Incompressible surfaces in handlebodies and boundary reducible 3-manifolds

João Miguel Nogueira [a,b,*,1], Henry Segerman [c,2]

[a] *Department of Mathematics, University of Texas at Austin, 1 University Station, C1200 Austin, TX 78712, USA*
[b] *CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal*
[c] *Department of Mathematics and Statistics, The University of Melbourne, VIC 3010, Australia*

A B S T R A C T

We study the existence of incompressible embeddings of surfaces into the genus two handlebody. We show that for every compact surface with boundary, orientable or not, there is an incompressible embedding of the surface into the genus two handlebody. In the orientable case the embedding can be either separating or non-separating. We also consider the case in which the genus two handlebody is replaced by an orientable 3-manifold with a compressible boundary component of genus greater than or equal to two.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we study the existence of incompressible, but generally boundary compressible, embeddings of compact surfaces with boundary in a boundary reducible 3-manifold[3] with a compressible boundary component of genus greater than or equal to two. In particular we study the case in which the 3-manifold is a handlebody of genus two, which we denote by H_2.[4] This naturally extends to the case of a handlebody of genus greater than two.

This type of question was first studied by Jaco [3] who proved that there is a non-separating, incompressible embedding of an orientable surface of any genus (and one or two boundary components) in H_2. Jaco then asked whether there exist separating, incompressible embeddings of surfaces with arbitrarily high genus in H_2. This question was answered in the affirmative independently by Eudave-Muñoz [1], Howards [2] and Qiu [4].

We extend Jaco's, Qiu's, Howards' and Eudave-Muñoz's results by proving that for any given compact surface with boundary there is an incompressible embedding in a handlebody of genus greater than or equal to two. (That is, for a compact surface of any genus and any number of boundary components greater than or equal to one there is such an embedding.) If the surface is orientable we can make this embedding either non-separating or separating. If the surface is non-orientable

---

[3] A boundary reducible 3-manifold is a 3-manifold with a compressible boundary component.
[4] Note that the disk is the only surface with an incompressible and boundary incompressible embedding in a handlebody.
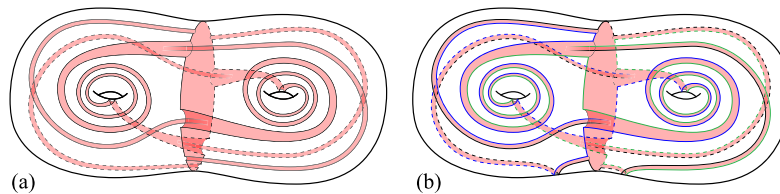
Fig. 1. Separating incompressible embeddings in H_2 of (a) a surface with genus two and one boundary component and (b) a torus with three boundary components. For clarity, when one "band" of the surface is nested within another we only draw the inner band as it enters and exits the tube formed between the outer band and ∂H_2.
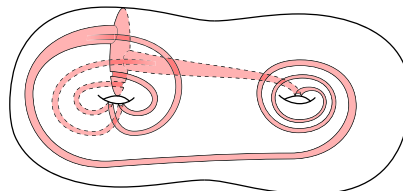

Fig. 2. Klein bottle with one boundary component incompressibly embedded in H_2.

we can only expect to get a non-separating embedding as the handlebody is orientable. Some examples of the incompressible embeddings that we construct can be seen in Figs. 1 and 2.

We make a further extension in Section 4, proving that we have similar incompressible embeddings in a 3-manifold with a compressible boundary component of genus two or greater. With the same conditions on the 3-manifold, unpublished work of Qiu [5] proves the existence of separating, incompressible embeddings of compact surfaces with any genus (and one or two boundary components).

In this paper we study the property of a surface having a π_1-injective embedding, which implies incompressibility.

In Section 2 we prove the following result:

Theorem 1.1. *For each compact surface S with boundary there is a non-separating π_1-injective embedding of S in H_2.*

We prove in Section 3 that when S is orientable there is also a separating embedding:

Theorem 1.2. *For each compact, orientable surface S with boundary there is a separating π_1-injective embedding of S in H_2.*

In Section 4, we extend these results to embeddings in an arbitrary boundary reducible 3-manifold M with a compressible boundary component of genus greater than or equal to two. We say that such a boundary component B is of **type ns** if there is a compression disk D for B such that ∂D does not separate B. We obtain the following general theorems:

Theorem 1.3. *Suppose M has a compressible boundary component of genus greater than or equal to two, of type ns. Then for each compact surface S with boundary there is a non-separating π_1-injective embedding of S in M. Furthermore, if S is orientable there is also a separating π_1-injective embedding of S in M.*

Theorem 1.4. *Suppose M has a compressible boundary component, B, of genus greater than or equal to two. Then for each compact, orientable surface S with boundary there is a π_1-injective embedding of S in M. Furthermore, if B has a (non-)separating compressing disk in M then S has a (resp., non-)separating π_1-injective embedding in M.*

Remark 1.5. Consider a manifold M which is the union of two copies of $T \times [0, 1]$ (where T denotes the torus) identified to each other along a disk in each $T \times \{0\}$. It is not hard to prove that M has a compressible boundary component of genus greater than or equal to two, but no such boundary component of type *ns*. Therefore, M is an example of a manifold that satisfies the conditions of Theorem 1.4 but not of Theorem 1.3.

In Sections 2 and 3 we use a similar standard technique to the one used by Qiu in [4]. We start with either a separating or a non-separating incompressible disk in H_2, and by adding certain boundary parallel bands to the disk we construct the surfaces with the desired properties. For the extension to the embedding in a 3-manifold with a compressible boundary component of genus greater than or equal to two, in Section 4, we first extend to compression bodies with a compressible boundary component of genus greater than or equal to two. Then, for the general case, we use a compression body in M obtained from a compressible boundary component of genus greater than or equal to two.

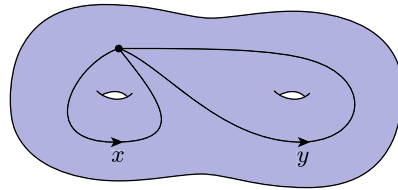In this paper, for a given topological space X, |X| denotes the number of connected components of X.

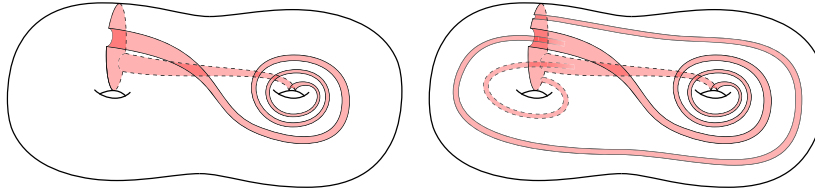**Fig. 3.** Basepoint and generators for $\pi_1(H_2)$.

Fig. 4. Incompressible surfaces embedded in the handlebody of genus 2, the first two steps. The second band added is nested within the previous one.
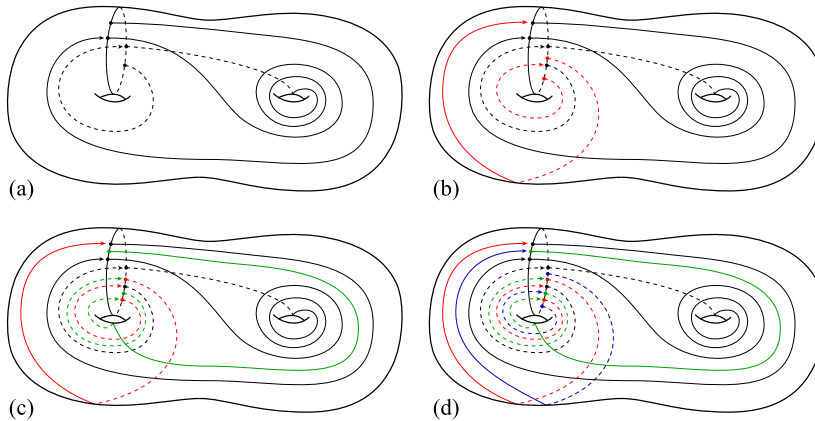

Fig. 5. Schematic diagrams of non-separating orientable surfaces within H_2.

2. The non-separating case

2.1. Orientable surfaces

The goal of this section is to construct a non-separating, π_1-injective embedding of each orientable compact surface with boundary, of genus g and number $b \geqslant 1$ of boundary components, in H_2.

First, fix a basepoint and generators for $\pi_1(H_2)$ as in Fig. 3. The surfaces we will construct consist of a single non-separating disk, embedded in the handlebody H_2, with various bands connecting parts of the boundary of the disk, contained within a small product neighborhood of ∂H_2. See Fig. 4. We will add bands in two stages: In the first stage we increase the genus of the surface to the desired value g, with the surface having only one boundary component. In the second stage we add bands to increase the number of boundary components to b.

In the first diagram of Fig. 4 we see the first added band, which we refer to as α_1. The ith band will be referred to as α_i, and we will abuse notation in also referring to the obvious corresponding generator of π_1 of the surface as α_i. In the second diagram of Fig. 4 we see added the band α_2, which is nested within α_1, in the following manner: The band α_2 starts at the boundary of the disk, goes around the handlebody (see Fig. 4) and returns to the disk, enters the tunnel formed by α_1 and follows it around, then exits, goes around the handlebody again and ends by joining the boundary of the disk. In general α_i and α_{i-1} will be related similarly.

In this example, the image of α_1 in $\pi_1(H_2)$ is y^3 and the image of α_2 is $y^{-1}x^{-1}y^3x$.

