z = 0 \). Example 1.2 shows that the condition that \( f \) has only zeros of multiplicity at least \( k \) is sharp in Theorem 1.1.

**Example 1.3.** Take a non-zero complex number \( a \) and fix an integers \( n \geq 2 \). Set
\[
\mathcal{F} = \left\{ f_m(z) = mz + a \left\lfloor \frac{m}{n} \right\rfloor z^k | m = 1, 2, \ldots, n \right\}.
\]

For distinct positive integers \( m, l \), we have \( f_m(f_{(k)})^n \) and \( f_l(f_{(k)})^n \) share a IM. However, the family \( \mathcal{F} \) is not normal at \( z = 0 \).

For the case \( k = 1 \), Example 1.3 shows that Theorem 1.1 is not true. However, according to its proof, it is true too if we add a condition that each \( f \in \mathcal{F} \) has only multiple zeros.

**Example 1.4.** Take \( D = \{z: |z| < 1\} \) and take
\[
\mathcal{F} = \left\{ f_m(z) = e^{mz} | m = 1, 2, \ldots \right\}
\]
or
\[
\mathcal{F} = \left\{ f_m(z) = mz^k + 1 | m = 1, 2, \ldots \right\}
\]

Obviously, any \( f_m \in \mathcal{F} \) has only zeros of multiplicity at least \( k \). For distinct positive integers \( m, l \), we have \( f_m(f_{(k)})^n \) and \( f_l(f_{(k)})^n \) share 0 IM. However, the families \( \mathcal{F} \) are not normal at \( z = 0 \).

Example 1.4 shows that the condition \( a \neq 0 \) in Theorem 1.1 is necessary.

**2. Preliminary lemmas**

Let \( D \) be a domain in \( \mathbb{C} \) and let \( \mathcal{F} \) be meromorphic functions defined in the domain \( D \). Then \( \mathcal{F} \) is said to be normal in \( D \), in the sense of Montel, if any sequence \( \{f_n\} \subset \mathcal{F} \) contains a subsequence \( \{f_{n_j}\} \) such that \( f_{n_j} \) converges spherically locally uniformly in \( D \), to a meromorphic function or \( \infty \). To prove Theorem 1.1, we will need the following Zalcman’s lemma (cf. [31]):

**Lemma 2.1.** Take a positive integer \( k \). Let \( \mathcal{F} \) be a family of meromorphic functions in the unit disc \( \Delta \) with the property that zeros of each \( f \in \mathcal{F} \) are of multiplicity at least \( k \). If \( \mathcal{F} \) is not normal at a point \( z_0 \in \Delta \), then for \( 0 < \alpha < k \), there exist a sequence \( \{z_n\} \subset \Delta \) of complex numbers with \( z_n \to z_0 \); a sequence \( \{f_n\} \) of \( \mathcal{F} \); and a sequence \( \{\rho_n\} \) of positive numbers with \( \rho_n \to 0 \) such that \( g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \) locally uniformly (with respect to the spherical metric) to a non-constant meromorphic function \( g(\xi) \) on \( \mathbb{C} \). Moreover, the zeros of \( g(\xi) \) are of multiplicity at least \( k \), and the function \( g(\xi) \) may be taken to satisfy the normalization \( g_i'(\xi) \leq g'(0) = 1 \) for any \( \xi \in \mathbb{C} \). In particular, \( g(\xi) \) has at most order 2.
This is Pang’s generalization (cf. [17,19,25]) of the Main Lemma in [30] (where \( \alpha \) is taken to be 0), with improvements due to Schwick [22] and Chen and Gu [5]. In Lemma 2.1, the order of \( g \) is defined by using the Nevanlinna’s characteristic function \( T(r, g) \):

\[
\text{ord}(g) = \limsup_{r \to \infty} \frac{\log T(r, g)}{\log r}.
\]

Here \( g^s \) denotes the spherical derivative

\[
g^s(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^s}.
\]

Lemma 2.2. Take positive integers \( n \) and \( k \) with \( n \geq 2 \) and take a finite non-zero complex number \( a \). If \( f \) is a rational but not a polynomial function and \( f \) has only zeros of multiplicity at least 2, then \( f(f^{(k)})^n - a \) has at least two distinct zeros.

