Minimal Algebra Resolutions for Cyclic Modules Defined by Huneke–Ulrich Ideals

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INTRODUCTION

In [H-U], Huneke and Ulrich defined a class of non-trivial deviation two Gorenstein ideals, which were the first large class of such ideals to be defined. These ideals were subsequently studied by Kustin [K1] and he constructed the minimal resolutions for these ideals. In this paper, we construct an algebra structure on the minimal resolutions of the cyclic modules defined by these ideals.

Throughout this paper $R$ will denote a commutative noetherian local ring with maximal ideal $m$ and $I$, an ideal of $R$. A free resolution

$$F: \cdots \rightarrow F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{d_1} R \rightarrow R/I \rightarrow 0$$

of $R/I$ is called minimal if $d(F) \subseteq m F$. $F$ is said to have an algebra structure if the differential graded $R$-module $F = \bigoplus_{i \geq 0} F_i$ admits a multiplication $F \times F \rightarrow F$ which gives $F$ the structure of an associative commutative differential graded $R$-algebra.

Let $F$ be the minimal resolution of $R/I$. $I$ is called Gorenstein if $F_n = R$ and $F_m = 0$ for $m > n$, where grade $I = n$. The deviation of $I$ is the minimal number of generators of $I$ minus the grade of $I$. Thus, the ideals of deviation zero are complete intersections and in this case, the minimal resolution is the well known koszul complex which has an algebra structure (the exterior algebra). Kunz [KZ] has shown that a Gorenstein ideal cannot have deviation one. All grade three Gorenstein ideals [B-F-3] and grade four Gorenstein ideals [K-M] have minimal algebra resolutions. In fact, all grade three Gorenstein ideals are ideals generated by the $(n-1)$ order pfaffians of an $n \times n$ alternating matrix. A Gorenstein ideal of grade $g$ is called
trivial if it is a hypersurface section of a Gorenstein ideal of grade \(g - 1\). All deviation two Gorenstein ideals of grade four are trivial [H-M, V-V].

The first known class of non-trivial deviation two Gorenstein ideals are those defined by Huneke and Ulrich [H-U], which can be described as follows: Let \(X\) be a \(2n \times 2n\) generic alternating matrix and \(Y\) a generic \(1 \times 2n\) matrix over \(R\). Then \(I_n = I_1(YX) + (\text{Pf}(X))\). \(I_n\) is of grade \(2n - 1\) and is in the linkage class of complete intersection. It can be shown (Corollary 2.8) that the assumption of \(X\) and \(Y\) being generic is not necessary. That is, even if \(X\) and \(Y\) are not generic, the ideals \(I_n = I_1(YX) + (\text{Pf}(X))\) are Gorenstein and of deviation two, provided that the grade \(I_n = 2n - 1\). We call all ideals of odd grade \(g\), given as above, by an alternating matrix \(X\) of order \(g + 1\) and a \(1 \times (g + 1)\) matrix \(Y\), Huneke–Ulrich ideals. In codimension five [L], these are essentially, all the non-trivial deviation two Gorenstein ideals which are in the linkage class of complete intersection.

In view of the fact that there are cyclic modules which do not possess any minimal algebra resolutions [AV], one may ask the following. Do the cyclic modules defined by ideals in the linkage class of a complete intersection have a minimal algebra resolution? Avramov, Kustin, and Miller [A-K-M, K-M-2] have answered this in the affirmative if the ideal is linked in one step or Gorenstein and linked in two steps to a complete intersection. In view of Lopez's [L] result, our Theorem 6.3 gives an affirmative answer to the question in the case of deviation two, grade five ideals. In general our result exhibits a non-trivial class of licci ideals whose cyclic modules do possess minimal algebra resolutions.

After establishing the notations, definition of Huneke–Ulrich ideals and some preliminary results in Section 1, we construct a minimal algebra resolution (Theorem 2.7, and Definitions 3.1, 3.4) for Huneke–Ulrich deviation two Gorenstein ideals in Sections 2 and 3 explicitly. This we do by first constructing an algebra structure on a non-minimal resolution \(F\). We then "restrict" it to the minimal resolution using Theorem 6.1, which gives a sufficient condition for an algebra structure on a non-minimal resolution to induce one on the minimal resolution. Some useful results on multilinear algebra are developed in Section 4. Sections 5 and 6 are mainly devoted to proving that the multiplication defined in Section 3 does indeed give an algebra structure on the minimal resolution of the Huneke–Ulrich deviation two ideals. We close with an intriguing binomial identity.

1. PRELIMINARIES

Let \(R\) be a commutative noetherian local ring. \(F\) is a free \(R\)-module of rank \(2n\) and \(F^*\) is the dual of \(F\). The diagonal map \(\Delta: \wedge F \to \wedge F \otimes \wedge F\) is the map of algebras given by, for \(a \in F\), \(\Delta(a) = a \otimes 1 + 1 \oplus a \in \wedge F \otimes \wedge F\).
Let $e_1, e_2, \ldots, e_{2n}$ be a basis for $F^*$. Choose $\eta = e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}$ and $\xi = e_1^* \wedge e_2^* \wedge \cdots \wedge e_{2n}^*$ as orientations for $F$ and $F^*$, respectively. If $I = (i_1, \ldots, i_t)$ is an ordered $t$-tuple of positive integers less than or equal to $2n$, then $e_I$ denotes the element $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} \in \wedge^t F$ and $e_I^*$ denotes $e_{i_1}^* \wedge \cdots \wedge e_{i_t}^* \in \wedge^t F^*$.

The algebra $\wedge F$ and $\wedge F^*$ are graded modules over each other [B-E-3], so that $\zeta(\eta) = (-1)^{n} \in R$. For $a, \ c \in \wedge F$; $b \in \wedge F^*$, $a(b)(c)$ denotes $(a(b))(c)$. Further $\wedge F$ is an algebra with divided powers and for $a \in \wedge F$ of even degree, for any integer $t \geq 0$, the divided power $a^{\ell(t)}$ is denoted by $a^{\ell(t)}$.

**Definition 1.1.** An ideal $I_n$ of $R$ of grade $2n-1$ is called a Huneke-Ulrich ideal if there exist $\phi \in \wedge^2 F$ and $Y \in F^*$, such that $I_n = Y(\phi)(F^*) + \phi^{(n)}(\xi)$.

**Remark 1.2.** We will see later, in Corollary 2.8, that if $\phi \in \wedge^2 F$ and $Y \in F^*$ are such that the ideal $I_n = Y(\phi)(F^*) + \phi^{(n)}(\xi)$ has grade $2n-1$, then $I_n$ must necessarily be Gorenstein and of deviation 2.

Now, let $X = (x_{ij})$ be a $2n \times 2n$ generic alternating matrix and $Y$ a generic $1 \times 2n$ matrix. The pfaffian of $X$ is a polynomial $Pf(X)$ in $x_{ij}$ such that $(Pf(X))^2 = \text{determinant of } X$. Consider the ring $S = \mathbb{Z}[X, Y]$. Huneke and Ulrich [H-U] have shown that the ideal $J_n$, generated by the entries of the matrix $XY$, together with the pfaffian of $X$ is a non-trivial deviation two Gorenstein prime ideal of grade $2n-1$. Suppose $\tilde{F}$ is a free $S$-module of rank $2n$ with basis $e_1, e_2, \ldots, e_{2n}$ and $\tilde{F}^*$ its dual with the dual basis $e_1^*, e_2^*, \ldots, e_{2n}^*$, and orientation $\xi = e_1^* \wedge e_2^* \wedge \cdots \wedge e_{2n}^*$. If we let $\tilde{\phi} = \sum_{j<i,j,j=1}^{2n} x_{ij} e_i \wedge e_j$ and $\tilde{Y} = \sum_{i=1}^{2n} y_i e_i^*$, then $J_n$ is precisely the ideal $\tilde{Y}(\tilde{\phi})(F^*) + \tilde{\phi}^{(n)}(\xi)$.

Thus $J_n$ is a Huneke-Ulrich deviation 2 Gorenstein ideal. Further, all such ideals are specializations of the ideal $J_n$ of the generic case. To see this, let $I_n$ be a Huneke-Ulrich deviation two Gorenstein ideal of grade $2n-1$ given by $\phi \in \wedge^2 F$ and $Y \in F^*$. Let $\phi = \sum_{i<j} a_{ij} e_i \wedge e_j$ and $Y = \sum_{i=1}^{2n} b_i e_i^*$. Define $s: S \to R$ by $s(x_{ij}) = a_{ij}$ and $s(y_i) = b_i$ for all $i$ and $j$. Then $s(\tilde{F}) = \tilde{F} \otimes_s R = F$, $s(\tilde{Y}) = Y$, $s(\tilde{\phi}) = \phi$, and hence $s(J_n) = I_n$. Henceforth, $\sim$ will always denote the generic case. So, $\tilde{R} = S$, $\tilde{I}_n = J_n$. Let $Y(\phi) = g \in F$; so that $I_n = g(F^*) + \phi^{(n)}(\xi)$.

Finally, recall that for any integer $n$ and $r$,

$$\binom{n}{r} = \frac{n!}{r! (n-r)!} \quad \text{if } n \geq r \geq 0$$

$$= 0 \quad \text{if } r < 0 \text{ or } 0 \leq n < r$$

$$= (-1)^{r} \binom{r-n-1}{r} \quad \text{if } n < 0.$$ 

In particular, $\binom{r}{r} = (\frac{1}{r})$ for all $r \geq 0$. 
2. The Minimal Resolution of $R/I_n$

We will define a complex $F$ which resolves $R/I_n$ giving a non-minimal resolution of $R/I_n$. The minimal resolution $M$ will then be obtained as a subcomplex of $F$.

DEFINITION 2.1. Given $\phi \in \bigwedge^2 F$ and $t$, a positive integer, $Q_t^\phi = Q_t : \bigwedge F \to \bigwedge F^*$ is an $R$-module homomorphism defined as follows. For $a \in \bigwedge^i F$,

$$Q_t(a) = \sum_{i + j \text{ even}} \left( \sum_{j=0}^{t-1} (-1)^{n+1+j(i+j)/2} \binom{n-i+j}{2} \frac{t-i-1}{2} \right) \phi^{(n-(i+j)/2)} \wedge a(\xi).$$

REMARK 2.2. (i) Clearly, $Q_t(a) = 0$ if $\deg a = i > t$, since in that case $t-i-1 < 0$.

(ii) $Q_t(a) \subseteq \bigoplus_{j=0}^{\min(t-1,2n-t)} \bigwedge^j F^* \bigoplus \bigwedge^{2n-\deg a} F^*$.

Proof. We note that $\phi^{(n-(i+j)/2)} \wedge a(\xi) \in \bigwedge^j F^*$ and that $t > \deg a = i$.

If $j > 2n - t$, then $(t-i-1)/2 > n-(i+j)/2 - 1$. Hence when $j > 2n - t$,

$$\left( \frac{n-i+j}{2}, \frac{t-i-1}{2} \right) = 0$$

unless $n-(i+j)/2-1 = -1$.

But, $n-(i+j)/2-1 = -1 \Rightarrow j = 2n-i$. So,

$$\sum_{j=0}^{t-1} \phi^{(n-(i+j)/2)} \wedge a(\xi)$$

$$= \sum_{j=0}^{\min(t-1,2n-t)} \phi^{(n-(i+j)/2)} \wedge a(\xi)$$

$$\in \bigoplus_{j=0}^{\min(t-1,2n-t)} \bigwedge^j F^* \bigoplus \bigwedge^{2n-i} F^*.$$
We will first construct the resolutions $\mathcal{F}$ and $\mathcal{M}$ in the generic case. Let $\mathcal{F}, \phi, \mathcal{Y}, \mathcal{g} = \mathcal{Y}(\phi)$ be as before.

**Definition 2.3.**

$$\mathcal{F} = \cdots \rightarrow \mathcal{F}_t \xrightarrow{\partial_t} \mathcal{F}_{t-1} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{R}$$

is the complex, where

$$\mathcal{F}_t = \bigwedge^t \mathcal{F}^* \oplus \bigwedge^{t-1} \mathcal{F} \oplus \bigwedge^{t-2} \mathcal{F}^* \oplus \cdots$$

and $\partial_t = \mathcal{F}_t \rightarrow \mathcal{F}_{t-1}$ is given by

$$\partial_t(a) = \mathcal{g}(a), \quad a \in \bigwedge^i \mathcal{F}^*$$

$$= \mathcal{g}(a) + \mathcal{Y} \wedge a, \quad a \in \bigwedge^i \mathcal{F}^*, i < t$$

$$= (-1)^{t+1} \left[ \mathcal{Y}(a) + Q_i(a) \right], \quad a \in \bigwedge^{t-1} \mathcal{F},$$

$$= (-1)^{t+1} \left[ \mathcal{Y}(a) + g \wedge a \right] + Q_i(a), \quad \text{if} \quad a \in \bigwedge^i \mathcal{F}, i < t - 1.$$ 

The terms $\mathcal{Y} \wedge a$ and $\mathcal{g} \wedge a$ are dropped in the top degrees, namely $t$ and $t - 1$, respectively. It can be seen by direct computations that $(\mathcal{F}, \partial)$ is a complex and $H_t(\mathcal{R}) = \mathcal{R}/I_n$. For any element $a \in \mathcal{F} = \bigoplus_{t > 0} \mathcal{F}_t \subseteq \bigwedge \mathcal{F}^* \oplus \bigwedge \mathcal{F}$ has 2 gradings. $\mathrm{deg}_F(a)$ denotes the degree in the graded module $\mathcal{F}$ and $\mathrm{deg} a$ denotes the degree in the exterior algebra. Thus for $a \in \bigwedge^i \mathcal{F} \subseteq \mathcal{F}_t$, $\mathrm{deg} a = i$, and $\mathrm{deg}_F(a) = t$.

