Global Asymptotic Stability in a Nonautonomous Lotka–Volterra Type System with Infinite Delay

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1. INTRODUCTION

In the following, we consider the nonautonomous Lotka–Volterra type system with infinite delays

\[
\dot{x}_i(t) = b_i(x_i(t)) \left[ r_i(t) - a_i(t)x_i(t) + \sum_{j=1}^{n} \sum_{l=1}^{m} b_{ijl}(t)x_j(t - \tau_{ijl}(t)) \right. \\
+ \left. \sum_{j=1}^{n} \int_0^\infty b_{ij}(t, s)x_j(t - s) \, ds \right], \quad i = 1, 2, \ldots, n, \tag{1.1}
\]

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together with the following assumptions:

(A1) \( b_1(x) \) is continuous such that \( b_1(0) = 0, b_1(x) > 0 \) for \( x > 0 \).

(A2) \( \tau_i(t), \ a_i(t), \) and \( b_{ij}(t) \) are bounded, continuous for \( t \geq 0; a_i(t) > 0 \) on \( 0 \leq t < \infty \).

(A3) \( \tau_{ij}(t) \) is continuously differentiable for \( t \geq 0; \) \( \tau_{ij}(t) \geq 0 \) and \( \tau_{ij}(t) < 1 \) on \( 0 \leq t < \infty \).

(A4) \( b_{ij}(t,s) \) is continuous with respect to \( t; |b_{ij}(t+s,s)| \) is integrable with respect to \( s \) on \( [0,\infty) \); and \( \sup_{t \geq 0} \int_0^\infty |b_{ij}(t,s)| \, ds < \infty \).

We consider initial conditions of the form

\[
x_i(\theta) = \varphi_i(\theta) \geq 0 \quad \text{on} \quad -\infty < \theta < 0; \quad \varphi_i(0) > 0; \quad \sup_{-\infty < \theta \leq 0} |\varphi_i(\theta)| < \infty,
\]

where \( \varphi_i \) \( (i = 1, 2, \ldots, n) \) is continuous on \((-\infty, 0]\).

Systems like (1.1)–(1.2) are important in the models of multi-species population dynamics. There are considerable works on the theory of global asymptotic stability of Lotka–Volterra type systems with delay that have been developed by [2, 3, 5, 7–11]. In addition to these, the books of Gopalsamy [4] and Kuang [6] are good sources for global attractivity of Lotka–Volterra systems. One of the reasons for considering this topic is that most of the stability results for Lotka–Volterra systems with delay are focused on global asymptotic stability of a positive steady state and it seems to us that there are several interesting open problems in more general equations without steady state solutions. Secondly, in his recent work [7], Kuang established sufficient conditions for the global asymptotic stability of (1.1)–(1.2) without infinite delay and time delays as well. Doing this, he does not assume that the system has a saturated equilibria. However, in the case of infinite delay, his method does not work—even if it has a saturated equilibria. So, this difficulty has been left as an open problem by him.

The main purpose of this work is to give sufficient conditions for global asymptotic stability of (1.1)–(1.2). The sufficient conditions obtained in this paper are weaker than those available in [7]. The method used herein is new and more general because it can be also applied to system (1.1) without infinite delay and time delays.

In this paper we do not deal with the existence and uniqueness of the solutions. We assume that some additional conditions are satisfied for the functions \( b_i \) such that the solutions of (1.1) and (1.2) exist on \([0, \infty)\). It is worth noting that the uniqueness of the solutions are not needed in our results.
In [3], Gopalsamy obtained sufficient conditions for the global attractivity of a positive steady state of the following constant coefficients system with infinite delay,

\[
\frac{dx_i(t)}{dt} = x_i(t) \left( b_i + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} k_{ij}(t - s) x_j(s) \, ds \right), \quad i = 1, 2, \ldots, n,
\]

which is a special case of (1.1) providing that \( a_{ij} = 0 \) for \( i \neq j \). In this case, our hypothesis (iii) with \( c_i = 1 \) \((i = 1, 2, \ldots, n)\) immediately reduces to the relevant hypothesis by Gopalsamy [3]:

\[
a_{ii} < 0; \quad |a_{ii}| > \sum_{j=1}^{n} |b_{ij}| + \sum_{j=1}^{n} |c_{ij}| \int_{0}^{\infty} |k_{ij}(s)| \, ds, \quad i = 1, 2, \ldots, n.
\]