Proof. Assume, to the contrary, that \( f(f^{(k)})^n - a \) has at most one zero. Set

\[
f(z)(f^{(k)}(z))^n = A \frac{(z - \alpha_1)^{m_1} \cdots (z - \alpha_s)^{m_s}}{(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_t}},
\]

where \( A \) is a non-zero constant. Since \( f \) has only zeros with multiplicity at least 2, we find

\[
m_i \geq 2 \quad (i = 1, 2, \ldots, s); \quad n_j > n(k + 1) \quad (j = 1, 2, \ldots, t).
\]

For simplicity, we denote

\[
M = m_1 + m_2 + \cdots + m_s \geq 2s,
\]

\[
N = n_1 + n_2 + \cdots + n_t > n(k + 1)t.
\]

By (1), we obtain

\[
\{f(z)(f^{(k)}(z))^n\}' = \frac{(z - \alpha_1)^{m_1-1} \cdots (z - \alpha_s)^{m_s-1}}{(z - \beta_1)^{n_1+1} \cdots (z - \beta_t)^{n_t+1}} g(z).
\]

where \( g(z) \) is a polynomial with \( \deg(g) \leq s + t - 1 \). Next we may distinguish two cases.

Case 1. The function \( f(f^{(k)})^n - a \) has exactly one zero. Now we can write

\[
f(z)(f^{(k)}(z))^n = a + \frac{B(z - z_0)^l}{(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_t}} = \frac{P(z)}{Q(z)},
\]

where \( l \geq 1 \) is a positive integer, \( B \) is a non-zero constant, \( P \) and \( Q \) are polynomials of degree \( M \) and \( N \), respectively. Also \( P \) and \( Q \) have no common factors. Obviously, we have \( z_0 \neq \alpha_i \) (\( i = 1, \ldots, s \)) since \( a \neq 0 \). Differentiating (5), we obtain

\[
\{f(z)(f^{(k)}(z))^n\}' = \frac{(z - z_0)^l-1 g_1(z)}{(z - \beta_1)^{n_1+1} \cdots (z - \beta_t)^{n_t+1}},
\]

where \( g_1 \) is a polynomial of the form

\[
g_1(z) = B(l - N)z^l + B_{l-1}z^{l-1} + \cdots + B_0
\]

in which \( B_0, \ldots, B_{l-1} \) are constants.

Case 1.1. \( l \neq N \). By using (5), we obtain \( \deg(P) \geq \deg(Q) \), that is, \( M \geq N \). Since \( z_0 \neq \alpha_i \), then (4) and (6) imply

\[
\sum_{i=1}^{s} (m_i - 1) = M - s \leq \deg(g_1) = t,
\]

and so \( M \leq s + t \). By using (2) and (3), we obtain

\[
M \leq s + t < \frac{M}{2} + \frac{N}{n(k + 1)} \leq \left\{ \frac{1}{2} + \frac{1}{n(k + 1)} \right\} M
\]

which is a contradiction since \( n \geq 2, k \geq 1 \).

Case 1.2. \( l = N \). We further distinguish two subcases:
Case 1.2.1. $M \geq N$. By (4) and (6), we obtain
\[ M - s \leq \deg(g_1) \leq t. \]
Similar to Case 1.1, we obtain a contradiction $M < M$.

Case 1.2.2. $M < N$. By using (4) and (6) again, we obtain
\[ l - 1 \leq \deg(g) \leq s + t - 1, \]
and hence
\[ N = l \leq \deg(g) + 1 \leq s + t < \left(\frac{1}{2} + \frac{1}{n(k+1)}\right)N \leq N \]
which is a contradiction.