**Definition 2.4.** $\mathcal{M}$ is the complex

$$\begin{array}{c}
0 \rightarrow \mathcal{M}_{2n-1} \rightarrow \cdots \rightarrow \mathcal{M}_t \xrightarrow{\partial_t} \mathcal{M}_{t-1} \\
\rightarrow \cdots \rightarrow \mathcal{M}_1 \rightarrow \mathcal{R},
\end{array}$$

where

$$\mathcal{M}_t = \mathcal{F}_t, \quad 0 \leq t < n$$

$$= \mathcal{F}_{2n-t-1}^*, \quad t > n$$

and $\partial_t = \partial_t$ on $\mathcal{F}_t$ restricted to $\mathcal{M}_t$.

**Lemma 2.5.** $\mathcal{M}$ in Definition 2.4 is well-defined and is minimal in the sense $d(\mathcal{M}) \subseteq m\mathcal{M}$, where $m = (X, Y)S$. 

Proof. Since, for $t > n$, 
\[
\tilde{F}_{2n-t-1}^* = \bigoplus_{i=0}^{2n-t-1} \tilde{F}_{i} \oplus \bigoplus_{i=0}^{2n-t-2} \tilde{F}_{i+1}^* \subseteq \tilde{F}_t,
\]
\[
\hat{M}_t \subseteq \tilde{F}_t \text{ for all } t. \text{ Further by Remark 2.2 and definition of } d_t, Q_t(\hat{M}_t \cap \wedge F) \subseteq \hat{M}_{t-1}, d_t(\hat{M}_t) \subseteq \hat{M}_{t-1}. \text{ Hence } d_t \text{ restricted to } \hat{M}_t \text{ makes } \hat{M} \text{ a subcomplex of } F. \text{ Also, if } \alpha \in \wedge F \subseteq \tilde{M}_t \text{ then } \deg \alpha = i \leq \min\{t, 2n-t-1\}.
\]
\[
\text{If } j = \min\{t-1, 2n-t\}, \text{ then } i+j < 2n. \text{ Thus for all } \alpha \in \wedge F \subseteq \tilde{M}_t, 
\]
\[
\phi^{(n-i+1/2)}(\alpha) \subseteq m \wedge \tilde{F}^*
\]
for all $j \leq \min\{t-1, 2n-t\}$. Hence $Q_t(\alpha) \subseteq m \hat{M}_{t-1}$ for all $t$. Since all other maps in $d_t$ are minimal, this proves that the complex $\hat{M}$ is minimal. 

The exactness of $F$ and $\hat{M}$ follows once we identify them as complexes $F$ and $\hat{M}$ in $[K]$. This we do via the isomorphism $T: F^* \to \hat{F}$, where $T(e_i^*) = e_i$ for all $i$. It is then clear that the free modules in the complexes $\hat{F}$ and $\hat{M}$ are isomorphic to those in $F$ and $M$ of $[K]$, respectively. With some work (using the identities in 4), it can be seen that the maps are the same. So $\hat{F}$ and $\hat{M}$ are isomorphic to $F$ and $M$ in $[K]$. Thus by theorem of $[K]$ we get

**Theorem 2.6.** $F$ and $\hat{M}$ are exact. Further, $H_0(\hat{F}) = H_0(\hat{M}) = R/I$, and $\hat{M}$ is the minimal resolution of $R/I$. 

Now, let $I_n$ be a Huneke–Ulrich deviation 2 ideal of $R$, given by a free module $F$, $\phi \in \wedge^2 F$, $Y \in F^*$. Let $s: R \to R$ be the specialization of $R$ to $R$, where $s(\tilde{S}) = \phi$, $s(\tilde{Y}) = Y$. Let $F = \hat{F} \otimes_R R$ and $M = \hat{M} \otimes_R R$ be specializations of $\hat{F}$ and $\hat{M}$, respectively.

**Theorem 2.7.** Let $I_n$ be a Huneke–Ulrich deviation two Gorenstein ideal given by $\phi \in \wedge^2 F$ and $Y \in F^*$. Then $M$ is the minimal free resolution of $R/I_n$. 

Proof. $(\hat{M}, \hat{d})$ is the minimal resolution of $I_n$ of $R$ in the generic case. Further, $I_n$ is prime and Gorenstein [H-U]. Hence by [B-E-1-S-1], the radical of the fitting ideals, $\sqrt{I(\hat{d})} = I_n$ for all $t$. Also, if rank $\hat{d}_t = r_t$, then 
\[
\sqrt{I_n(d)} = \sqrt{s(I_n(\hat{d}_t))} = s(\sqrt{I(\hat{d}_t)}) = s(I_n) = I_n
\]
for all $t$. But depth $I_n \geq 2n - 1$ by hypothesis. Thus depth $\sqrt{I(\hat{d}_t)} \geq 2n - 1$. By the exactness criterion [B-E-1], $M$ is exact.
**Corollary 2.8.** The ideal \( I_n = g(F^*) + \phi^{(n)}(\xi) \) is Gorenstein of deviation two for all \( \phi \) and \( Y \), provided grade \( I_n \geq 2n - 1 \).

**Proof.** By Theorem 2.7, \( M \) is the minimal free resolution of \( I_n \), length of \( M = 2n - 1 \) and \( M_{2n-1} = R \). So \( I_n \) is Gorenstein. \( I_n \) has deviation two because rank \( M_1 = \text{rank} \ F + 1 = 2n + 1 = \text{grade} \ I_n + 2 \).

### 3. Definition of the Multiplication on \( R/I_n \)

Let \( \phi, Y, g = Y(\phi) \), and \( F \) be as before and \( I_n = g(F^*) + \phi^{(n)}(\xi) \). We will first define a multiplication \( \mu : F \to F \) and for any \( a, b \in F \), we write \( a \cdot b \) or \( ab \) for \( \mu(a, b) \). It is enough to define \( \mu \) on the basis elements.

**Definition 3.1.** The multiplication \( \mu \) is determined by the following rules. Let \( a, b \in F \subseteq \wedge F \oplus \wedge F^* \).

1. If \( a, b \in \wedge F \), \( a \cdot b = 0 \).
2. If \( a, b \in \wedge F^* \) with \( a \in \wedge^i F^* \subseteq F_i \), \( b \in \wedge^j F^* \subseteq F_j \), define
   
   \[
   a \cdot b = \left( \frac{t - i + s - j}{2} \right) a \wedge b \in \wedge^{i + j} F^* \subseteq F_{i+j}.
   \]

   This is well defined because \( a \in \wedge^i F^* \subseteq F_i \) implies \( t - i \) is even and similarly \( b \in \wedge^j F^* \subseteq F_j \) implies \( s - j \) is even.

   It only remains to define \( a \cdot b \) when one of them is in \( \wedge F \) and the other in \( \wedge F^* \). We will define \( a \cdot b \) where \( a \in \wedge F \) and \( b \in \wedge F^* \).

   In view of (2), \( \wedge F^* \subset F \) is generated as a divided power algebra over \( R \) by the basis \( E = \{ e_1^*, e_2^*, ..., e_n^*, h \} \) where \( e_1^* \in F^* \subseteq F_1 \) and \( h = 1 \in F_2 \); so that \( \deg e = h = 2 \). Order \( E \) by setting \( e_1^* < e_2^* < \cdots < e_n^* < h \). Now let \( a \in \wedge^i F \subseteq F_i \).

3. If \( b = e^* \in F_1 \),
   
   \[
   a \cdot b = -e^*(a \wedge \phi^{(t-i+1)/2}) - e^*(a) \in \wedge^i F \oplus \wedge^{i-1} F \subseteq F_{i+1}.
   \]

4. If \( b = h = 1 \in F_2 \),
   
   \[
   a \cdot b = a \cdot h = (n - t + \frac{t - i - 1}{2}) a - (n - t - 1) a \wedge \phi^{(t-i+1)/2} \in \wedge^i F \oplus \wedge^{i+1} F \subseteq F_{i+2}.
   \]
(5) If \( b \in \bigwedge F^* \subseteq F_s \), then

\[ b = r b_1 \cdot b_2 \cdots \cdot b_k, \quad r \in R, \quad b_i \in E, \quad \text{and} \quad b_1 \leq b_2 \leq \cdots \leq b_k. \]

Then

\[ ab = (\cdots ((ab_1)b_2) \cdots )b_k \in F_{1+s}. \]

(6) Finally if \( b \in \bigwedge F \) and \( a \in \bigwedge F^* \), we define

\[ a \cdot b = (-1)^{\text{deg}(a) \cdot \text{deg}(b)} b \cdot a. \]

Definition 3.1 gives a multiplication on \( F \). The multiplication is clearly graded commutative by rules (1), (2), and (6). We will now proceed to show that this multiplication is associative. To begin with, Lemmas 3.2 and 3.3 establish that the order in which \( b_i \)'s are multiplied in (5), does not matter.

**Lemma 3.2.** Let \( a \in \bigwedge F \subseteq F_t \). If \( b = h \in F_2 \), \( c = e \in F^* \), then \((ab)c = (ac)b\).

**Proof.**

\[
(ab)c = \left[\left(n-t+\frac{t-i-1}{2}\right)a -(n-t-1)a \wedge \phi^{(t-i+1)/2}\right]c, \quad \text{by rule (4)}
\]

\[
= \left[-\left(n-t+\frac{t-i-1}{2}\right)e(a \wedge \phi^{(t+2-i+1)/2})
+ (n-t-1)e(a \wedge \phi^{(t-i+1)/2} \wedge \phi)
- (n-t-1)e(a \wedge \phi^{(t-i+1)/2})
+ \left(n-t+\frac{t-i-1}{2}\right)e(a)\right], \quad \text{by rule (3)}
\]

\[
= \left[e(a \wedge \phi^{(t-i+3)/2})\left[(n-t-1)\left(\frac{t-i+3}{2}\right) - \left(n-t+\frac{t-i-1}{2}\right)\right]
- (n-t-1)e(a \wedge \phi^{(t-i+1)/2}) + \left(n-t+\frac{t-i-1}{2}\right)e(a)\right]
\]

\[
e(a \wedge \phi^{(t-i+3)/2})(n-t-2)\left(\frac{t-i+1}{2}\right)
- (n-t-1)e(a \wedge \phi^{(t-i+1)/2}) + \left(n-t+\frac{t-i-1}{2}\right)e(a). \]
On the other hand applying rules (3) and (4) in that order,

\[(ac)b = \left[ -e(a \wedge \phi^{(r-i+1/2)}) + e(a) \right]b \]

\[= \left[ -(n-t-2)e(a) \wedge \phi^{(t+1-i+1/2)} \right. \]
\[+ (n-t-2)e(a \wedge \phi^{(t-i+1/2)}) \wedge \phi \]
\[- \left( (n-t-1) + \left( \frac{t+1-t-1}{2} \right) \right) e(a \wedge \phi^{(t-i+1/2)}) \]
\[+ \left( n-t-1 + \frac{t+1-i+1-1}{2} \right) e(a) \]
\[= \left[ (n-t-2) \left( \frac{t-i+1}{2} \right) e(a \wedge \phi^{(t-i+3/2)}) \right. \]
\[ - (n-t-1) e(a \wedge \phi^{(t-i+1/2)}) \left. + \left( n-t + \frac{t-i-1}{2} \right) e(a) \right] \]

Thus \((ac)b\).

LEMMA 3.3. If \(a \in \wedge^i F \subseteq F_r\), \(b = e^*_r \in F_1\), \(c = e^*_m \in F_1\), then

\[(ab)c = -(ac)b.\]

Proof. Applying rule (3) repeatedly, we get

\[(ab)c = -e^*_m(e^*_r(a) \wedge \phi^{(t-i+3/2)}) + e^*_m(e^*_r(a \wedge \phi^{(t-i+1/2)}) \wedge \phi)\]

\[- e^*_m(e^*_r(a \wedge \phi^{(t-i+1/2)})) + e^*_m(e^*_r(a))\]

\[= -e^*_r \wedge e^*_m(a \wedge \phi^{(t-i+3/2)}) \left( \frac{t-i+1}{2} \right) \]
\[+ e^*_r \wedge e^*_m(a \wedge \phi^{(t-i+1/2)}) - e^*_r \wedge e^*_m(a).\]

Clearly, \((ab)c = -(ac)b\) as \(e^*_r \wedge e^*_m = -e^*_m \wedge e^*_r\).