We pass to a schematic picture in Fig. 5, where in place of a band we draw only its core curve, with a dot where the band joins the disk and an arrow where it travels inside the tunnel formed by the previous band. In Fig. 5(a) we see the same situation as in the second diagram of Fig. 4. Note that there are two parts of α_2 outside of the α_1 tunnel, one of which
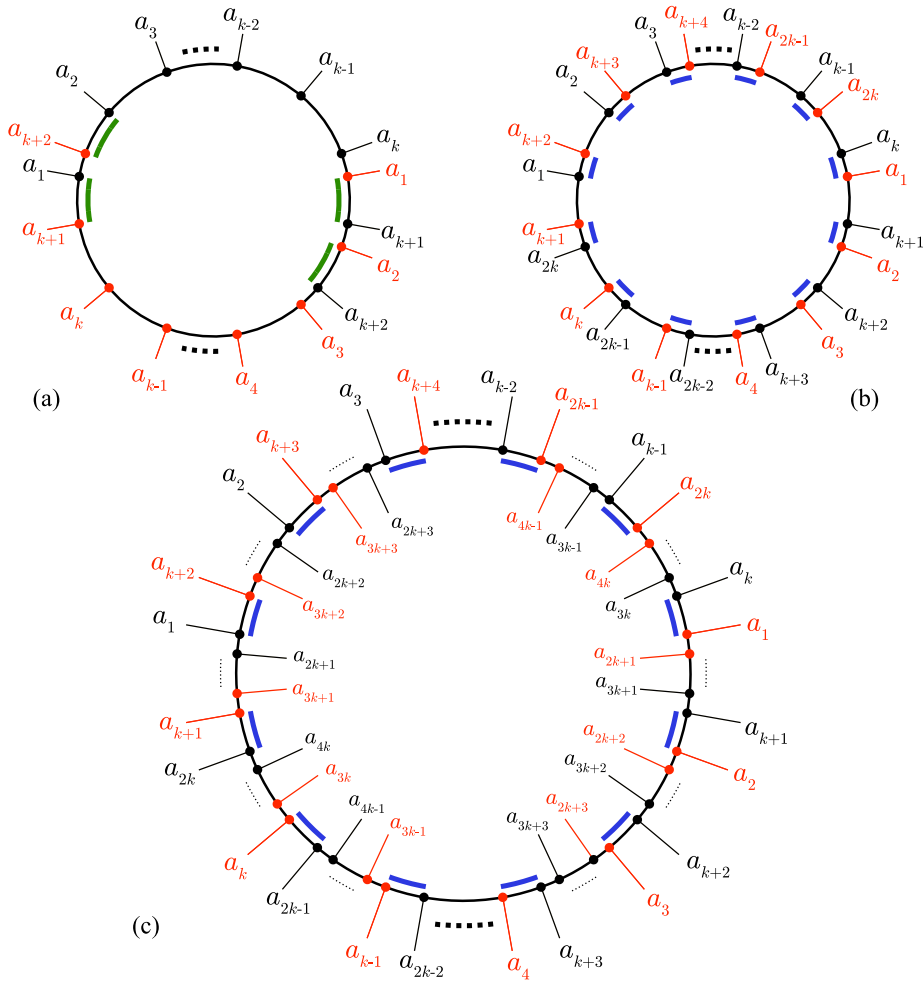
Fig. 6. Orderings of band ends around the disk. Fig. 5(c) corresponds to (a) in this figure, with $k = 2$.

corresponds to $y^{-1}x^{-1}$ (on the front side of the handlebody) and the other to x (on the back side). We could, if we chose, continue with building α_3 in the same way as α_2. That is, one part would go around $y^{-1}x^{-1}$ before entering α_2, and on exiting the other part would go around x. We could continue this indefinitely, the front side curves growing concentrically and the back side curves shrinking concentrically.

Instead however, in Fig. 5(b) we make what we call a **changeover** in the front side curve, looping it around to the back side. The back side curve does not changeover yet. In Fig. 5(c) we continue what was the front side curve with an x loop on the back side, and have no choice in this because the curves cannot cross if the surface is to be embedded. We do however make a changeover in the back side curve, looping it through the x hole and around the front side of the handlebody. We refer to these two changes in path as a **pair of changeovers**. In Fig. 5(d) we continue with yet another band, in this case the front side curve is the first of a new pair of changeovers.

We pass once more to a yet more schematic picture in Fig. 6. Here we show only the pattern of band ends as they meet the disk. We label the ends of α_k as a_k and a'_k, where a_1 is the end on the front side of the handlebody, a'_1 on the back, and in following α_i from a_i (a'_i) we enter the α_{i-1} tunnel at a_{i-1} (resp., a'_{i-1}).

In Fig. 6(a) we have waited until α_{k+1} (for some $k \geqslant 2$) to make the first changeover of the first pair of changeovers, and then α_{k+2} for the second changeover. This is the point at which we switch from increasing genus to increasing the number of boundary components, as will be proven later in this section. After a pair of changeovers we don't have a choice of where to add the ends of the next $k - 2$ bands in the boundary of the starting disk. On the band after adding these $k - 2$ bands, say α_{jk+1} for some positive j, we have a choice for the ends of the band. We choose to continue with another pair of changeovers by adding α_{jk+1} as the first changeover and α_{jk+2} as the second. Again, for the next $k - 2$ bands we don't have a choice of where to add its ends on the disk. We continue this process until we have added the desired number of bands. The continuation of the pattern obtained by the ends of the bands on the starting disk is as shown in Fig. 6(b) and (c).

Let $S_{n,k}$ be the surface described above using n bands and starting the first changeover by adding the band α_{k+1}, for some $k \geqslant 2$.
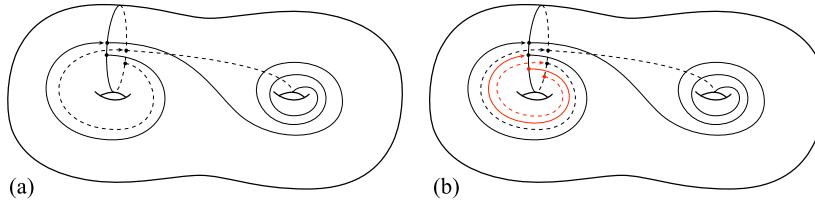
Fig. 7. Schematic diagrams of non-separating orientable surfaces within H_2 with genus zero. The notation is the same as used in Fig. 5.

Lemma 2.1. *Let S be a surface with boundary. Let S' be the surface obtained from S by adding a band $B = [0, 1] \times [0, 1]$, gluing the two ends $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$ of B to segments of ∂S.*

(a) *If the two ends of B are glued to different components of ∂S then $|\partial S'| = |\partial S| - 1$.*
(b) *If the two ends of B are glued to the same component of ∂S and B has an even number of half twists then $|\partial S'| = |\partial S| + 1$.*
(c) *If the two ends of B are glued to the same component of ∂S and B has an odd number of half twists then $|\partial S'| = |\partial S|$.[5]*

Proof. These statements are not difficult to verify. □

Proposition 2.2. *Let $k = 2g$, for some natural number $g \geqslant 1$. Then $S_{n,k}$ has genus g and $n - 2g + 1$ boundary components.*

Proof. Denote by Σ_s the surface obtained from adding the first s bands in the construction of $S_{n,k}$. All bands added in the construction in this section have an even number of half twists because ∂H_2 is orientable and the ends of each band meet the boundary of the disk from the same direction.

The surface Σ_0 is the starting disk so $|\partial \Sigma_0| = 1$. Let i be such that $2i < k$.[6] Assume that $|\partial \Sigma_{2i}| = 1$. By Lemma 2.1(b) $|\partial \Sigma_{2i+1}| = 2$. By construction, the ends of the band α_{2i+2} are on the different boundary components of Σ_{2i+1}. Therefore, by Lemma 2.1(a), $|\partial \Sigma_{2i+2}| = 1$. By induction on i, $|\partial \Sigma_k| = 1$.

By the construction of $S_{n,k}$ we can check that the band ends of the bands after the first pair of changeovers, $\alpha_{k+1}, \ldots, \alpha_n$, are on the same boundary component. This means, by Lemma 2.1(b), that each of these bands added increases the number of boundary components by one. Therefore, the number of boundary components of $S_{n,k}$ is $n - k + 1$. The Euler characteristic of $S_{n,k}$ is $1 - n$ so we conclude that $S_{n,k}$ has genus g. □

We require a special case to obtain a surface with genus zero and arbitrarily many boundary components. We start with the same non-separating disk as before and again denote the bands as α_i. See Fig. 7 for the positions of the first three bands. We continue adding bands as follows: the band α_i goes around x on the front side of the handlebody, goes through the tunnel formed by α_{i-1}, goes around x^{-1} on the back of the handlebody and ends at the disk. The back side and front side curves on the handlebody shrink concentrically. We denote the surface obtained by adding n bands following this procedure by $S_{n,0}$.

Proposition 2.3. *The surface $S_{n,0}$ has genus 0 and $n + 1$ boundary components.*

Proof. The argument is similar to the proof of Proposition 2.2. □

Suppose that $i : S_{n,k} \to H_2$ is the inclusion map as constructed above and $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ the homomorphism induced by i. As in Fig. 3, we denote by x and y the two generators of $\pi_1(H_2)$. We will now prove that the embedded surface in H_2 given by i is incompressible by showing that i_* is injective.

For $k = 0$, from the construction of $S_{n,0}$ we have that

$$i_*(\alpha_s) = x^{-(s-1)} y^3 x^{s-1}, \quad \text{for } 1 \leqslant s \leqslant n.$$

For $k \geqslant 2$ assume that $n = qk + r$, where $0 \leqslant r < k$.
Following the construction of the surface, the generators of $\pi_1(S_{n,k})$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$, and we have:

$$i_*(\alpha_1) = y^3,$$
$$i_*(\alpha_2) = y^{-1} x^{-1} y^3 x,$$
$$\vdots$$

---

[5] Note that the parity of the number of half twists is well defined for a band whose ends are attached to the same boundary component of S.
[6] So we are considering surfaces obtained in the construction before the first pair of changeovers.

$$i_*(\alpha_k) = \left(y^{-1}x^{-1}\right)^{k-1}y^3x^{k-1},$$

$$i_*(\alpha_{k+1}) = x^{-1}i_*(\alpha_k)x,$$

$$i_*(\alpha_{k+2}) = x^{-1}i_*(\alpha_{k+1})(xy),$$

$$\vdots$$

$$i_*(\alpha_{(2j)k}) = \left(x^{-k}\left(y^{-1}x^{-1}\right)^k\right)^{j-1}x^{-k}\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^{j-1}x^k(xy)^{k-1},$$

$$i_*(\alpha_{(2j)k+1}) = \left(y^{-1}x^{-1}\right)i_*(\alpha_{(2j)k})(xy),$$

$$i_*(\alpha_{(2j)k+2}) = \left(y^{-1}x^{-1}\right)i_*(\alpha_{(2j)k+1})x,$$

$$\vdots$$

$$i_*(\alpha_{(2j+1)k}) = \left(\left(y^{-1}x^{-1}\right)^k x^{-k}\right)^j\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^j x^{k-1},$$

$$i_*(\alpha_{(2j+1)k+1}) = x^{-1}i_*(\alpha_{(2j+1)k})x,$$

$$i_*(\alpha_{(2j+1)k+2}) = x^{-1}i_*(\alpha_{(2j+1)k+1})(xy),$$

$$\vdots$$

Either q is even and we have:

$$\vdots$$

$$i_*(\alpha_{qk}) = \left(x^{-k}\left(y^{-1}x^{-1}\right)^k\right)^{\frac{q}{2}-1}x^{-k}\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^{\frac{q}{2}-1}x^k(xy)^{k-1},$$

$$i_*(\alpha_{qk+1}) = \left(y^{-1}x^{-1}\right)i_*(\alpha_{qk})(xy),$$

$$i_*(\alpha_{qk+2}) = \left(y^{-1}x^{-1}\right)i_*(\alpha_{qk+1})x,$$

$$\vdots$$

$$i_*(\alpha_{qk+r}) = \left(y^{-1}x^{-1}\right)^r\left(x^{-k}\left(y^{-1}x^{-1}\right)^k\right)^{\frac{q}{2}-1}x^{-k}\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^{\frac{q}{2}-1}x^k(xy)^k x^{r-1}.$$

Or q is odd and we have:

$$\vdots$$

$$i_*(\alpha_{qk}) = \left(\left(y^{-1}x^{-1}\right)^k x^{-k}\right)^{\frac{q-1}{2}}\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^{\frac{q-1}{2}}x^{k-1},$$

$$i_*(\alpha_{qk+1}) = x^{-1}i_*(\alpha_{qk})x,$$

$$i_*(\alpha_{qk+2}) = x^{-1}i_*(\alpha_{qk+1})(xy),$$

$$\vdots$$

$$i_*(\alpha_{qk+r}) = x^{-r}\left(\left(y^{-1}x^{-1}\right)^k x^{-k}\right)^{\frac{q-1}{2}}\left(y^{-1}x^{-1}\right)^{k-1}y^3\left(x^k(xy)^k\right)^{\frac{q-1}{2}}x^k(xy)^{r-1}.$$

For the proof that i_* is injective we use the following lemma. For a reduced word w, we denote by $L(w)$ the length of the word.