Case 2. The function $f(f^{(k)})^n - a$ has no zero. We also have (4) and (5) with $l = 0$. Proceeding as in the proof of Case 1, we also have a contradiction. Now Lemma 2.2 is proved.

**Lemma 2.3.** Take positive integers $n$ and $k$ with $n, k \geq 2$ and take a finite non-zero complex number $a$. If $f$ is a non-constant meromorphic function such that $f$ has only zeros of multiplicity at least $k$, then $f(f^{(k)})^n - a$ has at least two distinct zeros.

**Proof.** If $f$ is a polynomial, we obtain immediately that $f(f^{(k)})^n$ has multiple zeros since $f$ has only zeros of multiplicity at least $k$ which means particularly $\deg(g) \geq k$, and hence $f(f^{(k)})^n - a$ has at least one zero. If $f(f^{(k)})^n - a$ has only a unique zero $z_0$, then there exist a non-zero constant $A$ and an integer $l \geq 2$ such that
\[ f(z)(f^{(k)}(z))^n = a + A(z - z_0)^l, \]
which, however, has only simple zeros since $a \neq 0$. This is a contradiction.

If $f$ is a rational but not a polynomial function, it follows from Lemma 2.2. If $f$ is transcendental, this is a direct consequence of a result due to Zhang and Song [34], and Alotaibi [1].

**Lemma 2.4.** Let $n \geq 2$ be a positive integer and let $a$ be a finite non-zero complex number. If $f$ is a non-constant meromorphic function, then $f(f')^n - a$ has at least one zero.

**Proof.** If $f$ is a non-constant polynomial, then $f(f')^n - a$ is also a non-constant polynomial, and hence it has at least one zero.

Next we assume that $f$ is rational with one pole at least. Write
\[ f(z) = A\frac{(z - \alpha_1)^{m_1} \cdots (z - \alpha_s)^{m_s}}{(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_t}}, \tag{7} \]
where $A$ is a non-zero constant, and $m_i, n_j$ are positive integers. Set
\[ m_1 + m_2 + \cdots + m_s = M, \tag{8} \]
\[ n_1 + n_2 + \cdots + n_t = N. \tag{9} \]
By (7), we have
\[ f'(z) = \frac{P_1(z)}{Q_1(z)}, \tag{10} \]
where
\[ P_1(z) = (z - \alpha_1)^{m_1-1} \cdots (z - \alpha_s)^{m_s-1}h(z), \]
\[ Q_1(z) = (z - \beta_1)^{n_1+1} \cdots (z - \beta_t)^{n_t+1}, \]
in which $h(z)$ is a polynomial of the form
\[ h(z) = A(M - N)z^{s+t-1} + \cdots. \]
Thus by (7) and (10), we obtain
\[ f(f')^n = \frac{P}{Q}. \tag{11} \]
where
and
\[ M_i = (n+1)m_i - n, \quad N_j = (n+1)n_j + n. \]

Suppose, to the contrary, that \( f(f')^n - a \) has no zero. Then
\[
f(f')^n = a + \frac{B}{Q} = \frac{P}{Q},
\]
where \( B \) is a non-zero constant, which implies particularly \( P = aQ + B \), and so \( \text{deg}(P) = \text{deg}(Q) \). Now we claim \( M > N \), otherwise, if \( M \leq N \), then
\[
\text{deg}(P_1) = M - s + \text{deg}(h) \leq M - s + (s + t - 1) < N + t = \text{deg}(Q_1),
\]
and hence
\[
\text{deg}(P) = n \text{deg}(P_1) + M < n \text{deg}(Q_1) + N = \text{deg}(Q).
\]
This is a contradiction, and so the claim is proved.
Therefore, we have
\[
\text{deg}(h^n) = n(s + t - 1)
\]
since \( M > N \), and hence
\[
\begin{align*}
\text{deg}(P) &= \sum_{i=1}^{s} M_i + \text{deg}(h^n) = \sum_{i=1}^{s} [(n+1)m_i - n] + n(s + t - 1) = (n+1)M + nt - n, \\
\text{deg}(Q) &= \sum_{j=1}^{t} N_j = \sum_{j=1}^{t} (n+1)n_j + n = (n+1)N + nt,
\end{align*}
\]
which further implies
\[
M - N = \frac{n}{n+1}
\]
since \( \text{deg}(P) = \text{deg}(Q) \). This is impossible since \( M - N \) is an integer. Therefore, \( f(f')^n - a \) has zeros.