On the other hand, Wang and Zhang [11] considered an infinite delay Lotka–Volterra system which is also a special case of (1.1). Indeed, it can be obtained from (1.1) when \( b_i(x(t)) = x_i(t), \ l_{ij} = 1, \) and \( b_{ij}(t, s) = b_{ij}(t)k_{ij}(s) \) in (1.1). They also investigated the same problem of a positive equilibrium. It should be noted that their sufficient conditions can be verified when the parameters are given.

Our method is based on finding a bounded function \( K_i > 0, \ i = 1, 2, \ldots, n, \) satisfying an equation which is a suitable perturbation of Eq. (1.1). In some special cases our conditions can be easily checked and reduced to some well-known results. The Lyapunov functional applied in the proof of our main result is based on an essential modification of the Lyapunov function which was introduced by Kuang [7].

2. MAIN RESULTS

Now we state our main result below.

**Theorem 1.** In addition to (A.1)–(A.4), assume further that

(i) all solutions of (1.1)–(1.2) are positive on \( 0 \leq t < \infty; \)
(ii) all solutions of (1.1)–(1.2) are bounded on $0 \leq t < \infty$;

(iii) there exist $c_i > 0$, $i = 1, 2, \ldots, n$, such that for $i = 1, 2, \ldots, n$

$$\liminf_{t \to \infty} \left[ c_i a_i(t) - \sum_{j=1}^{n} c_j \left( \sum_{l=1}^{l_{ji}} \frac{|b_{ji}(\psi^{-1}_{ji}(t))|}{1 - \hat{\tau}_{ji}(\psi^{-1}_{ji}(t))} \right) + \int_{0}^{\infty} |b_j(t + s, s)| \, ds \right] > 0,$$

where $\psi^{-1}_{ji}(t)$ is the inverse function of $\psi_{ji}(t) = t - \tau_{ji}(t)$;

(iv) there exists $K_i(t) > 0$, $i = 1, 2, \ldots, n$, continuously differentiable and bounded on $0 \leq t < \infty$ such that

$$\dot{K}_i(t) = b_i(K_i(t)) \left[ r_i(t) - a_i(t)K_i(t) \right. + \sum_{j=1}^{n} \sum_{l=1}^{l_{ji}} b_{ji}(t)K_j(t - \tau_{ji}(t)) + \sum_{j=1}^{n} \int_{0}^{\infty} b_j(t, s)K_j(t - s) \, ds \left. \right] + b_i(K_i(t))k_i(t), \quad (2.1)$$

where $k_i(t)$ is bounded on $[0, \infty)$ and $\int_{0}^{\infty} |k_i(t)| \, dt < \infty$.

Then for any solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of Eq. (1.1) one has

$$\lim_{t \to \infty} |x_i(t) - K_i(t)| = 0, \quad i = 1, 2, \ldots, n. \quad (2.2)$$

Proof. Consider the function $V(t) = V(t, x_1(t), \ldots, x_n(t))$ defined by

$$V(t) = \sum_{i=1}^{n} c_i V_i^{(1)}(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{l_{ji}} V_{ij}^{(2)}(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}^{(3)}(t), \quad (2.3)$$

for $t \geq 0$, where

$$V_i^{(2)}(t) = \left[ \int_{K_i(t)}^{x_i(t)} \frac{ds}{b_i(s)} \right] (i = 1, 2, \ldots, n), \quad (2.4)$$

$$V_{ij}^{(2)}(t) = \int_{t - \tau_{ij}(t)}^{t} \frac{|b_{ji}(\psi^{-1}_{ji}(s))|}{1 - \hat{\tau}_{ij}(\psi^{-1}_{ij}(s))} |x_j(s) - K_j(s)| \, ds$$
\[ (j = 1, 2, \ldots, n; \ l = 1, 2, \ldots, l_{ij}), \quad (2.5) \]

\[
V^{(3)}_{ij}(t) = \int_0^\infty \int_{t-s}^t |b_{ij}(u+s,s)||x_j(u) - K_j(u)| \, du \, ds, \quad (2.6)
\]

in which \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) is a solution of (1.1)-(1.2), and \( K(t) = (K_1(t), K_2(t), \ldots, K_n(t))^T \) is a function such that (iv) holds.