LEMMA 3.4. If \(a \in \wedge F \subseteq F_r\), \(b \in \wedge F^* \in F_s\), and \(b = b_1b_2 \cdots b_k \) where \(b_i \in E\), then \(ab = ((ab_1)b_2) \cdots b_k\) even if \(b_i \in E\) are not in ascending order with respect to the ordering on \(E\), defined in 3.1. Hence we can omit the parentheses.

Proof. Follows from repeated applications of Lemmas 3.2 and 3.3.

As a result of Lemmas 3.2 and 3.3 we now have closed forms for \(ab\) if \(a \in \wedge F\) and \(b \in \wedge F^*\) are basis elements. We record these as Corollaries 3.5, 3.6 for future reference.
Corollary 3.5. Let \( a \in \bigwedge^i F \subseteq F_t \) and \( b \in \bigwedge^i F^* \subseteq F_s \), then

\[
a \cdot b = \left[ \sum_{k=0}^{s-1} (-1)^{s(s+1)/2+k+1} b(a \wedge \phi^{((t-i+1)/2+k)}) + \frac{t-i-1}{2} + k \right]
\]

\[
+ \left( \frac{t-i-1}{2} + k \right) b(a)
\]

Corollary 3.6. If \( a \in \bigwedge^i F \subseteq F_t \), \( b = 1 \in F_s \), \( s \) even, then

\[
a \cdot b = \left[ \sum_{k=0}^{s/2-1} (-1)^{s+1} \left( \frac{n-t-1-k}{2} \right) \left( \frac{t-i-1}{2} + k \right) a \wedge \phi^{((t-i+1)/2+k)} + \left( \frac{n-t+\frac{t-i-1}{2}}{2} \right) \right]
\]

Both corollaries can be proved by induction on \( s \), the deg \( F \). The details are omitted. Now we return to the associativity of the multiplication.

Theorem 3.7. The multiplication \( \mu : F \times F \rightarrow F \) defined in 3.1 is associative on \( F \).

Proof. Let \( a \in F_t \), \( b \in F_s \), and \( c \in F_n \) be basis elements in \( F \). Since \( ab = 0 \) for any \( a, b \in \bigwedge F \), it suffices to prove \( (ab)c = a(bc) \) in two cases: (i) \( a, b, c \in \bigwedge F^* \) and (iii) \( a \in \bigwedge F, b, c \in \bigwedge F^* \).

(i) If \( a \in \bigwedge^i F^* \subseteq F_t \), \( b \in \bigwedge^i F^* \subseteq F_s \), and \( c \in \bigwedge^k F^* \subseteq F_r \), then it is easy to see that

\[
abc = \left( \frac{t+s+r-i-j-k}{2} \right) a \wedge b \wedge c.
\]

Hence \( (ab)c = a(bc) \).
(ii) Now let \( a \in \wedge^{1} F \subseteq F_{t} \), \( b, c \in \wedge^{*} F^{*} \) be basis elements of \( F \). Then \( b = b_{1} b_{2} \cdots b_{t} \) and \( c = c_{1} c_{2} \cdots c_{m} \) where

\[
b_{i}, c_{i} \in E = \{ e_{1}^{*}, \ldots, e_{2n}^{*}, h = 1 \in F_{2} \}, \quad \text{for all } i.
\]

\[
a(bc) = a(b_{1} \cdots b_{t} c_{1} c_{2} \cdots c_{m}).
\]

By Lemma 3.4,

\[
a(bc) = \left( (\cdots ((a b_{1}) b_{2}) \cdots) b_{t} \right) c_{1} \cdots c_{m} = (ab)c.
\]

This proves that \( \mu \) is associative on \( F \). 

To define a multiplication on \( M \), we observe that \( i: M \to F \), the inclusion splits as a map of complexes. However, the trivial projection \( p: F \to M \), given by \( p(a) = 0 \) if \( a \) is homogeneous and \( a \notin M \) and \( p(a) = a \) if \( a \in M \) is not a map of complexes. For, if \( a \in \wedge^{*} F^{*} \subseteq F_{n} \), \( p(a) = 0 \), but \( p(d_{n}(a)) = d_{n}(a) \neq 0 \). The complex map \( F \to F^{*} M \) which gives the necessary splitting for \( 'i' \) is defined as follows:

**Definition 3.8.** For all \( t \geq 0 \), \( \pi': F_{t} \to F_{t} \) is given by

\[
\pi'(a) = a \quad \text{if } a \in M_{t} \text{ or } a \in \wedge^{*} F
\]

\[
= (-1)^{(t-1)/2} a^{*} + (d_{t+1}(a(\eta)) - a(\eta)^{*}) a \quad \text{if } a \in \wedge^{*} F^{*}, a \notin M_{t};
\]

where \( a(\eta)^{*} \in \wedge^{*} F^{*} \) is obtained via \( F \to F^{*} \) which takes \( e_{i} \) to \( e_{i}^{*} \) for all \( i \). Define \( \pi: F_{t} \to M_{t} \), by \( \pi = p \circ \pi' \).

**Proposition 3.9.** \( \pi \) is a map of complexes and \( \pi \circ i = id_{M} \), so that \( i: M \to F \) splits as a map of complexes.

**Proof.** Let \( G_{t} = F_{t} \) for all \( t \) and let \( a \) be an arbitrary homogeneous element in \( G_{t} \). Define \( g_{t}: G_{t} \to G_{t-1} \) by

\[
g_{t}(a) = d_{t}(a), \quad \text{if } a \in M_{t},
\]

\[
= a^{*}, \quad \text{if } a \in \wedge F, a \notin M,
\]

\[
= 0, \quad \text{if } a \notin M_{t}, a \in \wedge F^{*}.
\]

Clearly, \( g_{t+1} \circ g_{t} = 0 \). Hence \( G = \bigoplus_{t \geq 0} G_{t} \) is a complex.

**Claim \( \pi' \).** \( F_{t} \to F_{t} = G_{t} \) gives an isomorphism of complexes \( F \) and \( G \) and hence, in particular, is a map of complexes.
Proof of Claim. We first show that $\pi'$ is an isomorphism. Define $(\pi')^{-1}: G_i \to F_i$ for any homogeneous $a \in G_i$, by

$$(\pi')^{-1}(a) = a, \quad \text{if } a \in M_i \text{ or } \wedge F_i$$

$$= d_{t+1}(a^*), \quad \text{if } a \in \wedge F^*, a \not\in M_i.$$ 

Then clearly $\pi' \circ (\pi')^{-1}$ and $(\pi')^{-1} \circ \pi'$ are identity on $M_i$ and $\wedge F_i$. We just need to check for $a \in \wedge F^*, a \not\in M_i$.

Let $a \in \wedge i F^*, a \not\in M_i$, so that $t \geq i > 2n - t - 1$ and $t \geq n$ and $t - i$ is even. Then

$$\pi'( (\pi')^{-1}(a) ) = \pi'(d_{t+1}(a^*))$$

$$= \pi'(d_{t+1}(a^*) - Q_{t+1}(a^*)) + \pi'(Q_{t+1}(a^*))$$

$$= d_{t+1}(a^*) - Q_{t+1}(a^*) + \pi'(Q_{t+1}(a^*)).$$

By Remark 2.2, the only term in $Q_{t+1}(a^*)$ not in $M_i$ is

$$(-1)^{n+1 + (2n - i)(2n - i + 1)/2} (-1)^{(t - i)/2} a^*(\xi)$$

$$= (-1)^{(t - i)/2 + 1} a^*(\xi) \in \wedge 2n - i F^*.$$ 

So

$$\pi'( (\pi')^{-1}(a) ) = d_{t+1}(a^*) + (-1)^{(t - i)/2} a^*(\xi) + (-1)^{(t - i)/2 + 1} \pi'(a^*(\xi))$$

$$= d_{t+1}(a^*) + (-1)^{(t - i)/2} a^*(\xi) + (-1)^{n+1} [d_{t+1}(a^*(\xi))(\eta) - (a^*(\xi))(\eta)]$$

$$+ (-1)^{(t - i)/2 + 1} a^*(\xi)$$

$$= a.$$

Similarly $\pi'(\pi(i)) = a$ for $a \in \wedge F^*, a \not\in M_i$. Thus $\pi' \circ \pi^{-1} = \pi'^{-1} \circ \pi' = id$ and hence $\pi'$ is an isomorphism.

Now to show that $\pi'$ is a map of complexes, it is enough to show that $(\pi')^{-1}$ is. For $a \in M_i$, $(\pi')^{-1}(a) = a$, $d_i(a) = g_i(a) \subseteq M_{t-1}$. So $(\pi')^{-1}(g_i(a)) = d_i((\pi')^{-1}(a))$. If $a \not\in M_i$ and $a \in \wedge F^*$, then $g_i(a) = 0$ and $d_i[(\pi')^{-1}(a)] = d_i[d_{t+1}(a^*)] = 0$. So $(\pi')^{-1} \circ g_i = d_i \circ (\pi')^{-1}$. If $a \not\in M_i$ and $a \in \wedge F$, $d_i((\pi')^{-1}(a)) = d_i(a)$ and

$$(\pi')^{-1}_i (g_i(a)) = (\pi')^{-1}_i (a^*) = d_i(a^*) = d_i(a).$$

Again, $(\pi')^{-1} \circ g_i = d_i \circ (\pi')^{-1}$. So $(\pi')^{-1}$ commutes with the boundary maps. This proves the claim.

Now $p: G_i \to M_i$, the trivial projection is a map of complexes. Let $a \in G_i$, If $a \in M_i$, then $d_i(p(a)) = d_i(a) = p(g_i(a)) = p(g_i(a))$. If $a \not\in M_i$ and $a \in \wedge F_i$, the trivial projection is a map of complexes.
\[d_i(p(a)) = 0\] and \(p(g_i(a)) = p(0) = 0\). Thus \(d_i \circ p = p \circ g_i\). So \(P\) is a complex map. Hence \(\pi = p \circ \pi'\) is also a map of complexes. Finally if \(a \in M_i\), \(\pi \circ i(a) = \pi(a) = a\). Hence \(\pi \circ i = \text{id}_M\) and \(i : M \to F\) splits as a map of complexes.

Now we define the multiplication on \(M\).

**Definition 3.10.** For any \(a, b \in M\), define \(a \cdot b = ab = \pi(\iota(a) \cdot \iota(b))\).

Even though the multiplication \(\mu\) defined on \(F\) is associative, it is not immediate that the multiplication on \(M\) defined in 3.10 is associative. We will prove this later. First we will show that \(\mu\) satisfies the differential condition making \(F\) an associative commutative differential graded algebra. We need some results from multilinear algebra.

We now state our main theorem.

**Theorem 3.11.** Let \(R\) be a noetherian local ring and \(I\) be a Huneke–Ulrich ideal. Then the minimal resolution \(M\) of \(R/I\) admits an algebra structure.

4. A Digression Into Pfaffians and Some Binomial Identities

For \(a \in \bigwedge^p F\) and \(b \in \bigwedge^q F^*\), the module operation \(b(a)\) is defined as follows: If \(\Lambda_{q, p - q(a)} = \sum a_i \otimes a'_i\) is the diagonalization of \(a\) into \(\bigwedge^q F \otimes \bigwedge^{p - q} F\), then \(b(a) = \sum b(a_i) \cdot a'_i\). The following properties of this module operation are useful.

**Proposition 4.1 [B-E-3].** Let \(a \in \bigwedge F\), \(b \in \bigwedge F^*\), and \(c \in \bigwedge F\). Then

1. If \(\Lambda(b) = \sum_i b_i \otimes b'_i \in \bigwedge F^* \otimes \bigwedge F^*\), \(b(a \wedge c) = \sum_i (-1)^{\deg a \cdot \deg b'_i} b_i(a) \wedge b'_i(c)\).

2. If \(\Lambda(a) = \sum_i a_i \otimes a'_i \in \bigwedge F \otimes \bigwedge F\), \(a(b)(c) = \sum (-1)^{\deg b + 1} \deg a'_i a_i \wedge b(a'_i \wedge c)\).

3. If \(\deg a = 1\), then

\[a(b)(c) = a \wedge b(c) + (-1)^{\deg b + 1} b(a \wedge c)\.

4. If \(\deg b = 2n = \text{rank } F^*\),

\[a(b)(c) = (-1)^v c(b)(a),\]

where

\[v = (\deg a + \deg c) + \deg a \cdot \deg c.\]
(5) If \( \deg c = 2n \),
\[
a(b)(c) = a \land b(c).
\]

(6) If \( \deg a = 1 \), then for all \( k \geq 1 \),
\[
a(\phi^{(k)}) = a(\phi) \land \phi^{(k-1)}.
\]

**Proof.** Parts (1), (2), and (6) can be verified by induction. The rest (3), (4), (5) follow from (2).

We would like to expand a Pfaffian of a given order in terms of pfaffians of lower order. More generally, we have the following propositions whose proof can be found in \([S-2]\).