Lemma 2.4. *Let F_2 be the free group with two generators, g and h, and with identity element e. Consider*

$$A = \left\{v_1 g^p v_1', v_2 g^p v_2', \ldots, v_n g^p v_n'\right\} \quad \text{with } p \geqslant 3,$$

where $v_i g^p v_i'$, for $i = 1, \ldots, n$, is a reduced word on g and h, and v_i, v_i' are words where g, or g^{-1}, alternate with powers of h. Furthermore, v_i ends with a power of h and v_i' starts with a power of h. Assume also that $L(v_{i+1}) > L(v_i)$ and $L(v_{i+1}') > L(v_i')$, for $i = 1, 2, \ldots, n-1$. Take w to be a reduced word on the elements of A. That is, w is expressed as a product of elements of A or inverses of those elements, with no consecutive cancelling terms. Then $w \neq e$.

Proof. Let $w = w_1 w_2 \cdots w_s$ where either $w_i \in A$ or $w_i^{-1} \in A$ and $s \in \mathbb{N}$. If $s = 1$ then $w = w_1$ and $w \neq e$. Let $s \geqslant 2$ and assume $w = e$. Then as $w \in F_2$ there are two consecutive letters in w that are inverses of each other. However w_i is a reduced word, which means that two consecutive canceling letters cannot be part of the same w_i. So, cancelling pairs in w are defined by letters of different w_i's. As either $w_i \in A$ or $w_i^{-1} \in A$ we have

$$w_i = u_i g^{\epsilon_i p} u_i', \quad i \in \{1, \ldots, s\}, \ \epsilon_i \in \{-1, 1\},$$

where u_i or u_i^{-1}, and u_i' or $u_i'^{-1}$ are elements of $\{v_j, \ v_j': \ j = 1, \ldots, n\}$.
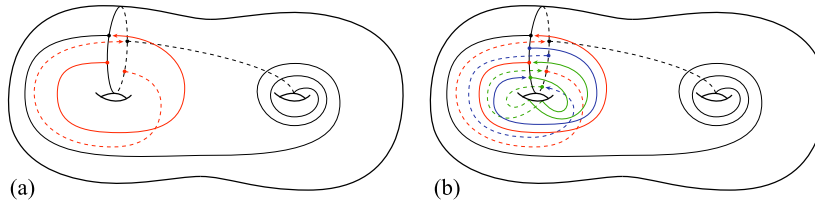
Fig. 8. Schematic diagrams of non-orientable surfaces within the handlebody. These diagrams show the first stage, which adds non-orientable genus.

Take $c_i = u_i' u_{i+1}$ from which we can write $w_i w_{i+1} = u_i g^{\epsilon_i p} c_i g^{\epsilon_{i+1} p} u_{i+1}'$.

If $c_i = e$ then $L(u_i') = L(u_{i+1})$. If the signs of ϵ_i and ϵ_{i+1} are opposite then it follows that $w_i = w_{i+1}^{-1}$ which contradicts the assumption on w. Therefore, when $c_i = e$ we have ϵ_i equal to ϵ_{i+1}, so the powers $g^{\epsilon_i p}$ and $g^{\epsilon_{i+1} p}$ don't cancel each other. We have

$$w = u_1 g^{\epsilon_1 p} u_1' u_2 g^{\epsilon_2 p} u_2' \cdots u_i g^{\epsilon_i p} u_i' u_{i+1} g^{\epsilon_{i+1} p} u_{i+1}' \cdots u_s g^{\epsilon_s p} u_s'$$
$$= u_1 g^{\epsilon_1 p} c_1 g^{\epsilon_2 p} c_2 \cdots c_{i-1} g^{\epsilon_i p} c_i g^{\epsilon_{i+1} p} c_{i+1} \cdots c_{s-1} g^{\epsilon_s p} u_s'.$$

Canceling each c_i as much as we can, for $i = \{1, \ldots, s-1\}$, either

a) $c_i = e$, when u_i and u_i' cancel each other, and, as above, $\epsilon_i = \epsilon_{i+1}$ which means the respective powers of g aren't canceled; or
b) $c_i = h^{t_1} \cdots h^{t_2} g^{\pm 1}$, which can only happen if part of u_i' cancels the whole of u_{i+1}; or
c) $c_i = g^{\pm 1} h^{t_1} \cdots h^{t_2}$, which can only happen if part of u_{i+1} cancels the whole of u_i'; or
d) $c_i = h^{t_1} \cdots h^{t_2}$, which can only happen if part of u_i' (or u_{i+1}) cancels the whole of u_{i+1} (resp., u_i') or when none of u_i' and u_{i+1} is canceled;

where t_1 and t_2 represent some non-zero exponent.

As $p \geqslant 3$, the exponent $\epsilon_i p$ of g from the word w_i, with $i \in \{1, 2, \ldots, s\}$, in w is never entirely canceled. Then $L(w) \geqslant s$. Hence $w \neq e$. $\quad\square$

Proposition 2.5. *The homomorphism* $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ *is injective.*

Proof. Take $A = \{i_*(\alpha_1), i_*(\alpha_2), \ldots, i_*(\alpha_n)\}$ and, using the words from above, write $i_*(\alpha_i) = v_i y^3 v_i'$, for $i = 1, \ldots, n$. We have that the $v_i y^3 v_i'$, for $i = 1, \ldots, n$, are reduced words on x and y, and v_i, v_i' are words where y, or its inverse, alternates with powers of x. Furthermore, v_i ends with a power of x and v_i' starts with a power of x. Also $L(v_{i+1}) > L(v_i)$ and $L(v_{i+1}') > L(v_i')$, for $i = 1, 2, \ldots, n-1$. So, we are under the hypotheses of Lemma 2.4.

Let $w = i_*(\alpha)$ for some $\alpha \in \pi_1(S_{n,k})$ with $\alpha \neq e$. We can write W as a reduced word on the elements of A: $w = w_1 w_2 \cdots w_s$, where either $w_i \in A$ or $w_i^{-1} \in A$. Then by Lemma 2.4, $w \neq e$. Hence i_* is injective. $\quad\square$

The above results prove Theorem 1.1 for orientable surfaces.

2.2. Non-orientable surfaces

The setup for the non-orientable case is similar to the orientable case. We use the same generators for $\pi_1(H_2)$ as in Fig. 3. We again start with a single disk, as in Fig. 8. This time the first k bands will meet the disk from opposite directions, and so have an odd number of half twists, and the constructed surfaces are non-orientable.

As before we refer to the bands as α_i with ends a_i, a_i'. Also, as in the orientable case there are two stages, and again the first stage (adding the first k bands) produces (non-orientable) genus and the second stage produces boundary components.

We show an example of the first stage in Fig. 8. The band α_1 is similar to as in the orientable case, except that one end meets the disk from the other side, and so the image of it in $\pi_1(H_2)$ is xy^3. The band α_2 spirals into the x handle from both sides. We continue spiraling in, but we choose some point at which the two ends of the band cross over their positions at which they meet the disk. We continue adding bands, spiraling outwards again, until the ends a_i and a_i' are adjacent and below a_1 and a_1' in the figure. This happens at α_4 in Fig. 8, and in general will happen at an even subscript.

If we want the number of bands in the first stage, k, to be even, then we end the first stage here. If not then we add one further band continuing the spiral outwards around the x handle, so that the ends of the band are adjacent and above a_1 and a_1'. See Fig. 9(3b). In Fig. 9 the two ends of the band cross over earlier than in Fig. 8, and we see the start of the second stage for k equal to 1, 2 and 3.
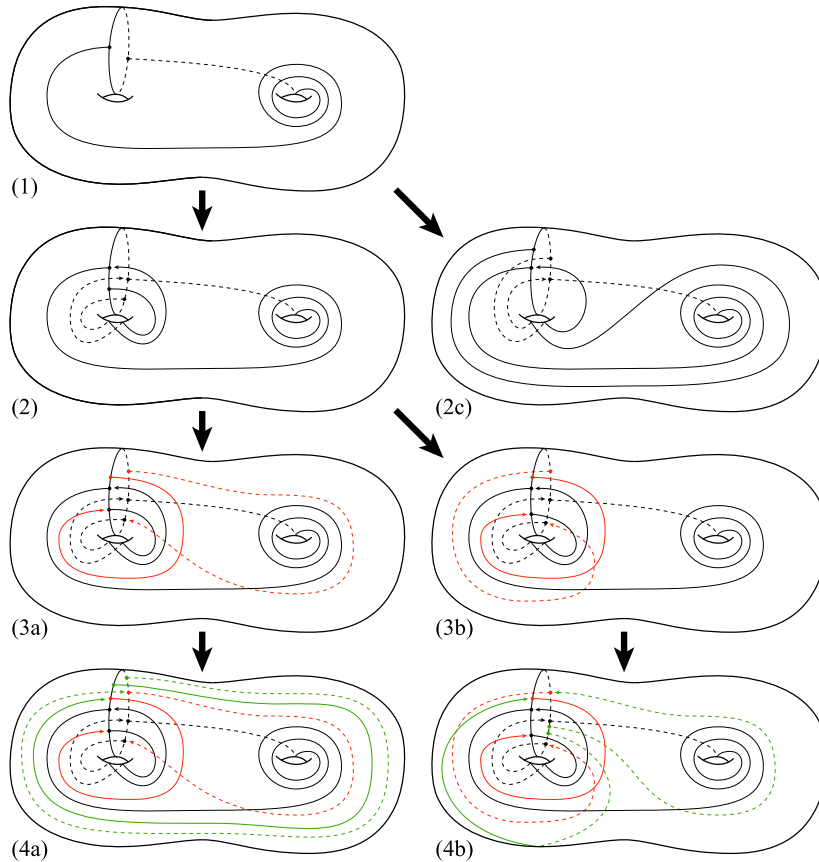
Fig. 9. Schematic diagrams of non-orientable surfaces within the handlebody. This shows the start of the second stage, for even k in (3a) and (4a), for $k = 1$ in (2c) and for other odd k in (4b).