Calculating the upper right derivative \( D^+V(t) \) of \( V(t) \) along the solutions of (1.1), we get

\[
D^+V(t) = \sum_{i=1}^n c_i D^+V_i(t)
\]

\[ + \sum_{i=1}^n c_i \sum_{j=1}^n \sum_{l=1}^{l_{ij}} \dot{V}_{ij}^{(2)}(t) + \sum_{i=1}^n c_i \sum_{j=1}^n \dot{V}_{ij}^{(3)}(t) \quad (2.7)
\]

a.e. \( t \geq 0 \) where

\[
D^+V_i^{(1)}(t) = \left( \frac{\dot{x}_i(t)}{b_i(x_i(t))} - \frac{\dot{K}_i(t)}{b_i(K_i(t))} \right) \text{sgn}(x_i(t) - K_i(t))
\]

\[ = -a_i(t)|x_i(t) - K_i(t)| \]

\[ + \sum_{j=1}^n \sum_{l=1}^{l_{ij}} b_{ij}(t)|x_j(t - \tau_{ij}(t)) - K_j(t - \tau_{ij}(t))| \]

\[ + \sum_{j=1}^n \int_0^\infty b_{ij}(t,s)|x_j(t - s) - K_j(t - s)| \, ds \]

\[-k_i(t)\text{sgn}(x_i(t) - K_i(t)), \quad \text{a.e.} \ t \geq 0, \]

\[
\dot{V}_{ij}^{(2)}(t) = \frac{|b_{ij}(\psi^{-1}_{ij}(t))|}{1 - \tau_{ij}(\psi^{-1}_{ij}(t))} |x_j(t) - K_j(t)|
\]

\[ -|b_{ij}(t)||x_j(t - \tau_{ij}(t)) - K_j(t - \tau_{ij}(t))|, \quad \text{a.e.} \ t \geq 0, \]

\[
\dot{V}_{ij}^{(3)}(t) = \int_0^\infty (|b_{ij}(t+s,s)||x_j(t) - K_j(t)|
\]

\[ -|b_{ij}(t,s)||x_j(t - s) - K_j(t - s)|) \, ds, \quad \text{a.e.} \ t \geq 0. \]
Putting these results in (2.7) and simplifying, we have

\[ D^+ V(t) \leq \sum_{i=1}^{n} -c_i a_i(t) |x_i(t) - K_i(t)| + \sum_{i=1}^{n} c_i |k_i(t)| 
+ \sum_{i=1}^{n} c_i \sum_{j=1}^{n} \sum_{l=1}^{l_i} \frac{|b_{ijl}(\psi_{ijl}^{-1}(t))|}{1 - \tau_{ijl}(\psi_{ijl}^{-1}(t))} |x_j(t) - K_j(t)| 
+ \sum_{i=1}^{n} c_i \left( \int_0^\infty |b_{ij}(t + s, s)| \, ds \right) |x_j(t) - K_j(t)| 
= \sum_{i=1}^{n} \left[ -c_i a_i(t) + \sum_{j=1}^{n} c_j \left( \sum_{l=1}^{l_i} \frac{|b_{ijl}(\psi_{ijl}^{-1}(t))|}{1 - \tau_{ijl}(\psi_{ijl}^{-1}(t))} 
+ \int_0^\infty |b_{ij}(t + s, s)| \, ds \right) \right] |x_i(t) - K_i(t)| 
+ \sum_{i=1}^{n} c_i |k_i(t)| \quad \text{for a.e. } t \geq 0. \] (2.8)

Integrating both sides of (2.8) with respect to \( t \),

\[ V(t) + \sum_{i=1}^{n} \int_0^t c_i a_i(\eta) 
- \sum_{j=1}^{n} c_j \left( \sum_{l=1}^{l_i} \frac{|b_{ijl}(\psi_{ijl}^{-1}(\eta))|}{1 - \tau_{ijl}(\psi_{ijl}^{-1}(\eta))} + \int_0^\infty |b_{ij}(\eta + s, s)| \, ds \right) \right] 
\cdot |x_j(\eta) - K_j(\eta)| \, d\eta 
\leq V(0) + \sum_{i=1}^{n} c_i \int_0^t |k_i(s)| \, ds, \quad t \geq 0. \] (2.9)