**Proposition 4.2 \([S-2]\).** For any \( a \in \bigwedge^i F^* \), and \( j \geq 0 \),
\[
a(\phi^{(i)}) = \sum_{k=1}^{[i/j]} (-1)^{k+1} a(\phi^{(i-k)}) \land \phi^{(k)} + \bigwedge^{(2j-i)} \bar{\phi}(\phi^{(i-j)}(a)),
\]
where \( \phi^{(n)} = 0 \) if \( n < 0 \), and \( \bar{\phi} : F^* \to F \) is given by \( \bar{\phi}(x) = x(\phi) \).

**Proposition 4.3 \([S-2]\).** For any \( a \in \bigwedge^i F \) and all integers \( i, j \), such that \( i + j \) is even
\[
\phi^{(i+j/2)} \land a(\xi)(\phi^{[(i+j)/2]})
= \sum_{w=0}^{j} \left( \begin{array}{c} n - \left[ \frac{i+t}{2} \right] \\ \frac{j-w}{2} \end{array} \right) \phi^{(n-(t-w)/2)}(\xi)(a) \land \phi^{[(i-w)/2]}.
\]

**Remark.** As a corollary to 4.3, if we let \( i=t=0 \) and \( j=2 \), \( \phi^{(n-1)}(\xi)(\phi) = \phi^{(n)}(\xi) \). If \( X \) is the alternating matrix represented by \( \phi \), then this is the well-known “Laplace expansion” for pfaffians, \( \sum_{j=1}^{2n} x_{ij} X_{jk} = \delta_{jk} Pf(X) \) where \( X_{jk} \) is the signed pfaffian of the submatrix of \( X \) obtained by deleting \( j \)th and \( k \)th rows and the corresponding columns.

For a sequence \( K=(k_1, ..., k_i) \), \( Pf(K) \) denotes the pfaffian of the submatrix of \( X \) made up of the rows and columns in \( K \). Also, \( X_K = (-1)^{v(K)} Pf((1, 2, ..., 2n) \setminus K) \), where \( v(K) = \sum_{i \in K} i + \sigma(K) + t(t-1)/2 \) and \( \sigma(K) \) is the permutation arranging \( k_1, ..., k_i \) in ascending order. With this notation, Propositions 4.2 and 4.3 give the following:

(a) For any two sequences \( I=(i_1, ..., i_p) \) and \( J=(j_1, ..., j_q) \) in ascending order
\[ Pf(I, J) = \sum_{t=0}^{q-2} \sum_{W \subset J \mid |W| = t} (-1)^{(t-q)/2 + 1} \text{sign}(\sigma) \; Pf(I, W) \; Pf(J \setminus W), \quad \text{if } p < q \]

\[ = \sum_{t=0}^{q-2} \sum_{W \subset J \mid |W| = t} (-1)^{(t-q)/2 + 1} \text{sign}(\sigma) \; Pf(I, W) \; Pf(J \setminus W) \]

\[ + (-1)^{(p-q-1)/2} \text{det}(I \times J), \quad \text{if } p = q \]

\[ = \sum_{t=0}^{q-2} \sum_{W \subset J \mid |W| = t} (-1)^{(t-q)/2 + 1} \text{sign}(\sigma) \; Pf(I, W) \; Pf(J \setminus W) \]

\[ + (-1)^{(q-(q-1))/2} \sum_{Z \subset I \mid |Z| = p-q} \text{sign}(\tau) \; Pf(Z) \; \text{det}((I \setminus Z) \times J), \quad \text{if } p > q, \]

where \( \text{det}(I \times J) \) is the determinant of the submatrix of \( X \) made up of the rows in \( I \) and columns in \( J \), \( \sigma \) is the permutation that arranges \((W, J \setminus W)\) in ascending order, and \( \tau \) is the permutation arranging \((Z, I \setminus Z)\) in ascending order.

(b)

\[ \sum_{K, |K| = r} X_{I_k} Pf(K, J) \]

\[ = \sum_{l=0}^{r} \sum_{W \subset I \cap J \mid |W| = l} (-1)^{q + r + p} \binom{n - (q + p)/2}{(r-l)/2} X_{I \setminus W} Pf(J \setminus W), \quad \text{if } I \cap J \neq \emptyset \]

\[ = \binom{n - (p + q)/2}{r/2} X_{I} Pf(J), \quad \text{if } I \cap J = \emptyset, \]

where \( \sigma \) and \( \tau \) are, respectively, the orders of the permutations arranging \((I \setminus W, W)\) and \((W, J \setminus W)\) in ascending order.

The following application of Proposition 4.2 is useful in our computations and we state it in the form in which it will be used.

**Proposition 4.4.** For any \( T \in \bigwedge^j F^* \), and any positive integer \( l \),

\[ \sum_{k=[(j-1)/2]}^{j-1} (-1)^{j+k+1} \binom{j-k+l-1}{l} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \]

\[ = \sum_{w=0}^{l} \sum_{t=0}^{l} (-1)^{w+t} \binom{l-t}{l-w} T(\phi^{(j+t+1)}) \wedge \phi^{(l-t)} \]

\[ = T(\phi^{(j+l+1)}). \]
**Proof.** By Proposition 4.2, for any $T \in \wedge^j F^*$, and $w \geq 0$

$$T(\phi^{(j+1+w)}) = \sum_{r=0}^{[j+1+w-1/2]} (-1)^{r+1} T(\phi^{(j+1+w-r)}) \wedge \phi^{(r)}$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1+w} (-1)^{j-k+w+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+w)}.$$ 

Therefore

$$(-1)^{w} T(\phi^{(j+1+w)}) \wedge \phi^{(l-w)}$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1} (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right)$$

$$+ \sum_{k=j}^{j-1+w} (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right)$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1} (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right)$$

$$+ \sum_{i=0}^{w-1} (-1)^{i+1} T(\phi^{(j+1+i)}) \wedge \phi^{(l-i)} \left(\begin{array}{c} l-i \\ l-w \end{array}\right).$$

Thus,

$$\sum_{t=0}^{w} (1)^t T(\phi^{(j+1+t)}) \wedge \phi^{(l-t)} \left(\begin{array}{c} l-t \\ l-w \end{array}\right)$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1} (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right).$$

Hence,

$$\sum_{w=0}^{l} (-1)^w \sum_{t=0}^{l} (-1)^t T(\phi^{(j+1+t)}) \wedge \phi^{(l-t)} \left(\begin{array}{c} l-t \\ l-w \end{array}\right)$$

$$= \sum_{w=0}^{l} (-1)^w \sum_{k=\lceil(j-1)/2\rceil}^{j-1} (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)} \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right)$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1} \left(\sum_{w=0}^{l} (-1)^w \left(\begin{array}{c} j-k+l \\ l-w \end{array}\right) \right) \left(\begin{array}{c} j-k+1 \\ l \end{array}\right)$$

$$= \sum_{k=\lceil(j-1)/2\rceil}^{j-1} \left(\begin{array}{c} j-k+l-1 \\ l \end{array}\right) (-1)^{j-k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l)}.$$
But

$$\sum_{w=0}^{l} (-1)^{w+t} \binom{l-w}{l-w} = (-1)^t \binom{l+t-1}{l} = \delta_{t,l}.$$ 

Therefore

$$\sum_{k=\lceil (j-1)/2 \rceil}^{j-1} (-1)^{j+k+1} T(\phi^{(k+1)}) \wedge \phi^{(j-k+l-1)} \binom{j-k+l-1}{l}$$

$$= \sum_{t=0}^{l} \sum_{w=0}^{t} (-1)^{w+t} \binom{l-t}{l-w} T(\phi^{(j+1+t)}) \wedge \phi^{(l-t)}$$

$$= T(\phi^{(j+1+l)}).$$

This finishes the proof of Proposition 4.4.

For any real number $k$, $\lfloor k \rfloor$ denotes the largest integer less than or equal to $k$. The following binomial identity is very useful. We state it in the same notation, in which it will be used.

**Proposition 4.5.** For all non-negative integers $n$, $s$, $p$, $l$, with $1 \leq s \leq 2n+1$ and $l-s$ odd,

$$\sum_{j=\lfloor (l-1)/2 \rfloor}^{s-1} (-1)^{\lfloor (l-1)/2 \rfloor} \binom{n-p-\lfloor \frac{j-1}{2} \rfloor -2}{s-1-j} \binom{n-p-\lfloor \frac{s}{2} \rfloor}{j-l} = (-1)^{s/2+1} \delta_{l,s-1},$$

where

$$\delta_{l,s-1} = \begin{cases} 0 & \text{if } l \neq s-1 \\ 1 & \text{if } l = s-1. \end{cases}$$

First we need two lemmas.

**Lemma 4.6.** For all $n \geq 0$, $l \geq 0$, and $p \geq 0$,

$$\sum_{j=0}^{n} (-1)^j \binom{n-p-j-1}{n-j} \binom{-p}{j-l} = 0 \quad \text{if } l \neq n$$

$$= (-1)^n \quad \text{if } l = n.$$

**Proof.** Let $C(n, p, l)$ be the sum $\sum_{j=0}^{n} (-1)^j \binom{n-p-j-1}{n-j}(-p)^{j-l}$, so that we must show that $C(n, p, l) = (-1)^n \delta_{l,n}$. When $n = 0$

$$C(0, p, l) = \binom{-p-1}{0} \binom{-p}{-l} = \delta_{l,0}.$$
Thus $C(0, p, l) = (-1)^0 \delta_{l,0}$ for all $p$ and $l$. If $p = 0$,

$$C(n, 0, l) = \sum_{j=0}^{n} (-1)^j \binom{n-j-1}{n-j} \binom{0}{j-l}$$

$$= (-1)^n \binom{n-n-1}{0} \binom{0}{n-l}$$

$$= (-1)^n \delta_{n,l}.$$

Hence the identity holds if either $p = 0$ or $n = 0$. By induction, assume

$$C(n, q, l) = (-1)^n \delta_{l,m} \quad \text{if either } m \leq n \text{ or } q \leq p.$$

Consider

$$C(n+1, p, l) = \sum_{j=0}^{n+1} (-1)^j \binom{n-p-j}{n+1-j} \binom{-p}{j-l}$$

$$= \sum_{j=0}^{n+1} (-1)^j \binom{n-p-j+1}{n+1-j} \binom{-p+1}{j-l}$$

$$- \sum_{j=0}^{n+1} (-1)^j \binom{n-p-j}{n-j} \binom{-p+1}{j-l}$$

$$- \sum_{j=0}^{n+1} (-1)^j \binom{n-p-j}{n+1-j} \binom{-p}{j-l-1}.$$

Since $l \geq 0$, this becomes

$$C(n+1, p, l) = \sum_{j=0}^{n+1} (-1)^j \binom{n-j-(p-1)}{n+1-j} \binom{-(p-1)}{j-l}$$

$$- \sum_{j=0}^{n} (-1)^j \binom{n-1-j-(p-1)}{n-j} \binom{-(p-1)}{j-l}$$

$$+ \sum_{j=0}^{n} (-1)^j \binom{n-p-j-1}{n-j} \binom{-p}{j-l}.$$

$$C(n+1, p, l) = C(n+1, p-1, l) - C(n, p-1, l) + C(n, p, l)$$

$$= (-1)^{n+1} \delta_{l,n+1} + (-1)^n \delta_{l,n}$$

$$+ (-1)^n \delta_{l,n} \quad \text{by induction hypothesis.}$$

Thus $C(n+1, p, l) = (-1)^{n+1} \delta_{l,n+1}$ as required.  

Lemma 4.7 is essentially Proposition 4.4, stated in a more pleasant form.
LEMMA 4.7. For all $n, t, p, k \geq 0$, $0 \leq t \leq n$,

$$\sum_{i=0}^{t} (-1)^{i+t} \binom{n-p-i-1}{t-i} \binom{n-p-t}{i-k} = 0 \quad \text{if } k \neq t$$

$$-1 \quad \text{if } k = t.$$

Proof. Let $B(n, t, p, k)$ denote the sum on the left. When $t = 0$, $B(n, 0, p, k) = \binom{n-p-1}{0} \binom{n-p}{-k} = 0$ if $k \neq 0$

$$= 1 \quad \text{if } k = 0.$$

Thus $B(n, 0, p, k) = \delta_{k,0}$. When $t = 1$,

$$B(n, 1, p, k) = -\left(\binom{n-p-1}{1} \binom{n-p-1}{-k}\right) + \left(\binom{n-p}{0} \binom{n-p-1}{1-k}\right)$$

$$= 0 \quad \text{if } k > 1$$

$$= 1 \quad \text{if } k = 1$$

$$= -(n-p-1) + (n-p-1) = 0 \quad \text{if } k = 0.$$

Thus $B(n, 1, p, k) = \delta_{k,1}$.