Finally, if \( f \) is transcendental, this is a direct consequence of a result due to Zhang and Song [34], and Alotaibi [1]. Lemma 2.4 is proved. \( \square \)

3. Proof of Theorem 1.1

Without loss of generality, we may assume that \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Suppose, to the contrary, that \( \mathcal{F} \) is not normal in \( D \). Without loss of generality, we assume that \( \mathcal{F} \) is not normal at \( z_0 = 0 \). Then, by Lemma 2.1, there exist a sequence \( \{z_j\} \) of complex numbers with \( z_j \to 0 \) \((j \to \infty)\); a sequence \( \{f_j\} \) of \( \mathcal{F} \); and a sequence \( \{\rho_j\} \) of positive numbers with \( \rho_j \to 0 \) such that
\[
g_j(\xi) = \rho_j^{-\frac{ak}{n}} f_j(z_j + \rho_j \xi)
\]
converges uniformly to a non-constant meromorphic function \( g(\xi) \) in \( \mathbb{C} \) with respect to the spherical metric. Moreover, \( g(\xi) \) is of order at most 2. By Hurwitz’s theorem, the zeros of \( g(\xi) \) have at least multiplicity \( k \).

On every compact subset of \( \mathbb{C} \) which contains no poles of \( g \), we have uniformly
\[
f_j(z_j + \rho_j \xi) \left( f_j^{(k)}(z_j + \rho_j \xi) \right)^n - a = g_j(\xi) \left( g_j^{(k)}(\xi) \right)^n - a \to g(\xi) \left( g^{(k)}(\xi) \right)^n - a
\]
with respect to the spherical metric. If \( g(\xi)^n \equiv a \), then \( g \) has no zeros. Of course, \( g \) also has no poles. Since \( g \) is a non-constant meromorphic function of order at most 2, then there exist constants \( c_i \) such that \( (c_1, c_2) \neq (0, 0) \), and
\[
g(\xi) = e^{c_1 \xi + c_2 \xi^2}.
\]
Obviously, this is contrary to the case \( g(\xi)^n \equiv a \). Hence \( g(\xi)^n \not\equiv a \).

By Lemma 2.3, the function \( g(\xi)^n - a \) has at least two distinct zeros. Let \( \xi_0 \) and \( \xi_0^* \) be two distinct zeros of \( g(\xi)^n - a \). We choose a positive number \( \delta \) small enough such that \( D_1 \cap D_2 = \emptyset \) and such that \( g(\xi)^n - a \) has no other zeros in \( D_1 \cup D_2 \) except for \( \xi_0 \) and \( \xi_0^* \), where
\[
D_1 = \{ \xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta \}, \quad D_2 = \{ \xi \in \mathbb{C} \mid |\xi - \xi_0^*| < \delta \}.
\]
By (15) and Hurwitz’s theorem, for sufficiently large \( j \) there exist points \( \xi_j \in D_1, \xi_j^* \in D_2 \) such that

\[
f_j(z_j + \rho_j \xi_j)(f_j^{(k)}(z_j + \rho_j \xi_j))^n - a = 0,
\]

\[
f_j(z_j + \rho_j \xi_j^*)(f_j^{(k)}(z_j + \rho_j \xi_j^*))^n - a = 0.
\]