By hypothesis (iii), there exists a constant \( \alpha_{i0} \) \((i = 1, \ldots, n)\) such that

\[ c_i a_i(t) - \sum_{j=1}^{n} c_j \left( \sum_{l=1}^{l_i} \frac{|b_{ijl}(\psi_{ijl}^{-1}(t))|}{1 - \tau_{ijl}(\psi_{ijl}^{-1}(t))} 
+ \int_0^\infty |b_{ij}(t + s, s)| \, ds \right) \geq \alpha_{i0} > 0 \] (2.10)

for large \( t \).
Moreover, since \( \int_0^\infty |k_i(t)| \, dt < \infty \), \( i = 1, \ldots, n \), it follows from (2.9) that
\[
V(t) + \sum_{i=1}^n \alpha_i \int_0^t |x_i(\eta) - K_i(\eta)| \, d\eta \leq C
\]
for some constant \( C > 0 \). Therefore, \( V(t) \) is bounded on \( 0 \leq t < \infty \), and also
\[
\int_0^\infty |x_i(\eta) - K_i(\eta)| \, d\eta < \infty, \quad i = 1, \ldots, n.
\]
By (ii) and (iv),
\[
|x_i(t) - K_i(t)|, \quad i = 1, \ldots, n, \text{ is bounded on } [0, \infty).
\]
On the other hand, from the hypotheses of the theorem, it is easy to see that \( \dot{x}_i(t) \) and \( K_i(t) \) \( (i = 1, \ldots, n) \) are bounded for \( t \geq 0 \). Therefore,
\[
|x_i(t) - K_i(t)|, \quad i = 1, \ldots, n, \text{ is uniformly continuous on } [0, \infty).
\]
From (2.12), (2.13), and (2.14), we get
\[
|x_i(t) - K_i(t)| \to 0 \quad \text{as } t \to \infty
\]
which means (2.2). The proof of the theorem is complete.

The following theorem gives the sufficient conditions for the solutions of (1.1)–(1.2) to be positive on \([0, \infty)\).

**Theorem 2.** Assume \((A_1)–(A_4)\) in system (1.1). If
\[
(i') \quad \int_0^1 \frac{du}{b_i(u)} = +\infty \quad (i = 1, \ldots, n),
\]
then a solution \((x_1(t), x_2(t), \ldots, x_n(t))^T\) of (1.1)–(1.2) satisfies
\[
x_i(t) > 0 \quad \text{for all } t \geq 0 \quad (i = 1, \ldots, n).
\]

**Proof.** Obviously, a solution \( x_i(t), \ i = 1, \ldots, n, \) of (1.1) with initial condition (1.2) satisfies
\[
x_i(0) > 0 \quad (i = 1, 2, \ldots, n).
\]
Suppose that (2.17) is not true. Then there exist \( i_0 \) and \( T_0 \) such that
\[
x_i(t) > 0 \quad \text{for } 1 \leq i \leq n \text{ and } 0 \leq t < T_0
\]
and 

\[ x_{i_0}(T_0) = 0. \]

Putting \( x_{i_0}(t) \) into (1.1) and then integrating with respect to \( t \), we get

\[
\int_0^t \frac{\dot{x}_{i_0}(\eta)}{b_{i_0}(x_{i_0}(\eta))} \, d\eta = \int_0^t \left[ r_{i_0}(\eta) - a_{i_0}(\eta) x_{i_0}(\eta) \right. \\
+ \sum_{j=1}^n \left. \sum_{l=1}^{l_{i,j}} b_{i,j,l}(\eta) x_j(\eta - \tau_{i,j,l}(\eta)) \right] \\
+ \sum_{j=1}^n \left[ \int_0^\infty b_{i,j}(\eta,s) x_j(\eta - s) \, ds \right] \, d\eta \quad \text{for } 0 \leq t < T_0.
\]

By (2.16), the limit of the left side of (2.19) is

\[
\lim_{t \to T_0} \int_0^t \frac{\dot{x}_{i_0}(\eta)}{b_{i_0}(x_{i_0}(\eta))} \, d\eta = \lim_{t \to T_0} \int_{x_{i_0}(0)}^{x_{i_0}(t)} \frac{du}{b_{i_0}(u)} = -\infty.
\]

On the other hand, the limit of the right side of (2.19) is a constant as \( t \to T_0 \). This is a contradiction and so \( x_i(t) > 0, i = 1, \ldots, n \), for \( t \geq 0 \).