The second stage is slightly different for the two cases: k even and k odd. We also have a different second stage for $k = 1$. Fig. 9(3a) and (4a) shows the continuation for even k. In each of these diagrams we add a band whose ends are adjacent to each other, and as they meet the disk from the same side they have an even number of half twists. The pattern continues with further bands that add parts that continue moving concentrically outwards around both handles of the handlebody. Each added band has adjacent ends that meet the disk from the same side.

Fig. 9(4b) shows the continuation for odd $k \geqslant 3$. Again we add a band whose ends are adjacent to each other and meet the disk from the same side, and so have an even number of half twists. Once again the pattern continues on with new bands with adjacent ends that meet the disk from the same side. This time it is a little less obvious where these bands will go, but it isn't hard to see that we can always continue the bands in a manner that spirals around the x handle in the anti-clockwise direction.

Fig. 9(2c) shows the continuation for $k = 1$. Similarly to the case for even k, the pattern continues with adding parts that move concentrically outwards around both handles of the handlebody.

Let $S_{n,k}$ be the surface described above using n bands and ending the first stage at step $k \geqslant 1$.

Proposition 2.6. *The surface $S_{n,k}$ has non-orientable genus k and $n - k + 1$ boundary components.*

Proof. Denote by Σ_s the surface obtained from adding the first s bands in the construction of $S_{n,k}$. In the construction of $S_{n,k}$ the ends of the first k bands meet the boundary of the disk from opposite directions and the ends of the remaining $n - k$ bands meet the boundary of the disk from the same direction. As ∂H_2 is orientable, the first k bands have an odd number of half twists and the next $n - k$ bands have an even number of half twists.

The starting disk is Σ_0 so $|\partial \Sigma_0| = 1$. Let i be such that $i < k$. Assume that $|\partial \Sigma_i| = 1$. As $i + 1 \leqslant k$, the band α_{i+1} has an odd number of half twists. By Lemma 2.1(c) we have that $|\partial \Sigma_{i+1}| = 1$. So, by induction on i we have that $|\partial \Sigma_k| = 1$.

By the construction of $S_{n,k}$ we can check that the band ends of $\alpha_{k+1}, \ldots, \alpha_n$ are on the same boundary component. Also, these bands have an even number of half twists. So, from Lemma 2.1(b) each of these bands added increases the number of boundary components by one. Therefore the number of boundary components of $S_{n,k}$ is $n - k + 1$. As the Euler characteristic of $S_{n,k}$ is $1 - n$ the surface $S_{n,k}$ has non-orientable genus k. $\square$

Suppose that $i : S_{n,k} \to H_2$ is the inclusion map as constructed above and $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ the induced homomorphism by i. As mentioned before, we denote by x and y the two generators of $\pi_1(H_2)$.

Following the construction of the surface, the generators of $\pi_1(S_{n,k})$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$.

We have:

$$i_*(\alpha_1) = xy^3,$$
$$i_*(\alpha_2) = x(xy^3)x,$$
$$\vdots$$
$$i_*(\alpha_k) = x^{k-1}(xy^3)x^{k-1},$$
$$\vdots$$

Either k is even and we have:

$$\vdots$$
$$i_*(\alpha_{k+1}) = y^{-1}x^{k-1}(xy^3)x^{k-1}x,$$
$$i_*(\alpha_{k+2}) = (y^{-1}x^{-1})y^{-1}x^k y^3 x^k (xy),$$
$$\vdots$$
$$i_*(\alpha_{k+(n-k)}) = (y^{-1}x^{-1})^{n-k-1}y^{-1}x^k y^3 x^k (xy)^{n-k-1}.$$

Or k is odd and we consider the two cases.

For $k = 1$ we have:

$$\vdots$$
$$i_*(\alpha_2) = x(xy^3)xy^{-1}x^{-1},$$
$$i_*(\alpha_3) = (xy)x^2 y^3 xy^{-1}x^{-1}(y^{-1}x^{-1}),$$
$$i_*(\alpha_4) = (xy)^2 x^2 y^3 x(y^{-1}x^{-1})^3,$$
$$\vdots$$
$$i_*(\alpha_n) = (xy)^{n-2}x^2 y^3 x(y^{-1}x^{-1})^{n-1}.$$

For $k \geqslant 3$ we have:

$$\vdots$$
$$i_*(\alpha_{k+1}) = yx^{k-1}(xy^3)x^{k-1}x,$$
$$i_*(\alpha_{k+2}) = x^{-1}yx^k y^3 x^k x,$$
$$i_*(\alpha_{k+3}) = x^{-2}yx^k y^3 x^k x^2,$$
$$\vdots$$
$$i_*(\alpha_{k+(n-k)}) = x^{-(n-k-1)}yx^k y^3 x^k x^{(n-k-1)}.$$

Proposition 2.7. *The homomorphism $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ is injective.*

Proof. Take $A = \{i_*(\alpha_1), i_*(\alpha_2), \ldots, i_*(\alpha_n)\}$ and, using the words from above, write $i_*(\alpha_i) = v_i y^3 v_i'$, for $i = 1, \ldots, n$. We have that the $v_i y^3 v_i'$, for $i = 1, \ldots, n$, are reduced words on x and y, and v_i, v_i' are words where y, or its inverse, alternates with powers of x. Furthermore, v_i ends with a power of x and v_i' starts with a power of x. Also $L(v_{i+1}) > L(v_i)$ and $L(v_{i+1}') > L(v_i')$, for $i = 1, 2, \ldots, n-1$. So, we are under the hypothesis of Lemma 2.4.

Let $w = i_*(\alpha)$ for some $\alpha \in \pi_1(S_{n,k})$ with $\alpha \neq e$. We can write W as a reduced word on the elements of A: $w = w_1 w_2 \cdots w_s$, where either $w_i \in A$ or $w_i^{-1} \in A$. Then by Lemma 2.4, $w \neq e$. Hence i_* is injective. $\square$

From the constructions and results in Section 2, in particular Propositions 2.5 and 2.7, we obtain Theorem 1.1.
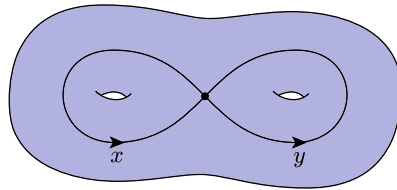
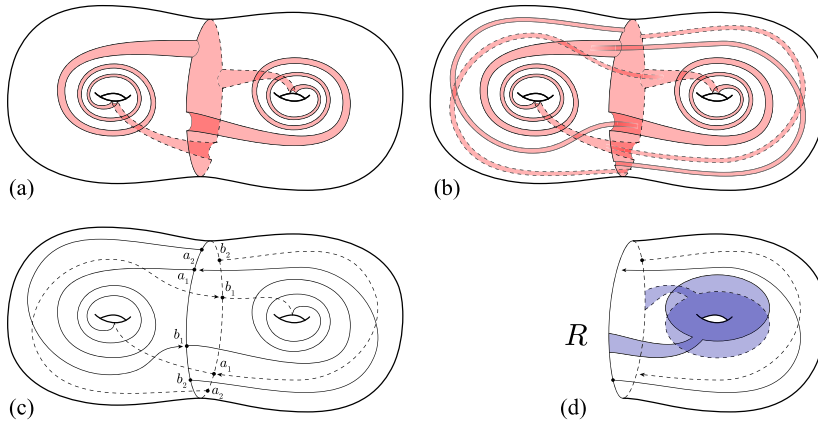**Fig. 10.** Basepoint and generators for $\pi_1(H_2)$, separating case.

Fig. 11. Diagrams for the construction of separating surfaces.

3. The separating case

In this section we construct a separating π_1-injective embedding of each compact, orientable surface with boundary, of genus g and number $b \geqslant 1$ of boundary components, in H_2. A separating surface in an orientable manifold must necessarily be orientable, so we consider only this case. We fix a basepoint and generators as in Fig. 10, which differs very slightly from the non-separating case.

Once again the surfaces consist of a single disk with various bands attached to its border contained within a small product neighbourhood of ∂H_2. See Fig. 11(a) and (b). We construct each surface in a number of steps. Each step consists of adding two bands to the border of the disk. We refer to the bands added in step i as α_i and β_i. In Fig. 11(a) band α_1 is to the left and β_1 to the right.

In Fig. 11(b) we have the second step. Similarly to in the non-separating cases, new bands start at the disk, travel around the boundary of the handlebody until they return to the disk, travel through the tunnel formed by a previous band, re-emerge at the other end of the tunnel, travel around the boundary of the handlebody and finally meet the disk. In step $i + 1$, α_{i+1} travels through the tunnel formed by β_i and β_{i+1} travels through the tunnel formed by α_i. This means that the α_i bands always meet the disk approaching from the left, and the β_i bands from the right.

In Fig. 11(c) we see the second step in schematic form, with the end points of the bands labelled. The band α_i has endpoints a_i and a'_i and β_i has endpoints b_i and b'_i. The end a_1 of α_1 is on the front side of the handlebody as we view it, a'_1 is on the back side and similarly for b_1 and b'_1. For further bands, similarly to in the non-separating cases, we choose the label a_{i+1} to be the end of α_{i+1} nearer to the end of β_i labelled b_i. Here "nearer" means in traveling along the band α_{i+1} towards the tunnel formed by β_i. Similarly for the labelling of b_{i+1} and b'_{i+1}.

As we add further bands, we will never add new bands that pass through the subset of the right side of the boundary of the handlebody, R_β, shown shaded in Fig. 11(d). The region R_β contains the core curve of β_1. The complementary region of R_β in the half of the boundary of the handlebody shown in Fig. 11(d) is a disk. This disk has boundary containing all of the boundary of the starting disk of our surfaces apart from a small neighbourhood of the endpoints of β_1. Thus, we can encode the data of the curves on the right side of Fig. 11(c) as curves drawn on a disk with the same boundary as the starting disk of our surfaces. We represent this disk in Fig. 12. The solid curves we draw on this disk will always cross the line between b_1 and b'_1 (not shown in the figure), as they have to travel around the handle. The curves cannot cross each other, for otherwise the corresponding surface would have self-intersections.