On the other hand, when $n = 0$ or 1, $t$ must be $\leq 1$ and hence the identity holds for $n = 0$ or 1. Hence by induction, assume the identity for all integers $\leq n$. Consider

$$B(n+1, t, p, k) = \sum_{i=0}^{t} (-1)^{i+t} \binom{n-p-i}{t-i} \binom{n-p-t+1}{i-k}$$

$$= \sum_{i=0}^{t} (-1)^{i+t} \binom{n-p-i-1}{t-i} \left[ \binom{n-p-t}{i-k} + \binom{n-p-t}{i-k-1} \right]$$

$$+ \sum_{i=0}^{t} (-1)^{i+t} \binom{n-p-i-1}{t-i-1} \binom{n-p-t+1}{i-k}$$

$$= B(n, t, p, k) + \sum_{i=0}^{t} (-1)^{i+t+1} \binom{n-p-i-2}{t-i-1}$$

$$\times \binom{n-p-t}{i-k} - B(n, t-1, p, k)$$

$$= B(n, t, p, k) + B(n-1, t-1, p, k) - B(n, t-1, p, k).$$

So

$$B(n+1, t, p, k) = B(n, t, p, k) - B(n, t-1, p, k) + B(n-1, t-1, p, k).$$
Thus, if \( t \leq n \), then
\[
B(n + 1, t, p, k) = \delta_{k,t} - \delta_{k,t-1} + \delta_{k,t-1}
\]
\[
= \delta_{k,t}
\]
as required.

Now, when \( t = n + 1 \)
\[
B(n + 1, t, p, k) = B(n + 1, n + 1, p, k)
\]
\[
= \sum_{t \geq 0} (-1)^{n+1+t} \binom{n-p-t}{n+1-t}(-p)^{n-p-t}
\]
\[
= \delta_{k,n+1},
\]
by Lemma 4.6. Thus \( B(n + 1, t, p, k) = \delta_{k,n+1} \) for all \( p, k \) and \( t \). This finishes the induction.

**Proof of Proposition 4.5.** Since \( l-s \) is odd and \( j-l \) is even, we have \( j \equiv l \equiv s-1 \pmod{2} \). Also since \( s-1-j \) and \( j-l \) are even,
\[
\frac{s-1-j}{2} = \frac{s}{2} - \frac{j+1}{2} = \left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor
\]
and
\[
\frac{j-l}{2} = \frac{j+1}{2} - \frac{l+1}{2} = \left\lfloor \frac{j+1}{2} \right\rfloor - \left\lfloor \frac{l+1}{2} \right\rfloor.
\]
Let \( i = \left\lfloor (j+1)/2 \right\rfloor \), \( t = \left\lfloor s/2 \right\rfloor \), and \( k = \left\lfloor (l+1)/2 \right\rfloor \). Then \( j \leq s-1, i \leq t \). With this transformation the sum on the left of the identity becomes
\[
\sum_{j=0}^{s-1} (-1)^{[(j+1)/2]-1} \binom{n-p-\left\lfloor \frac{j+1}{2} \right\rfloor -1}{s-(j+1)/2} \binom{n-p-\left\lfloor \frac{s}{2} \right\rfloor}{j-1/2}
\]
\[
= \sum_{i=0}^{l} (-1)^{i+1} \binom{n-p-i-1}{t-i} \binom{n-p-l}{i-k}
\]
\[
= (-1)^{t+1} \delta_{k,t},
\]
by Lemma 4.7.

But
\[
k = t \iff \left\lfloor \frac{l+1}{2} \right\rfloor = \left\lfloor \frac{s}{2} \right\rfloor
\]
\[
\iff l = s-1,
\]
since \( l - s \) is odd. So, \((-1)^{l+1} \delta_{k,s} = (-1)^{[s/2]+1} \delta_{l,s-1}\). This proves our identity.

The following binomial identity is a special case of Lemma 4.7.

**Corollary 4.8.** For all \( w \) and \( t \),

\[
\sum_{j=0}^{t} (-1)^{j+t} \binom{w-j}{t-j} = \begin{cases} 0 & \text{if } t > 0 \\ 1 & \text{if } t = 0. \end{cases}
\]

5. **The Algebra Structure on \( F \)**

In order to establish that the multiplication \( \mu \) defined in 3.1 gives an algebra structure on the complex \( F \), it remains to see that \( \mu \) satisfies the boundary condition, i.e., for any \( a, b \in F \), \( d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b) \). It suffices to check this in the generic case since the multiplication on the complex \( F \) is "natural." We need to go up to the generic situation only for the case where both \( a \) and \( b \) are in \( \wedge F \). In this case the product \( a \cdot b \) is trivial but showing \( d(a)b + (-1)^{\deg a} a \cdot d(b) = 0 \) is considerably complicated. This is treated in Case 3 of Proposition 5.3 and Lemmas 5.4 and 5.5.

To begin with we have the following two lemmas.

**Lemma 5.1.** If \( a \in \wedge^i F \subseteq F_i, b = e^* \in F^* \subseteq F_1 \), then

\[
d_{t+1}(a \cdot b) = d_t(a)b + (-1)^t ad_t(b).
\]

**Proof.** By rule (3) of 3.1,

\[
a \cdot b = -e^*(a \wedge \phi^{(t-i+1)/2}) + e^*(a) \in \wedge^t F \oplus \wedge^{t-i} F \subseteq F_{t+1}.
\]

So

\[
d_{t+1}(ab) = (-1)^{t+1} Y(e^*(a \wedge \phi^{(t-i+1)/2})) - Q_{t+1}(e^*(a \wedge \phi^{(t-i+1)/2}))
+ (-1)^t [Y(e^*(a)) + g \wedge e^*(a)] + Q_{t+1}(e^*(a)).
\]

But

\[
Y(e^*(a \wedge \phi^{(t-i+1)/2}))
= -[e^*(Y(a) \wedge \phi^{(t-i+1)/2}) + e^*(g \wedge a \wedge \phi^{(t-i-1)/2})].
\]
So,

\[ d_{t+1}(ab) = (-1)^t [e^*(Y(a) \wedge \phi^{((t-1)/2)}) + e^*(g \wedge a \wedge \phi^{((t-1)/2)})] \]

\[ + (-1)^t [Y(e^*(a)) + g \wedge (e^*(a)) - Q_{t+1}(e^*(a \wedge \phi^{((t-1)/2)}))] \]

\[ + Q_{t+1}(e^*(a)). \]  \hspace{1cm} (5.1.1)

Now,

\[ d_i(a)b + (-1)^i ad_i(b) \]

\[ = (-1)^{i+1} (g \wedge a + Y(a)) + Q_i(a)b + (-1)^i ad_i(b) \]

\[ = (-1)^{i+2} e^*(g \wedge a \wedge \phi^{((t-1)/2)}) + Y(a) \wedge \phi^{((t-1)/2)}) \]

\[ + (-1)^{i+1} e^*(g \wedge a + Y(a)) + Q_i(a) \wedge e^* + (-1)^i ag(e^*). \] \hspace{1cm} (5.1.2)

Note that (5.1.2) holds even if \( i = t - 1 \). From (5.1.1) and (5.1.2)

\[ d_{t+1}(ab) - (d_i(a)b + (-1)^i ad_i(b)) \]

\[ = (-1)^i g \wedge e^*(a) + (-1)^t e^*(g \wedge a) \]

\[ + (-1)^{i+1} ag(e^*) + Q_{t+1}(e^*(a)) \]

\[ - Q_{t+1}(e^*(a \wedge \phi^{((t-1)/2)})) - Q_i(a) \wedge e^*. \]

Since \( e^*(g \wedge a) = e^*(g)a - g \wedge e^*(a) \), we get

\[ d_{t+1}(ab) - d_i(a)b + (-1)^{i+1} ad_i(b) \]

\[ = Q_{t+1}(e^*(a)) - Q_{t+1}(e^*(a \wedge \phi^{((t-1)/2)})) - Q_i(a) \wedge e^*. \]

So we just need to show that for any \( e^* \in F^* \),

\[ Q_{t+1}(e^*(a \wedge \phi^{((t-1)/2)})) + Q_i(a) \wedge e^* - Q_{t+1}(e^*(a)) = 0. \]

Now,

\[ Q_{t+1}(e^*(a \wedge \phi^{((t-1)/2)})) - Q_{t+1}(e^*(a)) \]

\[ = \sum_{j=0}^{t} (-1)^{n+1+j/2} (\phi^{(n-(t+j)/2)}) \wedge e^*(a \wedge \phi^{((t-1)/2)})) \]

\[ \left( \frac{n-i+j-1}{2} - 1 \right) \left( \frac{t-i+1}{2} \right) \]

\[ - \sum_{j=0}^{t} (-1)^{n+1+j/2} \left( \frac{n-i+j-1}{2} - 1 \right) \left( \frac{t-i+1}{2} \right) \]

\[ \phi^{(n-(t+j-1)/2)} \wedge e^*(a) \).
\[
\begin{align*}
&= \sum_{j=0}^{t} (-1)^{n+1+j(j+1)/2} \left( \frac{n-i+j+1}{2} \right) \left( \frac{t-i-1}{2} \right) (e^*(a) \land \phi^{(n-(i+j-1)/2)})(\xi) \\
&+ \sum_{j=0}^{t} (-1)^{n+1+j(j+1)/2+i} (a \land e^*(\phi^{(n-(i+j-1)/2)}))(\xi) \\
&\times \left( \frac{n-i+j+1}{2} \right) \left( \frac{t-i-1}{2} \right) \\
&= \sum_{j=0}^{t} (-1)^{n+1+j(j+1)/2} \left( \frac{n-i+j+1}{2} \right) \left( \frac{t-i-1}{2} \right) e^*(a) \land \phi^{(n-(i+j-1)/2)})(\xi) \\
&= \sum_{j=0}^{t} (-1)^{n+1+j(j+1)/2} \left( \frac{n-i+j+1}{2} \right) \left( \frac{t-i-1}{2} \right) e^* \land (a \land \phi^{(n-(i+j-1)/2)})(\xi), \\
&\quad \text{by (6) of Theorem 4.1.}
\end{align*}
\]

If \( t \) is odd, then \( j \geq 1 \) and if \( t \) is even and \( j=0 \), \( a \land \phi^{(n-(i-1)/2)} \in \wedge^{\frac{3n+1}{2}} F = 0 \). Thus we get

\[
\begin{align*}
Q_{t+1}(e^*(a) \land \phi^{(t-i+1)/2})) - Q_{t+1}(e^*(a))
&= \sum_{j=1}^{t} (-1)^{n+1+j(j+1)/2} \\
&\times \left( \frac{n-i+j+1}{2} \right) \left( \frac{t-i-1}{2} \right) e^* \land (a \land \phi^{(n-(i+j-1)/2)})(\xi) \\
&= \sum_{j=0}^{t} (-1)^{n+1+(j+1)(j+2)/2} \\
&\times \left( \frac{n-i+j+2}{2} \right) \left( \frac{t-i-1}{2} \right) e^* \land (a \land \phi^{(n-(i+j)/2)})(\xi)
\end{align*}
\]
LEMMA 5.1. Let  \( a \in \bigwedge^i F \subseteq F \), and  \( h = 1 \in F_2 \). Then

\[
d_{t+2}(ah) = d_t(a)h + (-1)^t a d_2(h).
\]

Proof. By rule (4) of 3.1,

\[
ah = \left( n - t + \frac{t - i - 1}{2} \right) a - (n - t - 1) a \wedge \phi^{((t-i+1)/2)}
\]

\[
e \in \bigwedge^i F \oplus \bigwedge^{t+1} F \subseteq F_{t+2}
\]

\[
d_{t+2}(ah) = \left( n - t + \frac{t - i - 1}{2} \right) \left[ (-1)^{t+3} \left( (Y(a) + g \wedge a) + Q_{t+2}(a) \right) \right]
\]

\[
- (n - t - 1) \left[ (-1)^{t+3} (Y(a) \wedge \phi^{((t-i+1)/2)}) \right]
\]

\[
+ Q_{t+2}(a \wedge \phi^{((t-i+1)/2)}).
\] (5.2.1)

And

\[
d_t(a)h + (-1)^t a d_2(h) = (-1)^{t+1} (Y(a) + g \wedge a)h + Q_t(a)h + (-1)^t a \cdot Y
\]

\[
= (-1)^{t+1} \left[ \left( n - t + \frac{t - i + 1}{2} \right) Y(a) \right]
\]

\[
- (n - t) Y(a) \wedge \phi^{((t-i+1)/2)} \right]
\]

\[
+ (-1)^{t+1} \left[ \left( n - t + \frac{t - i - 1}{2} \right) g \wedge a \right.
\]

\[
- (n - t) g \wedge a \wedge \phi^{((t-i-1)/2)} \right] + Q_t(a)h
\]

\[
+ (-1)^t Y(a) + (-1)^{t+1} Y(a \wedge \phi^{((t-i+1)/2)}).
\]
Thus
\[ d_t(a)h + (-1)^t a d_2(h) = (-1)^{t+1} \left[ \left( n - t + \frac{t - i - 1}{2} \right) (Y(a) + g \wedge a) \right] \]
\[ + (-1)^{t+2} \left[ (n - t - 1)(Y(a) \wedge \phi^{((t-i+1)/2)}) \right] \]
\[ + g \wedge a \wedge \phi^{((t-i-1)/2)} \right] + Q_t(a)h. \]  