The following theorem gives sufficient conditions for the solutions of (1.1)–(1.2) to be bounded on \([0, \infty)\).

**Theorem 3.** Assume that all assumptions, expect (ii), of Theorem 1 are satisfied. If

\[
(ii') \quad \int_1^{+\infty} \frac{1}{b_i(u)} \, du = +\infty
\]

for \( i = 1, 2, \ldots \) \( n \), then every solution of (1.1)–(1.2) is bounded on \( 0 \leq t < \infty \).

**Proof.** Let us consider the same function \( V(t) \) defined by (2.3) along any solution, say \((x_1(t), x_2(t), \ldots, x_n(t))'\), of (1.1)–(1.2). By (2.11), we know that

\[
V(t) \leq C \quad \text{for } t \geq 0,
\]

where \( C \) is a positive constant.

Therefore, we get

\[
\left| \int_{t_0}^{x_{i_0}(t)} \frac{ds}{b_i(s)} \right| \leq C' \quad \text{on } [0, \infty)
\]

for \( i = 1, 2, \ldots, n \), where \( C' > 0 \) is a constant.
Since $K_i(t)$ is bounded function on $[0, \infty)$, from (2.20), it follows that
\[ x_i(t) \text{ is bounded on } [0, \infty) \] (2.23)
for $i = 1, 2, \ldots, n$.

Now, we can state the following corollary.

**Corollary 4.** If, in addition to (A1)–(A4), (i)', (ii)', (iii), and (iv) are satisfied, then system (1.1) is globally asymptotically stable.

**Proof.** Clearly (i)', (ii)' imply (i), (ii) as required in Theorem 1, respectively. So, all conditions of Theorem 1 are satisfied.

**Remark 1.** The conclusion (2.2) implies, that for any two solutions $x_i(t)$ and $y_i(t)$, $i = 1, \ldots, n$, of (1.1) and (1.2), one has
\[ \lim_{t \to \infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n, \]
which means the global asymptotic stability of (1.1). This could be seen easily after writing the inequality
\[ 0 \leq |x_i(t) - y_i(t)| \leq |x_i(t) - K_i(t)| + |y_i(t) - K_i(t)| \]
for $i = 1, 2, \ldots, n$.

**Remark 2.** If $k_i(t) = 0$, $i = 1, 2, \ldots, n$, for $t \geq 0$, then Eq. (2.1) will be the same as (1.1). At the same time, (2.2) is reduced to the limit of the difference of two solutions of Eq. (1.1). Therefore, our method is more general than Kuang's one in [7]. It is worth noting that his method does not work in the infinite delay case.

**Remark 3.** In comparison with Kuang's assumption [7, iv]
\[ c_i m_{ii} > \sum_{j=1, j \neq i}^{n} c_j |m_{ji}|, \quad i = 1, \ldots, n, \]
which, in scalar case, has the form
\[ \lim_{t \to \infty} \inf a_i(t) > \lim_{t \to \infty} \sup |b_{11i}(t)| + \int_{-\gamma_{11}}^{0} \left( \lim_{t \to \infty} \sup |k_{11i}(t, s)| \right) ds, \quad (2.24) \]
we see that our assumption (iii) is weaker.