There is a similar region R_α (not shown) on the left side, containing α_1, and the above observations all go through similarly. We get a second disk for curves on the left side of the handlebody, and we draw them on the same picture in Fig. 12 as dashed lines. In the right diagram of Fig. 12 we continue adding bands, and can add as many as we want following this pattern: adding ends moving from the top of the disk down and from the bottom of the disk up but not yet meeting in the middle.
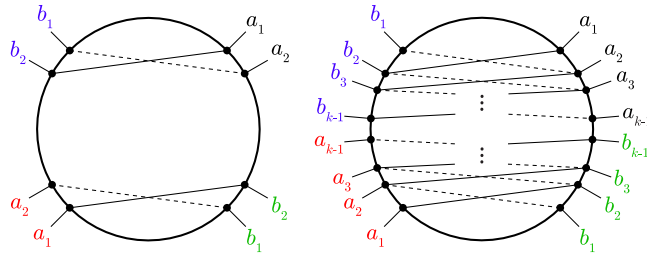
Fig. 12. On the left, band ends around the disk at the second step. The solid lines between a_1 and b_2, and between a'_1 and b'_2 correspond to the curves on the right side of Fig. 11(c). The dashed lines correspond to curves on the left side of that figure. On the right, we can continue this pattern as long as we want before the 'changeover'.
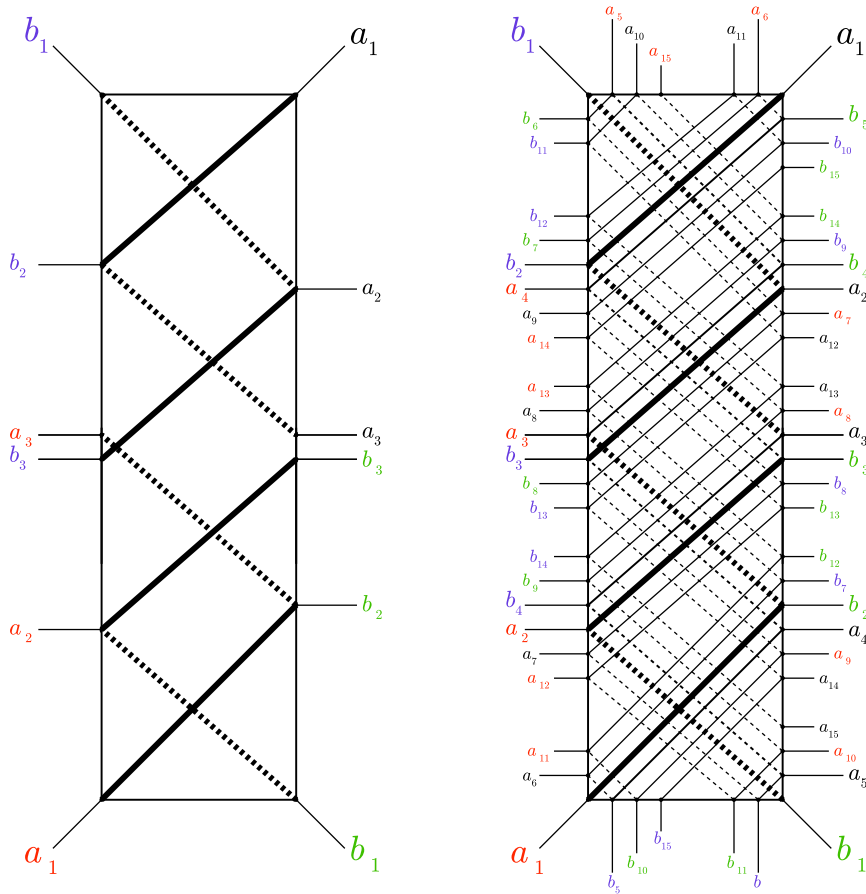

Fig. 13. Band ends around the disk at further steps. On the left, we continue the left diagram of Fig. 12 with the changeover at step 3. On the right, we continue the left diagram by adding bands after the changeover. The lines correspond to curves on the boundary of H_2 as in Fig. 12.

In Fig. 13 we modify the picture of the disk again to a rectangular form. Here a_1, a'_1, b_1 and b'_1 are at the corners of the rectangle so the solid lines cross over the diagonal line from top left to bottom right, and dashed lines cross over the diagonal from top right to bottom left. In the left diagram we have added up until α_3 and β_3, where we again have a form of **changeover** analogous to in the non-separating case: we cross the band end a'_3, of α_3, with the band end b_3, of β_3. Again, we can continue the pattern before the changeover, back and forth from the top and from the bottom as many times as we want, without meeting in the middle until, in this example, the third step. As in the non-separating case, we add genus up until the changeover, and add boundary components afterwards, again by adding pairs of bands. In general, the changeover occurs at step $k \geqslant 2$, where we follow a similar pattern: when adding the bands α_k and β_k we cross the band end a'_k of α_k with the band end b_k of β_k. After the changeover we no longer have a choice for where the ends of the bands are to be added and must follow the pattern. See Figs. 14, 15 and 16 for the general case when k is odd. When k is even we have a similar situation, in fact we can convert the figure to that case by reflecting the top and bottom end diagrams
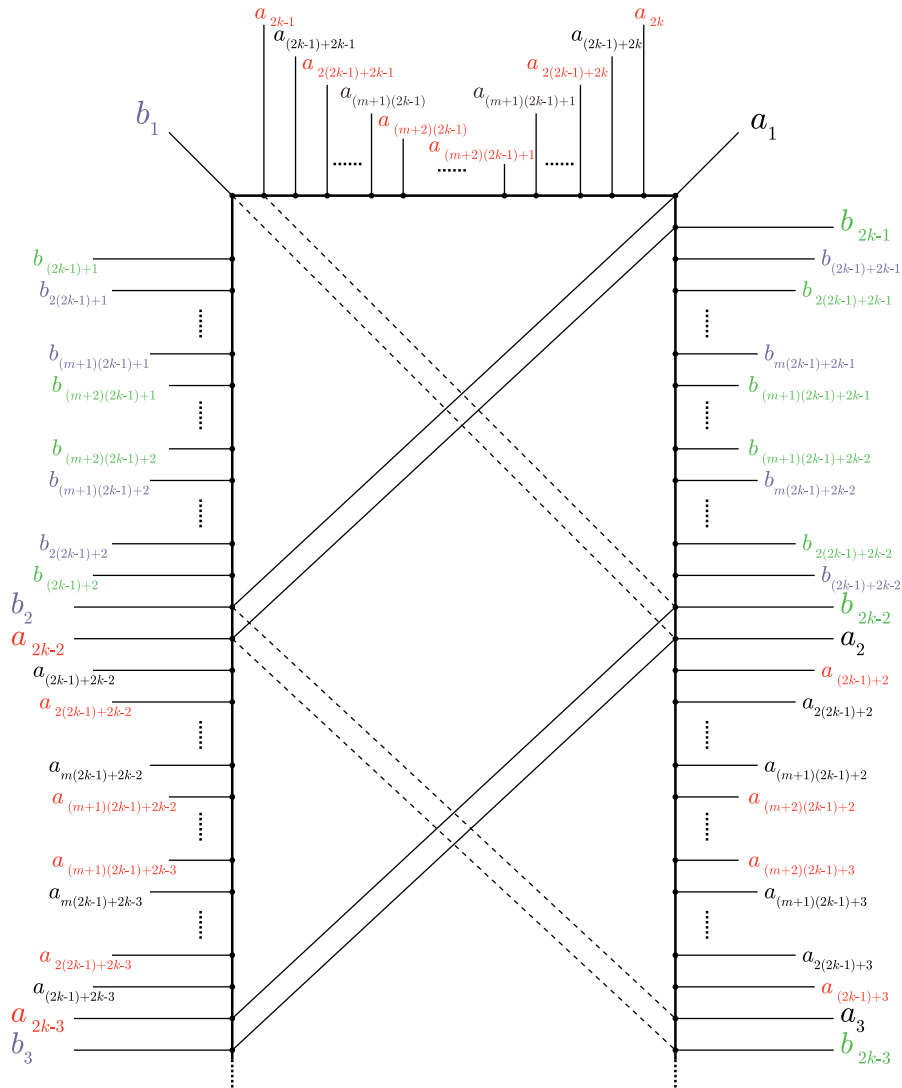
Fig. 14. The picture at the top of the disk for odd k.

along a vertical axis, swapping solid and dashed lines, and swapping all as with bs. The indices of the labels, whether or not they have a prime, and the entire middle diagram are all unchanged.

We can continue adding pairs of bands in the pattern shown in Figs. 14, 15 and 16 indefinitely, but for the construction of a particular surface we stop at some point. Let $S_{n,k}$ be the surface described above using a total of n pairs of bands and doing the changeover at step $k \geqslant 2$. At the very end of the construction we may need to add only one of the two next bands in the same pattern as the general construction, and we let $S'_{n,k}$ to be the surface obtained from $S_{n,k}$ by adding the extra band α_{n+1} to $S_{n,k}$.

Proposition 3.1. *The surface $S_{n,k}$ ($S'_{n,k}$) has genus $g = k - 1$ and $2n - 2g + 1$ (resp., $2n - 2g + 2$) boundary components.*

Proof. Denote by Σ_s the surface obtained from adding the first s steps (pairs of bands) in the construction of $S_{n,k}$. As in the proof of Proposition 2.2 all bands added have an even number of half twists. The surface Σ_0 is the starting disk, so $|\partial \Sigma_0| = 1$. Let i be such that $i < k - 1$. Assume that $|\partial \Sigma_i| = 1$. The next band added is α_{i+1}. As can be checked by construction, the band β_{i+1} has its ends on the two boundary components defined by α_{i+1}. Then $|\partial \Sigma_{i+1}| = 1$. By induction on i, we have $|\partial \Sigma_{k-1}| = 1$.