(5.2.2)

Again, we note that (5.2.2) holds for \( i = t - 1 \) also. From (5.2.1) and (5.2.2), we get
\[ d_{t+2}(ah) - d_t(a)h + (-1)^t a d_2(h) \]
\[ = \left( n - t + \frac{t - i - 1}{2} \right) Q_{t+2}(a) \]
\[ + (n - t - 1) Q_{t+2}(a \wedge \phi^{((t-i+1)/2)}) - Q_t(a)h. \]  

(5.2.3)

Thus
\[ d_{t+2}(ah) - d_t(a)h + (-1)^{t+1} a d_2(h) \]
\[ = \left( n - t + \frac{t - i - 1}{2} \right) \sum_{j=0}^{t+1} (-1)^{n+1+j(j+1)/2} \]
\[ \times \left( n - \frac{i + j}{2} \right) \phi^{(n-(i+j)/2)} \wedge a(\xi) \]
\[ - (n - t - 1) \sum_{j=0}^{t+1} (-1)^{n+1+j(j+1)/2} \]
\[ \times \phi^{(n-(t+1+j)/2)} \wedge a \wedge \phi^{((t-i-1)/2)}(\xi) \]
\[ - \sum_{j=0}^{t-1} (-1)^{n+1+j(j+1)/2} \phi^{(n-(i+j)/2)} \wedge a(\xi) \]
\[ \times \left( \frac{t+1-j}{2} \right) \phi^{(n-(i+j)/2)} \wedge a(\xi) \]
\[ = \sum_{j=0}^{t} (-1)^{n+1+j(j+1)/2} \phi^{(n-(i+j)/2)} \wedge a(\xi) W_j, \]
where

\[
W_j = \left( n - t + \frac{t - i - 1}{2} \right) \left( \frac{n - i + j - 1}{2} \right) - \left( n - t - 1 \right) \left( \frac{n - i + j}{2} \right) 
- \left( \frac{t + 1 - j}{2} \right) \left( \frac{n - i + j}{2} \right) 
+ \left( \frac{t + 1 - j}{2} \right) \left( \frac{n - i + j}{2} \right) 
\]

\[
= - \left[ \left( n - t + \frac{t - i - 1}{2} \right) \left( \frac{n - i + j - 1}{2} \right) - \left( t - i + 1 \right) \left( \frac{n - i + j}{2} \right) \right] 
\]

\[
= - \left[ \frac{n - i + j}{2} \left( \frac{n - i + j - 1}{2} \right) - \frac{t - i + 1}{2} \left( \frac{n - i + j}{2} \right) \right] 
= 0.
\]

Thus \( d_{r+2}(ah) = d_r(a)h + (-1)^r d_r(a)h. \)

\[\begin{proof}
\text{Proposition 5.3. The multiplication } \mu \text{ on } F, \text{ satisfies the boundary condition; viz. for } a \in F_r, b \in F_s, d_{r+s}(a \cdot b) = d_r(a)b + (-1)^r ad_s(b).
\end{proof}

Proof. Whenever there is no danger of confusion we will drop the subscript and write \( d \) for \( d_r \). Clearly, it is enough to show the condition for basis elements of \( \wedge F \oplus \wedge F^* \supseteq F \).

There are three possibilities:

Case 1. \( a, b \in \wedge F^* \).

Case 2. \( a \in \wedge F \) and \( b \in \wedge F^* \) or \( a \in \wedge F^* \) and \( b \in \wedge F \).

Case 3. \( a, b \in \wedge F \).
Case 1. \( a \in \wedge F^* \subseteq F_i, b \in \wedge F^* \subseteq F_s \). In this case \( \mu \) is just the multiplication in divided power algebra. Thus,

\[
d(a \cdot b) = d \left[ \left( \frac{t-i+s-j}{2} \right) \left( \frac{s-j}{2} \right) a \wedge b \right]
\]

\[
- \left( \frac{t-i+s-j}{2} \right) \left( \frac{s-j}{2} \right) \left[ Y \wedge a \wedge b + g(a \wedge b) \right] \quad \text{if} \quad i+j \neq t+s
\]

\[
= g(a \wedge b) \quad \text{if} \quad i+j = t+s.
\]

Also, if \( i+j \neq t+s \),

\[
d(a)b + (-1)^i \ ad(b)
\]

\[
= (Y \wedge a + g(a))b + (-1)^i a(Y \wedge b + g(b))
\]

\[
\left( \frac{t-i+s-j-2}{2} \right) \left( \frac{s-j}{2} \right) Y \wedge a \wedge b + \left( \frac{t-i+s-j}{2} \right) \left( \frac{s-j}{2} \right) g(a) \wedge b
\]

\[
+ (-1)^i \left( \frac{t-i+s-j-2}{2} \right) \left( \frac{s-j-2}{2} \right) a \wedge Y \wedge b + (-1)^i \left( \frac{t-i+s-j}{2} \right) \left( \frac{s-j}{2} \right) a \wedge g(b)
\]

\[
= \left( \frac{t-i+s-j}{2} \right) \left( \frac{s-j}{2} \right) \left[ Y \wedge a \wedge b + g(a \wedge b) \right].
\]

And if \( i+j = t+s \), then \( i = t, j = s \) and

\[
d(a)b + (-1)^i \ ad(b) = g(a)b + (-1)^i \ ag(b)
\]

\[
= g(a) \wedge b + (-1)^i \ a \wedge g(b)
\]

\[
= g(a \wedge b),
\]
since $t + i$ is even. Thus

$$d(ab) = d(a)b + (-1)^i ad(b)$$

in this case.

Case 2. Without loss of generality, we may take $a \in \bigwedge^i F \subseteq F_i$, and $b \in \bigwedge^j F* \subseteq F_j$. By rule (5), $b = rb_1 b_2 \cdots b_k$, where $b_i \in E = \{e_1^*, \ldots, e_n^*, h\}$. We write $b = b_1 b'_1$, where $b'_1 = rb_2, \ldots, b_k$, so that $\deg_F b_1 = 1$ or 2 and $\deg_F b'_1 < \deg_F b_1 = s$. By Lemmas 5.1 and 5.2, the boundary condition is verified if $s = 1$ or if $s = 2$ and $b = h$. By induction, assume the result is true if $\deg_F b < s$. Now, by associativity of $F$ (Theorem 3.7),

$$d(a \cdot b) = d(a(b_1 b'_1)) = d((ab_1)b'_1)$$

$$= d(ab_1)b'_1 + (-1)^{i + \deg_F b_1} ab_1 d(b'_1), \quad \text{by induction hypothesis}$$

$$= [d(a)b_1 + (-1)^i ad(b_1)] b'_1$$

$$+ (-1)^{i + \deg_F b_1} ab_1 d(b'_1), \quad \text{by Lemmas 5.1 and 5.2}$$

$$= d(a)(b_1 b'_1) + (-1)^i [ad(b_1)b'_1 + (-1)^{\deg_F b_1} ab_1 d(b'_1)]$$

$$= d(a)b + (-1)^i ad(b), \quad \text{by Case 1.}$$

Thus $d(ab) = d(a)b + (-1)^i ad(b)$ in this case also.

Case 3. Let $a \in \bigwedge^i F \subseteq F_i$ and $b \in \bigwedge^j F \subseteq F_j$. In this case the product $ab = 0$. Ironically, this case is much harder to settle. First we will do a quick reduction.

**Claim.** It suffices to show that $d(a)b + (-1)^i ad(b) = 0$ for $b = 1 \in F_1$.

**Proof of Claim.** Assume $d(a)b + (-1)^i ad(b) = 0$ if $b = 1 \in F_1$.

Let $b \in \bigwedge^j F \subseteq F_j$ be arbitrary. We must show that $d(a)b + (-1)^i ad(b) = 0$. We will go up to the generic case and show that this is indeed true in $F$. Since $F$ is just $\tilde{F} \otimes_R R$ and the multiplication $\mu$ is "natural" (in other words, one can write that $\mu = \tilde{\mu} \otimes_R 1$), it suffices to check in the generic case. We just need the fact that $\tilde{g}_{(m)}(\xi)$ is a nonzero divisor in $\tilde{R}$. From here on, we will omit the $\sim$. Now let $c = 1 \in F_1$.

$$(d(a)b) d(c) = d(a)(bd(c))$$

$$= d(a)((-1)^{s+1} d(b) c) \quad \text{by assumption}$$

$$= (-1)^{s+1} d(a) d(b) c$$

$$= -(d(a)c) d(b)$$

$$= +((-1)^{s+1} ad(c)) d(b)$$

$$= (-1)^{s+1} ad(b) d(c).$$
Thus \(d(a)b + (-1)^i \text{ad}(b)) d(c) = 0\). But \(d(c) = (-1)^{n+1} \phi^{(n)}(\xi)\) which is a nonzero divisor. Hence \(d(a)b + (-1)^i \text{ad}(b) = 0\). This proves the claim.

Thus we just need to show that
\[
d(a)b = (-1)^i + \text{ad}(b) \quad \text{ if } b = 1 \in F_1
\]
\[
= (-1)^i aPf(X),
\]
where \(X\) is the alternating matrix represented by \(\phi\).

This is what we do in the next two lemmas and the result is obtained in Lemma 5.5. 

**Lemma 5.4.** Let \(a \in \bigwedge^i F \subseteq F_s, b = 1 \in F_1, \) so that \(s - i\) is odd. Then
\[
d(a)b = \sum_{j=0}^{s-1} \left( \sum_{p=0}^{[i-1)/2]} \sum_{l=0}^{\text{min}(i,j)} (-1)^{p+n + [(j+1)/2]} \right) \phi^{(n-(i-l)/2)}(\xi)(a) \wedge \phi^{(p-[l/2])}(a).
\]

**Proof.**
\[
d(a)b = (-1)^i + (y(a) + g \wedge a)b + Q_s(a)b = Q_s(a)b.
\]

So,
\[
d(a) \cdot b = Q_s(a) \cdot b
\]
\[
= \sum_{j=0}^{s-1} (-1)^{n+1 + j(j+1)/2} \left( \sum_{p=0}^{i+j-1} \left( \sum_{l=0}^{\text{min}(i,j)} (-1)^{p+n + [(j+1)/2]} \right) \phi^{(n-(i-l)/2)}(\xi) \wedge a)(\xi) \cdot b.
\]

Applying Corollary 3.5 for \(\phi^{(n-(i+j)/2)} \wedge a(\xi) \in \bigwedge^F \subseteq F_s\) and \(b = 1 \in F_1\), where \(\left[ \phi^{(n-(i+j)/2)} \wedge a(\xi) \right] \cdot h^{(s-j-1)/2}\) is simply \(\phi^{(n-(i+j)/2)} \wedge a(\xi) \in F_s\), we get
\[ d(a) \cdot b = \sum_{j=0}^{s-1} (\sum_{i+j \text{ even}} (-1)^{(n+1+j(j+1)/2)} \times \sum_{k=[(j+1)/2]}^{j} (h)^{(r-j)}/2) \left( \begin{array}{c} n-i+j \\ 2 \\ \end{array} \right) \left( \begin{array}{c} s-i-1 \\ 2 \\ \end{array} \right) \times (-1)^{(j+1)/2+k} (\phi(n-(i+j)/2) \land a)(\xi)(\phi^k), \quad \text{by Corollary 3.5} \]

\[ = \sum_{j=0}^{s-1} (\sum_{k=[(j+1)/2]}^{j} (-1)^k) \times \left( \begin{array}{c} n-i+j \\ 2 \\ \end{array} \right) \left( \begin{array}{c} s-i-1 \\ 2 \\ \end{array} \right) T_j(\phi^k) \cdot h^{(s-j-1)/2}, \]

where \( T_j = (\phi(n-(i+j)/2) \land a)(\xi) \in \bigwedge_i F^* \).