We have an example below to explain this case.
Example 1. Consider the scalar equation of the form

\[
\dot{x}(t) = x(t) \left[ \left( \frac{\pi}{6} + |\cos(t + \tau)| \right) \left( \frac{\pi}{4} + \arctan t \right) \right.
- (\cos t) \left( \frac{\pi}{4} + \arctan(t - \tau) \right)
- \left( \frac{\pi}{6} + |\cos(t + \tau)| \right) x(t) + (\cos t) x(t - \tau) \right]
\]  

(2.25)

with the initial condition

\[
x(\theta) = \varphi(\theta) \geq 0 \text{ on } -\tau \leq \theta \leq 0; \quad \varphi(0) > 0; \quad \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)| < \infty,
\]

(2.26)

where \( \tau \) is a nonnegative constant and \( \varphi \in C[-\tau, 0] \).

Equation (2.25) is a kind of (1.1) providing that \( b_1(x) = x; \ r_2(t) = (\pi/6 + |\cos(t + \tau)|) (\pi/4 + \arctan t) - (\cos t)(\pi/4 + \arctan(t - \tau)); \ a_3(t) = \pi/6 + |\cos(t + \tau)|; \ b_{111}(t) = \cos t; \ b_{33}(t, s) = 0; \ \tau_{111}(t) = \tau. \) Therefore, the assumptions (A1), (A2), (A3), (A4), (i), (ii), and (iii) are satisfied for (2.25)--(2.26). Moreover, there is a suitable function of the form \( K(t) = \pi/4 + \arctan t \) such that

\[
\dot{K}(t) = K(t) \left[ \left( \frac{\pi}{6} + |\cos(t + \tau)| \right) \left( \frac{\pi}{4} + \arctan t \right) \right.
- (\cos t) \left( \frac{\pi}{4} + \arctan(t - \tau) \right)
- \left( \frac{\pi}{6} + |\cos(t + \tau)| \right) K(t) + (\cos t) K(t - \tau) \right]
+ \frac{K(t)}{(\pi/4 + \arctan t)(1 + t^2)}
\]

(2.27)

for \( t \geq 0 \).

This equation is the form of (2.1) in which \( k(t) = 1/(\pi/4 + \arctan t) \) \((1 + t^2)\) is bounded on \([0, \infty)\) and \( k(t) \in L_1[0, \infty) \). Therefore, the last hypothesis (iv) of Theorem 1 holds. Hence, Eq. (2.25) is globally asymptotically stable. We note that this property cannot be obtained by [7, Theorem 2.1] because \( \liminf_{t \to -\infty} a_3(t) \) and \( \limsup_{t \to -\infty} |b_{111}(t)| \) are equal to \( \pi/6 \) and 1, respectively. So, (2.24) does not hold. By the way, we can clarify our hypothesis (iii) of Theorem 1 as \( \liminf_{t \to -\infty} [a_3(t) - |b_{111}(t + \tau)|] = \pi/6. \)
Corollary 5. Under the hypotheses of Theorem 1, except (iv), if there exists a constant vector \( K = (K_1, K_2, \ldots, K_n)^T \) such that \( K_i > 0 \) and
\[
\int_0^\infty \left| r_i(t) - a_i(t) K_i + \sum_{j=1}^n K_j \left( \sum_{l=1}^{l_i} b_{ij}(t) + \int_0^\infty b_{ij}(t, s) \, ds \right) \right| \, dt < \infty
\]
(2.28)
for \( i = 1, 2, \ldots, n \), then
\[
x_i(t) - K_i \to 0 \quad \text{as} \quad t \to \infty.
\]
(2.29)

Proof. We can let \( K_1(t) \equiv K_1 \) and \( k_i(t) = -r_i(t) + a_i(t) K_i - \sum_{j=1}^n K_j (S_i \sum_{l=1}^{l_i} b_{ij}(t) + \int_0^\infty b_{ij}(t, s) \, ds) \) in (2.1) because \( K_i > 0 \) and \( k_i(t) \) is bounded and also \( k_i(t) \in L_{[0, \infty)} \) for \( i = 1, 2, \ldots, n \). So, the corollary is a simple consequence of Theorem 1.