At step k we do the changeover. By the construction of $S_{n,k}$, we can check that the bands ends of $\alpha_k, \ldots, \alpha_n$ and of $\beta_k, \ldots, \beta_n$ are on the same boundary component. This means, by Lemma 2.1(b), that each of these bands added increases the number of boundary components by one. Therefore, the number of boundary components of $S_{n,k}$ is $2n - 2(k - 1) + 1$. As the Euler characteristics of $S_{n,k}$ is $1 - 2n$ the surface $S_{n,k}$ has genus $k - 1$.
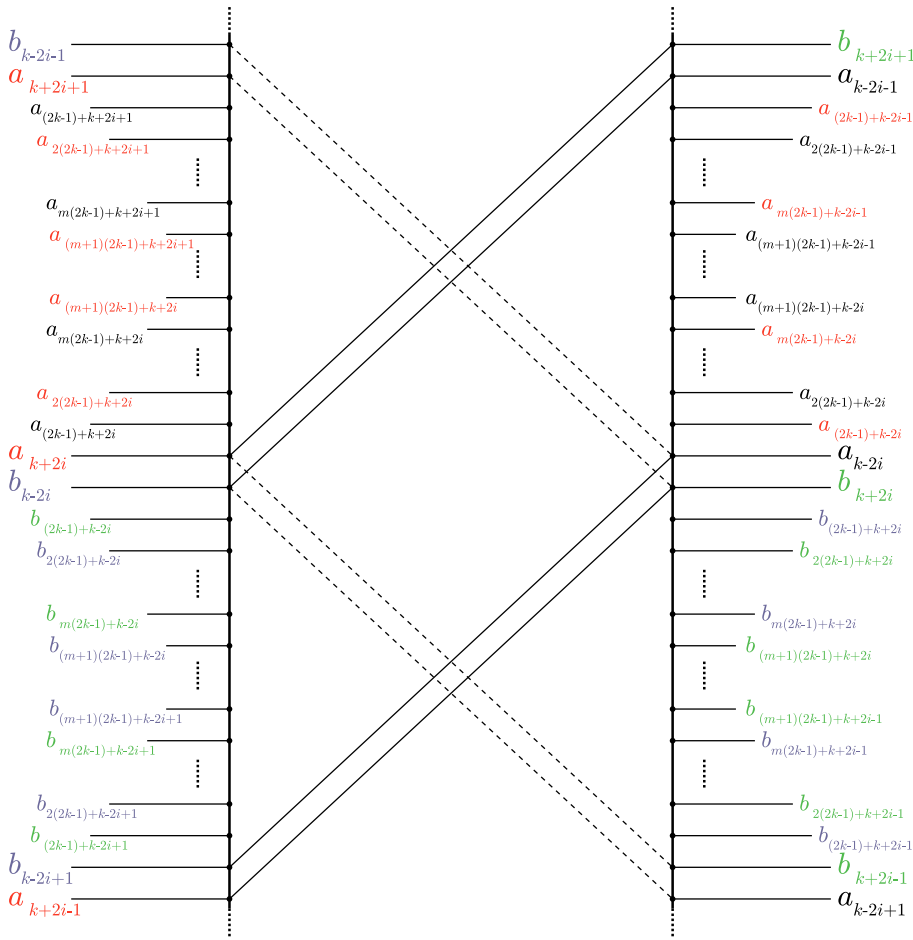
Fig. 15. The picture of a middle section of the disk for odd k.

The surface $S'_{n,k}$ is obtained from $S_{n,k}$ by adding the band α_{n+1} which increases the number of boundary components by one. ☐

In this way we can get any genus $g \geqslant 1$ and any number of boundary components $b \geqslant 1$. To get genus $g = 0$ we need a slightly different starting configuration. See Fig. 17(a). We start with the same disk as in the general separating case and the first pair of bands, α_1 and β_1 again go three times around the x and y handles. However the ends do not alternate around the disk: a_1 and a'_1 are adjacent. The pattern continues in Fig. 17(b), with schematic diagrams in Fig. 17(c) and (d) analogous to Fig. 12. We can continue indefinitely and there are never any choices to make. We denote the surface obtained by adding n bands following this procedure by $S_{n,0}$ (and, by $S'_{n,0}$ the surface obtained from $S_{n,0}$ by adding the band α_{n+1}).

Proposition 3.2. *The surface $S_{n,0}$ ($S'_{n,0}$) has genus 0 and $2n + 1$ (resp., $2n + 2$) boundary components.*

Proof. The argument is similar to the proof of Proposition 3.1. ☐

Suppose that $i : S_{n,k} \to H_2$ is the inclusion map as constructed above and $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ the induced map by i. As mentioned before, we denote by x and y two generators of $\pi_1(H_2)$. Following the surface construction the generators of $\pi_1(S_{n,k})$ are $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$.
If $k = 0$ we have:
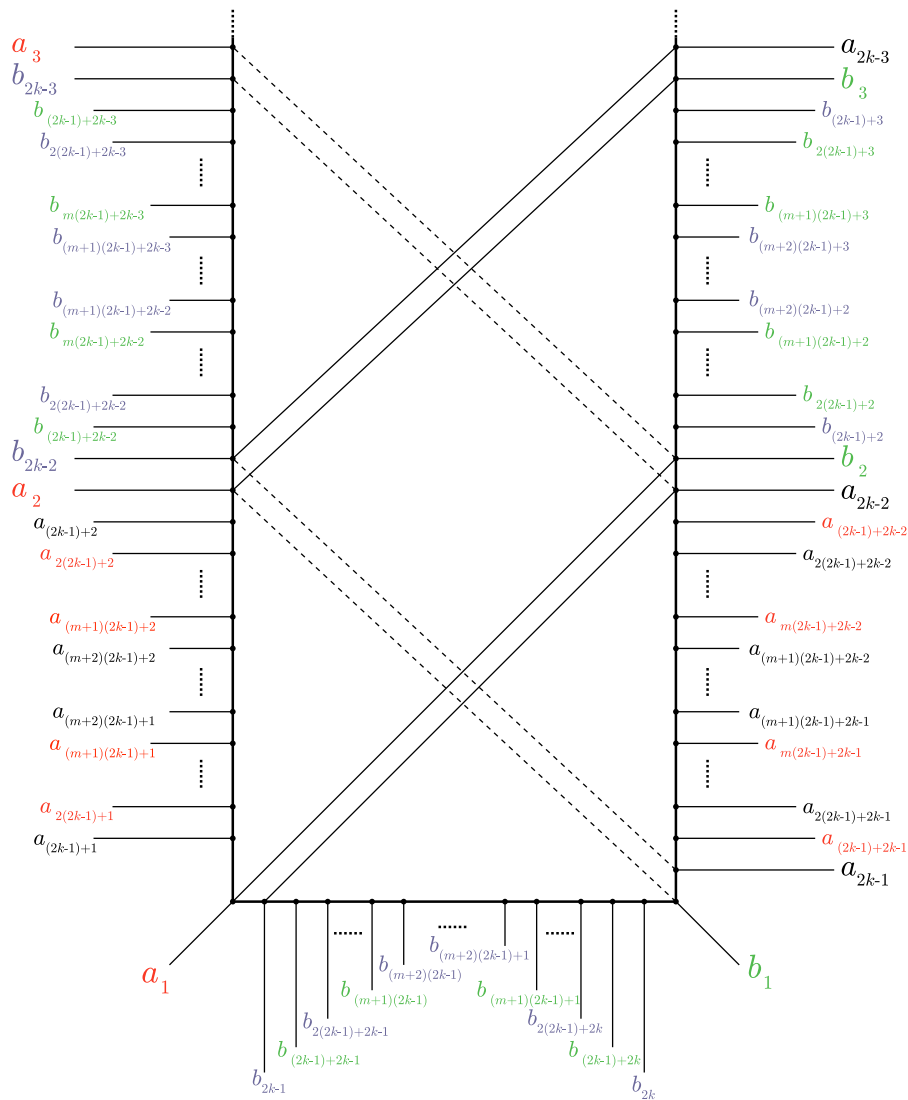
$$i_*(\alpha_1) = x^3,$$
$$i_*(\beta_1) = y^3,$$
$$i_*(\alpha_2) = x^{-1} i_*(\beta_1) x,$$

Fig. 16. The picture at the bottom of the disk for odd k.

$$i_*(\beta_2) = y^{-1} i_*(\alpha_1) y,$$
$$\vdots$$
$$i_*(\alpha_n) = x^{-1} i_*(\beta_{n-1}) x,$$
$$i_*(\beta_n) = y^{-1} i_*(\alpha_{n-1}) y.$$

If $k \geqslant 2$ assume that $n = q(2k-1) + r$, where $0 \leqslant r < 2k - 1$. When k is odd (similarly for even k) we have:

$$i_*(\alpha_1) = x^3,$$
$$i_*(\beta_1) = y^3,$$
$$i_*(\alpha_2) = x i_*(\beta_1) x,$$
$$i_*(\beta_2) = y i_*(\alpha_1) y,$$
$$\vdots$$
$$i_*(\alpha_{2k-2}) = x i_*(\beta_{2k-3}) x,$$
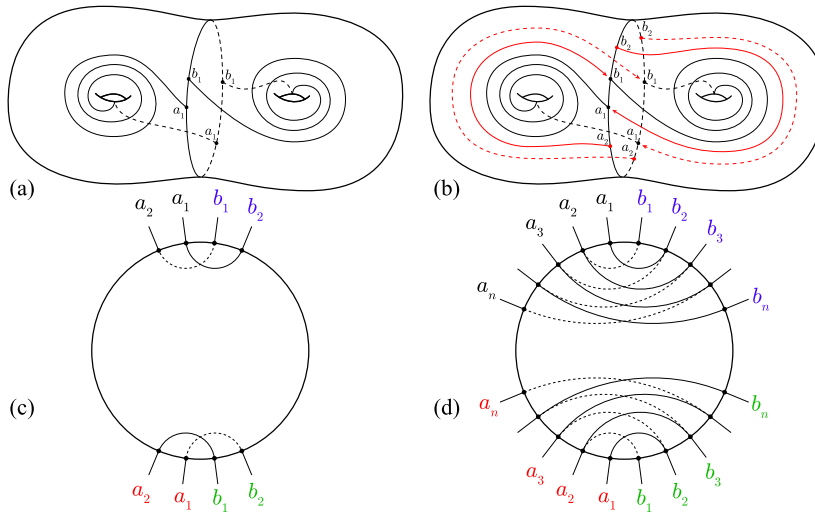$$i_*(\beta_{2k-2}) = y i_*(\alpha_{2k-3}) y,$$

Fig. 17. Diagrams for the construction of separating surfaces of genus zero.