By Corollary 3.6, this becomes

\[ da \cdot b = \sum_{j=0}^{s-1} \sum_{k=[(j+1)/2]}^{j} (-1)^{n+1+k} \times \left( \begin{array}{c} (j-1-j)/2-1 \\ \end{array} \right) \left( \begin{array}{c} n-i+j \\ 2 \\ \end{array} \right) \left( \begin{array}{c} s-i-1 \\ 2 \\ \end{array} \right) \left( \begin{array}{c} n-2-j-r \\ 2 \\ \end{array} \right) \times \left( \begin{array}{c} j-k+r \\ r \\ \end{array} \right) T_j(\phi^k) \land \phi^{j-k+r+1} \right) \]

\[ + \sum_{j=0}^{s-1} (\sum_{k=[(j+1)/2]}^{j} (-1)^k) \times \left( \begin{array}{c} n-i+j \\ 2 \\ \end{array} \right) \left( \begin{array}{c} s-i-1 \\ 2 \\ \end{array} \right) T_j(\phi^k). \quad (5.4.1) \]
But by Proposition 4.4,

\[ \sum_{k = [(j-1)/2]}^{j-1} (-1)^{j+k+1} T(\phi(k+1)) \wedge \phi(j-k+l) \binom{j-k+l-1}{l} \]

\[ = \sum_{w=0}^{l} \sum_{t=0}^{l} T(\phi(j+t+1)) \wedge \phi(t' \cdot t) \times (-1)^{w+t} \binom{l-t}{l-w}, \quad \text{for } T \in \wedge^{i} F^*. \tag{5.4.2} \]

However,

\[ \sum_{w=0}^{l} (-1)^{w+t} \binom{l-t}{l-w} = (-1)^l \binom{l-t-1}{l} = 0 \]

unless \( l = t \). Therefore

\[ \sum_{k = [(j-1)/2]}^{j-1} (-1)^{j+k+1} T_j(\phi(k+1)) \wedge \phi(j-k+l) \binom{j-k+l-1}{l} \]

\[ = (-1)^{l+1} T_j(\phi(j+l+1)) = T_j(\phi(j+l+1)). \]

Thus

\[ \sum_{k = [(j+1)/2]}^{j} (-1)^k \binom{j-k+r}{r} T_j(\phi(k)) \wedge \phi(j-k+r+1) \]

\[ = (-1)^l T_j(\phi(j+r+1)). \]

Therefore

\[\text{da} \cdot \text{b} = \sum_{j=0}^{s-1} \sum_{r=0}^{(s-1-j)/2-1} (-1)^{n+1+j+r+1} \]

\[ \times \binom{n - \frac{i+j}{2} - 1}{2} \binom{n - 2 - j - r}{2} \binom{s - i - 1}{2} \binom{s - 1 - j}{2} T_j(\phi(j+r+1)) \]

\[ + \sum_{j=0}^{s-1} \sum_{k = [(j-1)/2]}^{j-1} (-1)^{n+1+k+1} \]

\[ \times \binom{n - \frac{i+j}{2} - 1}{2} \binom{n - k - 2}{2} \binom{s - i - 1}{2} \binom{s - 1 - j}{2} T_j(\phi(k+1)). \]
Let \( k = j + r \), then \( r = (s - 1 - j)/2 - 1 \Rightarrow k = (s - 1 + j)/2 - 1 \) and

\[
d a \cdot b = \sum_{j=0}^{s-1} \sum_{i+j \text{ even}} (-1)^i \sum_{k=j}^{(s-1+j)/2-1} (-1)^k \]

\[
\times \left( \frac{n - \frac{i+j}{2} - 1}{2} \right) \left( \frac{n - \frac{2-k}{2}}{2} \right) T_j(\phi^{(k+1)})
\]

\[
+ \sum_{k=\lfloor (j-1)/2 \rfloor}^{j-1} (-1)^k \left( \frac{n - \frac{i+j}{2} - 1}{2} \right) \left( \frac{n - \frac{2-k}{2}}{2} \right) T_j(\phi^{(k+1)})
\]

\[
= \sum_{j=0}^{s-1} \sum_{i+j \text{ even}} (-1)^{n-j} \sum_{k=\lfloor (j-1)/2 \rfloor}^{(s-1+i)/2-1} \left( \frac{n - \frac{i+j}{2} - 1}{2} \right) \left( \frac{n - \frac{2-k}{2}}{2} \right) T_j(\phi^{(k+1)}).
\]

Now letting \( p = k - \lfloor (j-1)/2 \rfloor \), so that

\[
\frac{s-1+j}{2} - 1 - \left\lfloor \frac{j-1}{2} \right\rfloor = \left\lfloor \frac{s-1}{2} \right\rfloor.
\]

Hence

\[
(da) b = \sum_{j=0}^{s-1} \sum_{i+j \text{ even}} \sum_{p=0}^{\lfloor (s-1)/2 \rfloor} (-1)^{n+p+\lfloor (j-1)/2 \rfloor} \left( \frac{n - \frac{i+j}{2} - 1}{2} \right) \left( \frac{n - \frac{2-p}{2} - 1}{2} \right) \left( \frac{s - \frac{j-1}{2}}{2} \right)
\]

\[
\times \left( \frac{n - 2 - p - \left\lfloor \frac{j-1}{2} \right\rfloor}{2} \right) T_j(\phi^{(p+\lfloor (j-1)/2 \rfloor + 1)}).}

Now
\[
T_j(\phi^{(n+1+[i-1])/2})) = (\phi^{(n-(i+j)/2)} \land a)(\xi)(\phi^{(i+1+[j-1])/2}))
\]
\[
= (\phi^{(n-(i+j)/2)} \land a)(\xi)(\phi^{((i+m)/2)}),
\]
where \(m = 2p + 1\).

By the formula in Proposition 4.3,
\[
T_j(\phi^{(n+1+[i-1]/2)}) = \sum_{l=0}^{\min(i,j,2p+1)} \left( \begin{array}{c}
\frac{n - \left(\frac{i+2p+1}{2}\right)}{2} \\
\frac{j-l}{2}
\end{array} \right)
\]
\[
\times \phi^{(n-(i-1)/2)}(\xi)(a) \land \phi^{((2p+1-l)/2)}
\]
\[
= \sum_{l=0}^{\min(i,j)} \left( \begin{array}{c}
\frac{n-p - \left[\frac{i+1}{2}\right]}{2} \\
\frac{j-l}{2}
\end{array} \right)
\]
\[
\times \phi^{(n-(i-1)/2)}(\xi)(a) \land \phi^{((p-l)/2)}.
\]

Thus
\[
da \cdot b = \sum_{j=0}^{s-1} \sum_{p=0}^{\left[(s-1)/2\right]} \sum_{l=0}^{i} (-1)^{n+p+[j-1]} /2
\]
\[
\times \left( \begin{array}{c}
\frac{n-i+j-1}{2} \\
\frac{s-i-1}{2}
\end{array} \right) \left( \begin{array}{c}
\frac{n-p-2-\left[\frac{j}{2}\right]}{2} \\
\frac{s-1-j}{2}
\end{array} \right) \left( \begin{array}{c}
\frac{n-p-\left[\frac{i+1}{2}\right]}{2} \\
\frac{j-l}{2}
\end{array} \right)
\]
\[
\times \phi^{(n-(i-1)/2)}(\xi)(a) \land \phi^{((p-l)/2)}. \]

**Lemma 5.5.** Let \(a \in \wedge^i F \subseteq F_1\) and \(b = 1 \in F_1\).

Then \(d(a)b = (-1)^{s+n} a \cdot \phi^{(n)}(\xi)\).

**Proof.** We prove this by descending induction on \(\deg a = i\). Also, as explained in Case 3 of Proposition 5.3 it suffices to prove this in the generic case. We will omit the \(~\). Thus \(R = Z[X, Y]\).
Case 1. Let $i = s - 1$. Then by the Lemma 5.4,
\[
d(a) \cdot b = \sum_{j=0}^{s-1} \sum_{p=0}^{(s-1)/2} \sum_{l=0}^{j} (-1)^{p+n+[(j-1)/2]} \\
\times \left( n - p - \left[\frac{j-1}{2}\right] - 2 \right) \left( n - p - 1 - \left[\frac{s-2}{2}\right] \right) \\
\times \phi^{(n-(s-1-l)/2)}(\xi)(a) \wedge \phi^{(p-[l/2])}.
\]

By Proposition 4.5, this sum is zero except when $p = [(s-1)/2]$ and $l = s - 1$. Thus, we put $p = [(s-1)/2] = [l/2]$. Hence, $j = s - 1$ and $[(j-1)/2] = p + s \pmod{2}$.

As required.

Now let $a \in \bigwedge F \subseteq F_s$. If $s = 1$, the result holds by Case 1. So assume it is true for all deg $a < s$. Also since by Case 1, the result is true for $i = s - 1$, assume by induction it is true for all deg $a > i$. We may assume that $a \in \bigwedge F$ is a basis element. The trick is to write the $d(a) b$ in terms of products of the types for which we have already verified the boundary condition. In other words, we will write $a = \sum a_i b_i + c_i$ where $a_i \in \bigwedge F$, $b_i \in \bigwedge F^*$ or $b_i = 1 \in F_1$ and deg($c_i$) $> i$ and $r \in R$ is a nonzero divisor. Let $\hat{a} = a \in \bigwedge F \subseteq F_{s-2}$ and let $\hat{a} = a \in \bigwedge F \subseteq F_{s+1}$ and $b' \in \bigwedge F^* \subseteq F_2$ be such that $b'(a) = 0$. Recall $h = 1 \in F_2$. Now using Corollary 3.6 to compute $\hat{a} \cdot h$, we get
\[
\left( n - \frac{s+i-1}{2} \right) a = \hat{a} h - (n-s+1)\hat{a} \wedge \phi^{(s-i-1)/2}.
\]

If $2n + 1 = s + i$, computing $\hat{a} \cdot b'^{(s-i-1)/2}$ using Corollary 3.6, we get
\[
b'^{(s-i-1)/2}(\phi^{(s-i-1)/2})a = \hat{a} b'^{(s-i-1)/2} + \sum_{k=1}^{(s-i-1)/2} (-1)^{k+1} a \\
\wedge b'^{(s-i-1)/2}(\phi^{(k+(s-i-1)/2)}).
\]

This is enough to finish the proof as illustrated by the following computations. Recall that $R = \mathbb{Z}[X, Y]$.
\[
\hat{a} \cdot h = \left( n - s + 1 + \frac{s-i-1}{2} \right) a - (n-s+1)a \wedge \phi^{(s-i-1)/2} \in F_s.
\]
Thus we have
\[
\left( n - \frac{s + i - 1}{2} \right) d(a) \cdot b = 1 \left[ d(\hat{a} \cdot h + (n - s + 1)a \wedge \phi^{(s - i - 1/2)}) \right] b
\]
\[
= 1 \left[ d(\hat{a} \cdot h) b + (-1)^{s+n} \phi^{(n)}(\xi) \times (n - s + 1)a \wedge \phi^{(s - i - 1/2)} \right] \quad \text{by Case 1}
\]
\[
= 1 \left[ [d(\hat{a})h + (-1)^{s-2} \hat{a} \cdot d(h)] b + (-1)^{s+n} (n - s + 1) \phi^{(n)}(\xi) a \wedge \phi^{(s - i - 1/2)} \right]
\quad \text{by Case 2 of Proposition 5.3}
\]
\[
= [(-1)^{s-2+n} \hat{a} \phi^{(n)}(\xi) \cdot h + 0 + (-1)^{s+n} (n - s + 1) \phi^{(n)}(\xi) a \wedge \phi^{(s - i - 1/2)}]
\quad \text{by induction on } s \text{ and associativity}
\]
\[
= (-1)^{s+n} \phi^{(n)}(\xi) [\hat{a} \cdot h + (n - s + 1)a \wedge \phi^{(s - i - 1/2)}]
\]
\[
= \left( n - \frac{s + i - 1}{2} \right) (-1)^{s+n} \phi^{(n)}(\xi) a,
\]
by definition.

Thus \( d(a) b = (-1)^{s+n} \phi^{(n)}(\xi) a \) for all \( a \). This proves the Lemma 5 and completes the proof of Proposition 5.3.

6. ALGEBRA STRUCTURE ON \( M \)

Suppose we have a non-minimal resolution \( G \) and the minimal resolution \( M \) of a cyclic module, with \( G = M \otimes N \), as complexes. Then any algebra structure on \( G \) will induce a multiplication on \( M \) which will be commutative, satisfy the boundary condition, but, in general, not associative. Of course if the multiplication on \( G \), when restricted to \( M \), stays in \( M \), i.e., for \( a, b \in M \subseteq G \), \( a \ast b \in M \), then the induced multiplication will be associative. But this is clearly too much to hope. The following theorem gives a weaker sufficient condition for the induced multiplication on the minimal resolution to be associative.