Since, by the assumption in Theorem 1.1, \( f_1(f_j^{(k)})^n \) and \( f_j(f_j^{(k)})^n \) share a IM for each \( j \), it follows that

\[
f_1(z_j + \rho_j \xi_j)(f_1^{(k)}(z_j + \rho_j \xi_j))^n - a = 0,
\]

\[
f_1(z_j + \rho_j \xi_j^*)(f_1^{(k)}(z_j + \rho_j \xi_j^*))^n - a = 0.
\]

Letting \( j \to \infty \), and noting \( z_j + \rho_j \xi_j \to 0 \), \( z_j + \rho_j \xi_j^* \to 0 \), we obtain

\[
f_1(0)(f_1^{(k)}(0))^n - a = 0.
\]

Since the zeros of \( f_1(f_j^{(k)})^n - a \) have no accumulation points, in fact we have

\[
z_j + \rho_j \xi_j = 0, \quad z_j + \rho_j \xi_j^* = 0,
\]

or equivalently

\[
\xi_j = -\frac{z_j}{\rho_j}, \quad \xi_j^* = -\frac{z_j}{\rho_j}.
\]

This contradicts with the facts that \( \xi_j \in D_1, \xi_j^* \in D_2 \), \( D_1 \cap D_2 = \emptyset \). Theorem 1.1 is proved completely.

4. Notes

According to the proof of Theorem 1.1 and based on ideas from [4], we may modify Theorem 1.1 as follows:

**Theorem 4.1.** Take positive integers \( n \) and \( k \) with \( n \geq 2 \) and take a non-zero complex number \( a \). Let \( \mathcal{F} \) be a family of meromorphic functions in the plane domain \( D \) such that each \( f \in \mathcal{F} \) has only zeros of multiplicity at least \( k \). For each element \( f \) of \( \mathcal{F} \), if \( f(z)(f^{(k)}(z))^n = a \) implies \( |f^{(k)}(z)| \leq A \) for a positive number \( A \), then \( \mathcal{F} \) is normal in \( D \).

**Proof.** By using the notations in the proof of Theorem 1.1, and now noting that, by Hurwitz’s theorem, the zeros of \( g(\xi) \) have at least multiplicity \( k \), the function \( g(f^{(k)})^n - a \) has at least one zero \( \xi_0 \) based on Lemmas 2.4 and 2.3. Thus we have

\[
|g_j^{(k)}(\xi_j)| = \rho_j^{\frac{k}{n}} |f_j^{(k)}(z_j + \rho_j \xi_j)| \leq A \rho_j^{\frac{k}{n}}.
\]

Since Hurwitz’s theorem implies \( \xi_j \to \xi_0 \) as \( j \to \infty \), we obtain consequently

\[
g_j^{(k)}(\xi_0) = \lim_{j \to \infty} g_j^{(k)}(\xi_j) = 0.
\]

This contradicts \( g(\xi_0)(g^{(k)}(\xi_0))^n = a \neq 0 \). Theorem 4.1 is proved. \( \square \)

**Corollary 4.2.** Take positive integers \( n \) and \( k \) with \( n \geq 2 \) and take a non-zero complex number \( a \). Let \( \mathcal{F} \) be a family of meromorphic functions in the plane domain \( D \) such that each \( f \in \mathcal{F} \) has only zeros of multiplicity at least \( k \). If each element \( f \) of \( \mathcal{F} \) satisfies \( f(z)(f^{(k)}(z))^n \neq a \) for any \( z \in D \), then \( \mathcal{F} \) is normal in \( D \).

If we replace the form \( f(f^{(k)})^n \) (resp. \( g(g^{(k)})^n \)) in Theorem 1.1 by the form \( f^l(f^{(k)})^n \) (resp. \( g^l(g^{(k)})^n \)) for an integer \( l \geq 2 \), the conclusion holds too (see [23] on the result of value distribution).

**References**


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