$$i_*(\alpha_{2k-1}) = (xy)^{k-1}x^3(yx)^{k-1},$$

$$i_*(\beta_{2k-1}) = (yx)^{k-1}y^3(xy)^{k-1},$$

$$i_*(\alpha_{(2k-1)+1}) = x^{-1}i_*(\beta_{2k-1})x,$$

$$i_*(\beta_{(2k-1)+1}) = yi_*(\alpha_{2k-1})y^{-1},$$

$$i_*(\alpha_{(2k-1)+2}) = x^{-1}i_*(\beta_{(2k-1)+1})x^{-1},$$

$$i_*(\beta_{(2k-1)+2}) = y^{-1}i_*(\alpha_{(2k-1)+1})y^{-1},$$

$$\vdots$$

$$i_*(\alpha_{(2k-1)+2k-2}) = x^{-1}i_*(\beta_{(2k-1)+2k-3})x^{-1},$$

$$i_*(\beta_{(2k-1)+2k-2}) = y^{-1}i_*(\alpha_{(2k-1)+2k-3})y^{-1},$$

$$i_*(\alpha_{2(2k-1)}) = \left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}y^3(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1},$$

$$i_*(\beta_{2(2k-1)}) = \left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}x^3(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1},$$

$$i_*(\alpha_{2(2k-1)+1}) = x^{-1}i_*(\beta_{2(2k-1)})x,$$

$$i_*(\beta_{2(2k-1)+1}) = yi_*(\alpha_{2(2k-1)})y^{-1},$$

$$i_*(\alpha_{2(2k-1)+2}) = xi_*(\beta_{2(2k-1)+1})x,$$

$$i_*(\beta_{2(2k-1)+2}) = yi_*(\alpha_{2(2k-1)+1})y,$$

$$\vdots$$

$$i_*(\alpha_{(2j-2)(2k-1)+2k-2}) = xi_*(\beta_{(2j-2)(2k-1)+2k-3})x,$$

$$i_*(\beta_{(2j-2)(2k-1)+2k-2}) = yi_*(\alpha_{(2j-2)(2k-1)+2k-3})y,$$

$$i_*(\alpha_{(2j-1)(2k-1)}) = (xy)^{k-1}\left[x^{-1}\left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\right]^{j-1}x^3\left[(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1}x\right]^{j-1}(yx)^{k-1},$$

$$i_*(\beta_{(2j-1)(2k-1)}) = (yx)^{k-1}\left[y\left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\right]^{j-1}y^3\left[(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1}y^{-1}\right]^{j-1}(xy)^{k-1},$$

$$i_*(\alpha_{(2j-1)(2k-1)+1}) = x^{-1}i_*(\beta_{(2j-1)(2k-1)})x,$$

$$i_*(\beta_{(2j-1)(2k-1)+1}) = yi_*(\alpha_{(2j-1)(2k-1)})y^{-1},$$

$$i_*(\alpha_{(2j-1)(2k-1)+2}) = x^{-1}i_*(\beta_{(2j-1)(2k-1)+1})x^{-1},$$

$$i_*(\beta_{(2j-1)(2k-1)+2}) = y^{-1}i_*(\alpha_{(2j-1)(2k-1)+1})y^{-1},$$

$$\vdots$$

$$i_*(\alpha_{(2j-1)(2k-1)+2k-2}) = x^{-1}i_*(\beta_{(2j-1)(2k-1)+2k-3})x^{-1},$$

$$i_*(\beta_{(2j-1)(2k-1)+2k-2}) = y^{-1}i_*(\alpha_{(2j-1)(2k-1)+2k-3})y^{-1},$$

$$i_*(\alpha_{(2j)(2k-1)}) = \left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\left[y\left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\right]^{j-1}y^3$$
$$\times \left[(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1}y^{-1}\right]^{j-1}(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1},$$

$$i_*(\beta_{(2j)(2k-1)}) = \left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\left[x^{-1}\left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\right]^{j-1}x^3$$
$$\times \left[(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1}x\right]^{j-1}(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1},$$

$$i_*(\alpha_{(2j)(2k-1)+1}) = x^{-1}i_*(\beta_{(2j)(2k-1)})x,$$

$$i_*(\beta_{(2j)(2k-1)+1}) = yi_*(\alpha_{(2j)(2k-1)})y^{-1},$$

$$i_*(\alpha_{(2j)(2k-1)+2}) = xi_*(\beta_{(2j)(2k-1)+1})x,$$

$$i_*(\beta_{(2j)(2k-1)+2}) = yi_*(\alpha_{(2j)(2k-1)+1})y,$$

$$\vdots$$

Either q is odd and we have:

$$\vdots$$

$$i_*(\alpha_{(q-1)(2k-1)+2k-2}) = xi_*(\beta_{(q-1)(2k-1)+2k-3})x,$$

$$i_*(\beta_{(q-1)(2k-1)+2k-2}) = yi_*(\alpha_{(q-1)(2k-1)+2k-3})y,$$

$$i_*(\alpha_{q(2k-1)}) = (xy)^{k-1}\left[x^{-1}\left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\right]^{\frac{q-1}{2}}x^3\left[(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1}x\right]^{\frac{q-1}{2}}(yx)^{k-1},$$

$$i_*(\beta_{q(2k-1)}) = (yx)^{k-1}\left[y\left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\right]^{\frac{q-1}{2}}y^3\left[(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1}y^{-1}\right]^{\frac{q-1}{2}}(xy)^{k-1},$$

$$i_*(\alpha_{q(2k-1)+1}) = x^{-1}i_*(\beta_{q(2k-1)})x,$$

$$i_*(\beta_{q(2k-1)+1}) = yi_*(\alpha_{q(2k-1)})y^{-1},$$

$$i_*(\alpha_{q(2k-1)+2}) = x^{-1}i_*(\beta_{q(2k-1)+1})x^{-1},$$

$$i_*(\beta_{q(2k-1)+2}) = y^{-1}i_*(\alpha_{q(2k-1)+1})y^{-1},$$

$$\vdots$$

$$i_*(\alpha_{q(2k-1)+r}) = x^{-1}i_*(\beta_{q(2k-1)+(r-1)})x^{-1},$$

$$i_*(\beta_{q(2k-1)+r}) = y^{-1}i_*(\alpha_{q(2k-1)+(r-1)})y^{-1}.$$

Or q is even and we have:

$$\vdots$$

$$i_*(\alpha_{(q-1)(2k-1)+2k-2}) = xi_*(\beta_{(q-1)(2k-1)+2k-3})x,$$

$$i_*(\beta_{(q-1)(2k-1)+2k-2}) = yi_*(\alpha_{(q-1)(2k-1)+2k-3})y,$$

$$i_*(\alpha_{q(2k-1)}) = \left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\left[y\left(x^{-1}y^{-1}\right)^{k-1}x^{-1}(yx)^{k-1}\right]^{\frac{q}{2}-1}y^3$$
$$\times \left[(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1}y^{-1}\right]^{\frac{q}{2}-1}(xy)^{k-1}x\left(y^{-1}x^{-1}\right)^{k-1},$$

$$i_*(\beta_{q(2k-1)}) = \left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\left[x^{-1}\left(y^{-1}x^{-1}\right)^{k-1}y(xy)^{k-1}\right]^{\frac{q}{2}-1}x^3$$
$$\times \left[(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1}x\right]^{\frac{q}{2}-1}(yx)^{k-1}y^{-1}\left(x^{-1}y^{-1}\right)^{k-1},$$

$$i_*(\alpha_{q(2k-1)+1}) = x^{-1}i_*(\beta_{q(2k-1)})x,$$

$$i_*(\beta_{q(2k-1)+1}) = yi_*(\alpha_{q(2k-1)})y^{-1},$$

$$i_*(\alpha_{q(2k-1)+2}) = xi_*(\beta_{q(2k-1)+1})x,$$

$$i_*(\beta_{q(2k-1)+2}) = y i_*(\alpha_{q(2k-1)+1}) y,$$

$$\vdots$$

$$i_*(\alpha_{q(2k-1)+r}) = x i_*(\beta_{q(2k-1)+(r-1)}) x,$$

$$i_*(\beta_{q(2k-1)+r}) = y i_*(\alpha_{q(2k-1)+(r-1)}) y.$$

For the surface $S'_{n,k}$ we continue by adding the word $i_*(\alpha_{n+1})$ similarly.

Lemma 3.3. *Let F_2 be the free group with two generators, g and h, with identity element e. Consider*

$$A = \left\{ v_1 g^p v'_1, v_2 g^p v'_2, \ldots, v_{n_1} g^p v'_{n_1} \right\},$$

$$B = \left\{ u_1 h^p u'_1, u_2 h^p u'_2, \ldots, u_{n_2} h^p u'_{n_2} \right\}, \quad \text{with } p \geqslant 3,$$

where $v_i g^p v'_i$, for $i = 1, \ldots, n_1$, and $u_i h^p u'_i$, for $i = 1, \ldots, n_2$ are reduced words on g and h, and v_i, v'_i, u_i and u'_i are words where g, or its inverse, alternates with h, or its inverse. Furthermore, v_i (u_i) ends with h or h^{-1} (resp., g or g^{-1}), and v'_i (u'_i) starts with h or h^{-1} (resp., g or g^{-1}). Assume also that $L(v_{i+1}) > L(v_i)$, $L(v'_{i+1}) > L(v'_i)$, for $i = 1, 2, \ldots, n_1 - 1$ and $L(u_{i+1}) > L(u_i)$, $L(u'_{i+1}) > L(u'_i)$, for $i = 1, 2, \ldots, n_2 - 1$. Take w to be a reduced word on the elements of A or B. That is, w is expressed as a product of elements of A or B, or inverses of those elements, with no consecutive cancelling terms. Then $w \neq e$.

Proof. Let $w = w_1 w_2 \cdots w_s$ where each w_i, for $i = 1, \ldots, s$ is a reduced word on either the elements of A or the elements of B, with the condition that if w_i is a word on the elements of A (or B) then w_{i+1} is a word on the elements of B (resp., A). Note that from Lemma 2.4 we have $w_i \neq e$, for $i = 1, \ldots, s$.

If $s = 1$ then $w = w_1$ and $w \neq e$. Let $s \geqslant 2$. After reducing the w'_i as words of g and h, from Lemma 2.4 we have either:

$$w_i = c_i g^{p_i} d_i g^{p'_i} c'_i \quad \text{if } w_i \text{ is a word on the elements of } A, \quad \text{or}$$

$$w_i = c_i h^{p_i} d_i h^{p'_i} c'_i \quad \text{if } w_i \text{ is a word on the elements of } B,$$

where d_i is some reduced word on g and h, and c_i, c'_i, or its inverses, are in $\{u_i, u'_i, v_i, v'_i\}$ and also $|p_i|, |p'_i| \geqslant 2$, for $i = 1, \ldots, s$. Then we get either:

$$w_i w_{i+1} = c_i g^{p_i} d_i g^{p'_i} c'_i c_{i+1} h^{p_{i+1}} d_{i+1} h^{p'_{i+1}} c'_{i+1}, \quad \text{or} \tag{1}$$

$$w_i w_{i+1} = c_i h^{p_i} d_i h^{p'_i} c'_i c_{i+1} g^{p_{i+1}} d_{i+1} g^{p'_{i+1}} c'_{i+1}. \tag{2}$$

Let $l_i = c'_i c_{i+1}$. We also have, for $i = \{1, \ldots, s-1\}$,

a) $l_i = e$, when c'_i and c_{i+1} cancel each other; or
b) $l_i = h^{\pm 1} \cdots h^{\pm 1}$, when part of c'_i (c_{i+1}) cancels c_{i+1} (resp., c'_i) in case (1) (resp., case (2)); or
c) $l_i = g^{\pm 1} \cdots g^{\pm 1}$, when part of c'_i (c_{i+1}) cancels c_{i+1} (resp., c'_i) in case (2) (resp., case (1)); or
d) $l_i = h^{\pm 1} \cdots g^{\pm 1}$, when neither c'_i nor c_{i+1} in case (1) is canceled; or
e) $l_i = g^{\pm 1} \cdots h^{\pm 1}$, when neither c'_i nor c_{i+1} in case (2) is canceled.