**THEOREM 6.1.** Let \( \mathcal{F} = M \otimes N \) be a direct sum of complexes, and let \( P_M \) and \( P_N \) be the projections onto \( M \) and \( N \), respectively. Suppose \( \mathcal{F} \) has an algebra structure, given by a multiplication \( \ast \). Then \( \ast \) induces an algebra structure on \( M \), provided, for all \( a \in M \), the composition,

\[
M \times M \overset{\ast}{\longrightarrow} \mathcal{F} \overset{P_N}{\longrightarrow} N \overset{a \ast}{\longrightarrow} \mathcal{F} \overset{P_M}{\longrightarrow} M
\]
is zero. Here \( \ast \mid \) denotes the restriction of \( \ast \) to \( M \times M \subseteq \mathcal{F} \times \mathcal{F} \). (Here we suppress the inclusion \( i_M : M \rightarrow \mathcal{F} \) and \( i_N : N \rightarrow \mathcal{F} \).)
Define \( \mathcal{M} \times \mathcal{M} \to ^{\mathcal{M}} \mathcal{M} \) as follows. For \( a, b \in \mathcal{M} \),
\[
a \ast_M b = P_M(a \ast b).
\]
Since \( a \ast b = (-1)^{\deg a \cdot \deg b} b \ast a \), \( \ast_M \) is commutative. Also, \( \mathcal{F} = \mathcal{M} \oplus \mathcal{N} \) as complexes, \( P_M \) and the inclusion are maps of complexes. Hence
\[
d(a \ast_M b) = d(P_M(a \ast b)) = P_M(d(a \ast b))
= P_M(d(a) \ast b + (-1)^{\deg a} a \ast d(b))
= P_M(d(a) \ast b) + (-1)^{\deg a} P_M(a \ast d(b)).
\]
Since \( a, b \in \mathcal{M} \), \( d(a), d(b) \in \mathcal{M} \).
So, \( d(a \ast_M b) = d(a) \ast_M b + (-1)^{\deg a} a \ast_M d(b) \). So \( \ast_M \) satisfies the differential conditions. Now, consider, \( a, b, c \in \mathcal{M} \). Then
\[
a \ast_M (b \ast_M c) = P_M(a \ast (b \ast_M c))
= P_M(a \ast (P_M(b \ast c)))
= P_M(a \ast ((b \ast c) - P_N(b \ast c)))
= P_M(a \ast (b \ast c) - a \ast (P_N(b \ast c)))
= P_M(a \ast (b \ast c)) - P_M(a \ast (P_N(b \ast c))).
\]
Since the composition
\[
\mathcal{M} \times \mathcal{M} \xrightarrow{\cdot} \mathcal{F} \xrightarrow{r_N} \mathcal{W} \xrightarrow{a \ast} \mathcal{F} \xrightarrow{r_M} \mathcal{M}
\]
is zero,
\[
P_M(a \ast (P_N(b \ast c))) = 0.
\]
So
\[
a \ast_M (b \ast_M c) = P_M(a \ast (b \ast c)) \quad \text{for all } a, b, c \in \mathcal{M}.
\]
Thus
\[
a \ast_M (b \ast_M c) = P_M(a \ast (b \ast c))
= P_M((a \ast b) \ast c) \quad \text{by associativity of } \ast
= (a \ast_M b) \ast_M c.
\]
This shows that \( \ast_M \) is associative. Thus, \( \ast \) induces an algebra structure \( \ast_M \) on \( \mathcal{M} \).

Here we are in a similar situation. We have a non-minimal resolution \( \mathcal{F} \) and the minimal resolution \( M \) is a subcomplex of \( \mathcal{F} \). The multiplication of \( M \) defined in 3.10 is precisely the one induced by the multiplication \( \mu \) on \( \mathcal{F} \). We can now apply Theorem 6.1.

**Proposition 6.2.** The multiplication on \( \mathcal{F} \) induces an algebra structure on \( \mathcal{M} \).
Proof. By Proposition 3.9, the inclusion $i: M \to F$ splits as a map of complexes and $\pi: F \to M$, defined in 3.8, is the projection map. Write $F = M \oplus N$ with $\pi: F \to M$ the projection onto $M$ and $P_N: F \to N$ the projection onto $N$.

Note that

$$N_t = 0 \quad \text{for} \quad t < n$$

$$= \bigoplus_{i=2n-t}^{t} \wedge^i F^* \bigoplus_{i=2n-t+1}^{t-1} \wedge^i F \quad \text{if} \quad t \geq n.$$

So if $a \in N_t \subseteq F_t$, then $\deg a \geq 2n - t$. Also, the inclusion of $N \to F$, which splits is the map $(\pi')^{-1}$ restricted to $N$ (where $(\pi')^{-1}$ is defined in 3.9). Thus for $x \in N$, $a \in F$, $a \cdot x = a(\pi')^{-1}(x)$. Recall that $(\pi')^{-1}(x) = x$ if $x \in \wedge F$ or $x \in M$ and $(\pi')^{-1}(x) = d_{t+1}(x^*)$ if $x \in \wedge F^* \cap N_t$. For convenience, we will denote the multiplication in this special case by $\ast$. Thus if $a \in F$ and $x \in N$ the product is $a \ast x = a((\pi')^{-1}(x))$. Note that we continue to denote $\mu(a, b) = a \cdot b = ab$ for all $a, b \in F$. Now, by Theorem 6.1, we just need to verify that for any $a \in M$, the composition

$$M \times M \to \mu \to F \to P_N \to N \to \ast \to F \to \pi \to M$$

is zero. In other words, for any $a \in M$, and for all $b, c \in M$, we must show that

$$\pi(a \ast P_N(bc)) = 0.$$

First we make the following observation.

Claim. If $b \in \wedge F \cap N$, then $a \ast b \in N$ for all $a \in F$ and in particular $\pi(a \ast b) = 0$.

Proof of Claim. Let $b \in \wedge^i F$ be an element of $N_t \subseteq F_t$. Now, $a \ast b = a((\pi')^{-1}(b)) = ab$. Then if $a \in \wedge F$, $a \cdot b = 0 \in N$. So let $a \in \wedge^i F^* \subseteq F_t$. By the definition of $\mu$, $a \cdot b = \sum_{k=i-j}^{i+j} c_k$, where $c_k \in \wedge F$. Since $b \in N$, we have $i > 2n - t$. So $i, j > 2n - t, j \geq 2n - t - s$. Thus $\deg c_k \geq 2n - (t + s)$ for all $k$. Hence $a \cdot b \in N$ as claimed.

Now to show that $\pi(a \ast P_N(bc)) = 0$, we have three cases.

Case 1. Let $b, c \in \wedge F$. Then $bc = 0$ and hence $\pi(a \cdot P_N(bc)) = 0$.

Case 2. One of $b$ or $c$ is an element of $\wedge F$. Then $bc \in \wedge F$. So, $P_N(bc) = \pi(bc) - \pi(bc) \in \wedge F \cap N$. Then by our claim $\pi(a \ast P_N(bc)) = 0$.

Case 3. $b, c \in \wedge F^*$ and $a \in M_t$. If $bc \in M$, then $P_N(bc) = 0$ and we are through. So, let $bc \in \wedge^i F^* \subseteq F_s$ and $bc \notin M$, so that $j \geq 2n - s$. Then

$$P_N(bc) = \pi(bc) - \pi(bc) = (-1)^{s(1/2 + n)} bc(\eta)^* + w,$$

where $w \in \wedge F \cap N$. Then,
\[ a \ast P_N(bc) = (-1)^{(s-1)/2 + n} a((\pi')^{-1}(bc(\eta))) + a \ast w \]
\[ = (1)^{(s-1)/2 + n} a \cdot (d_{s+1}(bc(\eta))) + a \ast w \]
\[ = (-1)^{(s-1)/2 + n + t} (d_{s+s+1}(a \cdot (bc(\eta))) - d_{s}(a) \cdot (bc(\eta))) + a \ast w. \]

If \( a \in \bigcap F \), then \( a \cdot (bc(\eta)) = 0 \) and by the claim above,
\[ \pi(a \ast P_N(bc)) = (-1)^{t+n + s(s-1)/2} \pi(d_{s}(a) \cdot (bc(\eta))). \tag{6.2.1} \]

If \( a \in \bigcap F^* \subseteq F_1 \), then by the above claim,
\[ \pi(d_{s+s+1}(a((bc(\eta))))) = d_{s+s+1}(\pi(a \cdot bc(\eta))). \]

But, since \( 2n - j - i \geq 2n - s - t \), \( \pi(a \cdot bc(\eta)) = 0 \). So again,
\[ \pi(a \ast P_N(bc)) = (-1)^{t+n + s(s-1)/2} \pi(d_{s}(a) \cdot bc(\eta)). \tag{6.2.2} \]

Finally,
\[ d_{s}(a) \cdot bc(\eta) = \sum_{k} w_k, \quad w_k \in \bigcap^k F. \]

\( \text{Min}\{k | w_k \neq 0\} \) is at least \( 2n - j - t + 1 \) if \( a \in \bigcap F \) and \( 2n - j - i - 1 \) if \( a \in \bigcap F^* \). In either case,
\[ \min\{k | w_k \neq 0\} \geq 2n - j - t + 1 \geq 2n - s - t + 1 = 2n - (s + 1 + t - 1) + 1. \]

Hence, from (6.2.1) and (6.2.2), we get
\[ \pi(a \ast P_N(bc)) = \pi(d_{s}(a) \cdot bc(\eta)) = 0. \]

This verifies the required condition of Theorem 6.1. So \( M \) has an algebra structure. \( \square \)

Thus we have proved,

**Theorem 6.3.** Let \( R \) be a noetherian commutative local ring and \( I \) a Huneke–Ulrich ideal of grade \( 2n - 1 \) (given by \( F = R^{2n} \), \( \phi \in \bigwedge^2 F \), and \( y \in F^* \)). Then the minimal resolutions of \( R/I \) admit an algebra structure.

**Corollary 6.4.** Let \( R \) be a commutative noetherian local ring with maximal ideal \( m \) and residue field \( k \). Let \( I \) be a Huneke–Ulrich ideal as above. Then the tor algebra, \( \text{Tor}^R(R/I, k) \) decomposes as
\[ \text{Tor}^k(R/I, k) = H \bigoplus_{k=1}^\eta E_k, \]
where \( E \) is the divided power algebra over \( R \) generated by \( \{e_1^*, \ldots, e_{2n}^*, h\} \) with \( \deg e_1^* = 1 \) and \( \deg h = 2 \) and \( H \) is an \( E \)-module which is itself trivial as an \( R \)-algebra.

Finally, we have the following surprising binomial identity as a consequence of the algebra structure on \( F \).
Corollary 6.5. Let \( i, s, n, p, l \) be integers such that \( 0 \leq [i/2], [l/2], p \leq [(s - 1)/2] \leq n, \) and \( i \equiv l \equiv s - 1 \mod 2. \) Then

\[
\sum_{j=0}^{s-1} (-1)^{[(j-1)/2]} \left( \begin{array}{c} n-rac{i+j}{2} - 1 \\ s-l-1 \end{array} \right) \\
\times \left( \begin{array}{c} n-p-\left\lfloor \frac{i-1}{2} \right\rfloor \\ s-j-l-1 \end{array} \right) \\
\times \phi^{(n)}(\xi)(a) \wedge \phi^{(p-[l/2])}(\eta) = 0,
\]

if \( l \neq i \neq 2p \) or \( 2p + 1 \) and

\[
= (-1)^{p(l-i-1)/2}
\]

if \( l - i = 2p \) or \( 2p + 1. \)

Proof. In the complex \( F \) associated to a \( \phi \) and \( Y \) as above, consider an element \( a \in A^l F \subseteq F_s, \) and \( b = 1 \in F_1. \) Since \( \mu \) defines an algebra structure on \( F, \) \( d(a)b + (-1)^{i} ad(b) = d(a \cdot b) = 0. \) Hence \( d(a)b = (-1)^{s+1+n+1} a \cdot \phi^{(n)}(\xi). \) By Proposition 5.4, this becomes

\[
\sum_{j=0}^{s-1} \sum_{l=0}^{[(s-1)/2]} (-1)^{p+n+[l-(j-1)/2]} \left( \begin{array}{c} n-rac{i+j}{2} - 1 \\ s-l-1 \end{array} \right) \\
\times \left( \begin{array}{c} n-p-\left\lfloor \frac{i-1}{2} \right\rfloor \\ s-j-l-1 \end{array} \right) \\
\times \phi^{(n-(i-l)/2)}(\xi)(a) \wedge \phi^{(p-[l/2])}(\eta)
\]

Hence the only nonzero term in the sum on the left which survives corresponds to \( p = [l/2] \) and \( i = l. \) Thus, fixing \( l \) and \( p, \) we get

\[
\sum_{j=0}^{s-1} (-1)^{[(j-1)/2]} \left( \begin{array}{c} n-rac{i+j}{2} - 1 \\ s-l-1 \end{array} \right) \\
\times \left( \begin{array}{c} n-p-\left\lfloor \frac{i-1}{2} \right\rfloor \\ s-j-l-1 \end{array} \right) \\
\times \phi^{(n)}(\xi)(a) \wedge \phi^{(p-[l/2])}(\eta)
\]
\[ \begin{align*} 
&= 0, \quad \text{if } l \neq i \text{ or } \left\lfloor \frac{i}{2} \right\rfloor \neq p \\
&= (-1)^{s+p}, \quad \text{if } \left\lfloor \frac{l}{2} \right\rfloor = \left\lfloor \frac{i}{2} \right\rfloor = p. 
\end{align*} \]

REFERENCES