Remark 4. A result of Freedman and Wu [1] states that if \( a(t), b(t), c(t), \tau(t) \) are continuously differentiable, \( \omega \)-periodic functions, and \( a(t) > 0, b(t) > 0, c(t) > 0, \tau(t) \geq 0 \) for \( t \in (-\infty, \infty) \), and the equation
\[
a(t) - b(t) K(t) + c(t) K(t - \tau(t)) = 0
\]
(2.30)
has a positive, \( \omega \)-periodic, continuously differentiable solution \( K(t) \), then the scalar equation
\[
\dot{x}(t) = x(t) \left[ a(t) - b(t) x(t) + c(t) x(t - \tau(t)) \right]
\]
(2.31)
has a positive \( \omega \)-periodic solution \( Q(t) \). Moreover, if \( b(t) > c(t)(Q(t - \tau(t))/Q(t)) \) for all \( t \in [0, \omega] \), then \( Q(t) \) is globally asymptotically stable with respect to positive solutions of (2.31). We note that when \( K(t) \) is a constant in (2.1) and (2.30) as well, then Eq. (2.30) can be obtained from Eq. (2.1) with \( k(t) \equiv 0 \). Furthermore, the conditions of Theorem 1, when \( n = 1 \), are weaker than the ones mentioned above. For instance, in the example
\[
\dot{x}(t) = x(t) \left[ r(t) - x(t) + \frac{1}{2} x(t - \frac{\pi}{2}) \right]
\]
(2.32)
where \( r(t) = \cos t/(2 + \sin t) + 1 + \sin t + \cos t/2 \). The result of [1] does not work for (2.32) because \( r(t) \) is not positive on \([0, \infty)\). But our Theorem 1 where \( n = 1 \) can be applied to conclude that all solutions of (2.32) are
globally asymptotically stable. Indeed, in this case $K(t) = 2 + \sin t$ is a bounded and positive solution of the equation

$$
\dot{K}(t) = K(t) \left[ r(t) - K(t) + \frac{1}{2} K \left( t - \frac{\pi}{2} \right) \right],
$$

where $k(t) \equiv 0$. It is worth noting that solutions of Eq. (2.32) are asymptotically periodic functions.

Now, we show the following example which consists of infinite delay.

**Example 2.** Consider the scalar equation

$$
\dot{x}(t) = x(t) \left[ \left( \frac{3}{2} + \frac{1}{1 + t^2} - \frac{\pi}{\sqrt{3} + 4t} \right) 
+ \frac{\pi}{2\sqrt{1 + t}} + \frac{2}{\sqrt{3} + 4t} \arctan \frac{-1}{\sqrt{3} + 4t} - \cos t \right]
- \left( \frac{3}{2} + \frac{\pi}{2\sqrt{1 + t}} \right) x(t) + (\cos t) x(t - \tau) + \int_0^\infty \frac{x(t - s)}{s^2 - s + t + 1} ds.
$$

(2.33)

Taking $b_2(x) = x$, $r_2(t) = 3/2 + 1/(1 + t^2) - \pi/\sqrt{3} + 4t + \pi/2\sqrt{1 + t} + (2/\sqrt{3} + 4t) \arctan (-1/\sqrt{3} + 4t) - \cos t$, $a_2(t) = 3/2 + \pi/2\sqrt{1 + t}$, $b_{111}(t) = \cos t$, $\tau_{111}(t) = \tau$ ($\tau$ is a non-negative constant) and $b_{111}(t, s) = 1/(s^2 - s + t + 1)$ in (1.1), then the assumptions (A1), (A2), (A3), (A4), (i), (ii), and (iii) of Theorem 1 hold for (2.33) with (2.26). On the other hand, the function $K(t) = 1$ satisfies the equation

$$
\dot{K}(t) = K(t) \left[ r_1(t) - \left( \frac{3}{2} + \frac{\pi}{2\sqrt{1 + t}} \right) K(t) + (\cos t) K(t - \tau) 
+ \int_0^\infty \frac{K(t - s)}{s^2 - s + t + 1} ds \right] - \frac{1}{1 + t^2} K(t)
$$

for $t \geq 0$. It should be noted that this equation is the same as Eq. (2.1) with $k_1(t) = -1/(1 + t^2)$ which is bounded for $t \geq 0$ and belongs to $L_1(0, \infty)$. So the hypothesis (iv) is also satisfied. Hence, although 1 is not a saturated equilibrium of (2.33), $|x(t) - 1| \to 0$ as $t \to \infty$ where $x(t)$ is a solution of (2.33)–(2.26).
REFERENCES