Therefore, as $|p_i|, |p'_i| \geqslant 2$, for $i = 1, \ldots, s$ the powers p_i and p'_i are never canceled. This implies that $L(w) > s$ and $w \neq e$. □

Proposition 3.4. *The homomorphism $i_* : \pi_1(S_{n,k}) \to \pi_1(H_2)$ is injective.*

Proof. Take

$$A = \left\{ i_*(\alpha_{2i-1}) \mid 1 \leqslant 2i - 1 \leqslant n, \ i \in \mathbb{N} \right\} \cup \left\{ i_*(\beta_{2i}) \mid 2 \leqslant 2i \leqslant n, \ i \in \mathbb{N} \right\}$$

and

$$B = \left\{ i_*(\beta_{2i-1}) \mid 1 \leqslant 2i - 1 \leqslant n, \ i \in \mathbb{N} \right\} \cup \left\{ i_*(\alpha_{2i}) \mid 2 \leqslant 2i \leqslant n, \ i \in \mathbb{N} \right\}.$$

Using the words from above we can write $i_*(\alpha_{2i-1}) = v_{2i-1} x^3 v'_{2i-1}$, $i_*(\beta_{2i}) = v_{2i} x^3 v'_{2i}$, and $i_*(\beta_{2i-1}) = u_{2i-1} y^3 u'_{2i-1}$, $i_*(\alpha_{2i}) = u_{2i} y^3 u'_{2i}$. We have that $v_i x^3 v'_i$ and $u_i y^3 u'_i$, for $i = 1, \ldots, n$, are reduced words on x and y, where the words v_i, v'_i, u_i and u'_i are words where x, or its inverse, alternate with y, or its inverse. Furthermore, v_i (u_i) ends with y (resp., x), and v'_i (u'_i) starts with y (resp., x). Also, $L(v_{i+1}) > L(v_i)$, $L(v'_{i+1}) > L(v'_i)$, $L(u_{i+1}) > L(u_i)$ and $L(u'_{i+1}) > L(u'_i)$, for $i = 1, \ldots, n - 1$. So, we are under the hypothesis of Lemma 3.3.
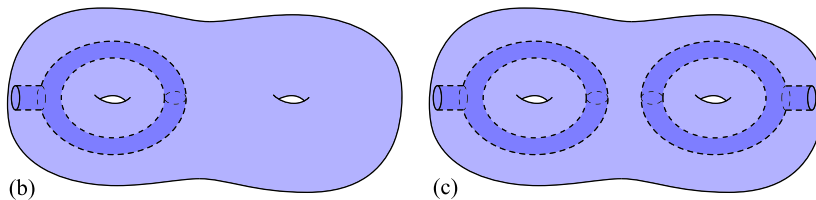
Fig. 18. Examples of the subspace H in cases (b) ($H = P_1 \cup_D Q$) and (c) ($H = P_1 \cup_D P_1$) of Lemma 4.1.

Let $w = i_*(\alpha)$ for some $\alpha \in \pi_1(S_{n,k})$ with $\alpha \neq e$. We can write w as a reduced word on the elements of $A \cup B$: $w = w_1 w_2 \cdots w_s$, for some $s \in \mathbb{N}$, where the w_i's alternate between being words in either the elements of A or the elements of B. Then by Lemma 3.3, $w \neq e$. Therefore i_* is injective. $\square$

The same argument, with the extra word $i_*(\alpha_{n+1})$ included, proves that $i_* : \pi_1(S'_{n,k}) \to \pi_1(H_2)$ is injective. A similar argument also works for the case when k is even.

The above construction and results, and in particular Proposition 3.4, prove Theorem 1.2.

4. Incompressible surfaces in boundary reducible 3-manifolds

Consider a closed surface F_-, not necessarily connected. Take $F_- \times [0,1]$ and add a finite number of 1-handles to $F_- \times \{1\}$. A 3-manifold obtained in this way is said to be a compression body. Dually a connected compression body can be obtained from a connected surface F_+, by taking $F_+ \times [0,1]$ and adding a finite number of 2-handles to $F_+ \times \{0\}$.

Let M be a compact 3-manifold with a compressible boundary component B of genus greater or equal than two.

Take the compressible boundary component B of M and consider $B \times [0,1] \subset M$ where we view B as $B \times \{1\}$. As B is compressible there is a compressing disk for $B \times \{0\}$ in M. We say that a (disjoint) collection of such compressing disks $\{D_1, D_2, \ldots, D_n\}$, for some $n \in \mathbb{N}$, is maximal if any other compressing disk for $B \times \{0\}$ in M either intersects or is isotopic to some disk in the collection.[7] Consider the compression body obtained from $B \times [0,1]$ by adding the 2-handles defined by the compressing disks in a maximal collection of compressing disks for $B \times \{0\}$ in M. We denote this compression body by C, $\partial_+ C = B$ and $\partial_- C = \partial C - B$. Let N be defined by $M = C \cup_{\partial_- C} N$. We have that $\partial_- C$ is incompressible in N, otherwise some isotopy class of compressing disks for $B \times \{0\}$ would be missing from the construction of C. In addition $\partial_- C$ is incompressible in C, because C is obtained from $\partial_- C \times [0,1]$ by adding a collection of 1-handles to $\partial_- C \times \{1\}$. This means that there is no compressing disk for $\partial_- C$ in C. We refer to such a compression body as a maximal compression body of B in M.

Consider a connected compression body C where $\partial_+ C$ has genus greater than or equal to two and such that $\partial_+ C$ is compressible in C. From now on C will denote a compression body with these properties.

Let T_s be a punctured torus with s punctures and take $P_s = T_s \times [0,1]$ and denote by Q a solid torus.

Lemma 4.1. *There are subspaces $H, C^* \subset C$ giving the decomposition $C = C^* \cup_J H$, where $C^* \cap H = J$ is a collection of annuli or disks properly embedded in C, such that*

(a) $H = Q \cup_D Q = H_2$, or
(b) $H = P_{s_1} \cup_D Q$, or
(c) $H = P_{s_1} \cup_D P_{s_2}$,

for some $s_i \in \{0,1\}$, $i = 1, 2$, D is a separating disk in H and C, and J is one disk or one annulus in case (a) *and* (b) *and a collection of two disks or annuli in case* (c) *(see Fig. 18).*

Proof. Let us take C obtained from a closed surface $F \times [0,1]$ by adding a collection of 1-handles to $F \times \{1\}$. Consider a graph G where the vertices correspond to the components of $F \times [0,1]$ and the edges correspond to the 1-handles. Here we assume that C is connected and so G is a connected graph.

Assume first that G has two or more cycles. We can slide the ends of two 1-handles along $F \times \{1\}$ and other 1-handles until they are both on the same component of $F \times [0,1]$. Then by cutting along a disk J we get $C = C^* \cup_J H$ where H is a handlebody of genus two. Note that in this case H contains an essential compressing disk that is separating, and an essential compressing disk that is non-separating, in both H and C.

Now assume that G has precisely one cycle. If all components of F are spheres then the genus of $\partial_+ C$ is one. This is a contradiction to the assumption that $\partial_+ C$ has genus greater than or equal to two. Therefore at least one component of F has genus greater than or equal to one.

---

[7] A maximal collection of compressing disks always exists for B because B is compressible in M (and has finite genus).

- If there is a component of $F \times [0,1]$ with genus one then we slide one 1-handle to be a loop on this vertex of G and if necessary, slide the other 1-handles around (altering the corresponding graph G) to leave only one 1-handle connecting this component to the other components. So, now there is a disk J in C such that $C = C^* \cup_J H$, where $H = P_0 \cup_D Q$.
- If, on the other hand, there is no component of F with genus one then take a component F_1 with positive genus. We slide one 1-handle to be a loop on this vertex of G and again leave only one 1-handle connecting this component to the other components. We take a simple closed curve a in F_1 separating a punctured torus from the rest of F_1 and we slide both ends of the 1-handle to this punctured torus. Taking $F \times [0,1]$ we get a separating annulus $A = a \times [0,1]$. By cutting along this annulus we get $C = C^* \cup_J H$ where $H = P_1 \cup_D Q$ and $J = A$.

Note that also in this case H contains an essential compressing disk that is separating, and an essential compressing disk that is non-separating, in both H and C.

Finally, assume that G is a tree. If the components of F are all spheres apart possibly from at most one torus component then the genus of $\partial_+ C$ is less or equal than one. As in the previous case, this gives a contradiction to the assumption that $\partial_+ C$ has genus greater than or equal to two. If the components of F are all spheres apart from a single surface of genus $g \geqslant 2$ then the genus of $\partial_+ C$ is also $g \geqslant 2$. Then $\partial_+ C$ is incompressible in C, as the existence of a compressing disk would mean that no genus g component of F exists. So, F must have more than one component of genus greater than or equal to one. Take two of these components, say F_1 and F_2, and slide 1-handles if necessary until each $F_i \times [0,1]$ is adjacent to two edges in G and one edge connects $F_1 \times [0,1]$ to $F_2 \times [0,1]$. We have different cases depending on if F_1 and F_2 are tori or are surfaces of higher genus:

- If F_1 and F_2 are both tori then cut the two 1-handles attached to each of them that does not connect them to each other. We denote by H the component with the 1-handle attaching F_1 and F_2. So, we get $C = C^* \cup_J H$ where $H = P_0 \cup_D P_0$ and J is a collection of two disks.
- Without loss of generality, we assume F_1 is a torus and F_2 has genus greater than or equal to two. We cut the 1-handle attached to F_1 but not to F_2 along a disk. Consider a simple closed curve a in F_2 cutting a punctured torus from F_2. We slide the 1-handle connecting F_1 and F_2 to this punctured torus and if necessary, slide other 1-handles until there are no other 1-handles attached to the punctured torus. Taking $F \times [0,1]$ we get an annulus $A = a \times [0,1]$. By cutting along this annulus and the disk we get $C = C^* \cup_J H$ where $H = P_0 \cup_D P_1$ and J is a collection of a disk and an annulus.
- Finally, assume that both F_1 and F_2 have genus greater than or equal to two. Take simple closed curves a_i in F_i cutting a punctured torus from F_i, $i = 1, 2$, and we slide the 1-handle connecting F_1 and F_2 to these punctured tori and all other 1-handles off of the punctured tori. By taking $F \times [0,1]$ we get the annuli $A_i = a_i \times [0,1]$, for $i = 1, 2$. We cut along this annuli and we get $C = C^* \cup_J H$ where $H = P_1 \cup_D P_1$ and $J = A_1 \cup A_2$.

Note that in this case H contains an essential compressing disk that is separating in both H and C (and no essential non-separating, compressing disk in either H or C). $\quad \square$

Let S be a surface embedded in $H_2 = Q \cup_D Q$ as in the previous sections. Let E be the starting disk of the construction of the embedding of S in H_2. Then, by construction, S is in a regular neighborhood $E \times [0,1] \cup \partial H_2 \times [0,1]$ of $E \cup \partial H_2$. We know that the embedding of S is constructed from a finite collection of bands parallel to ∂H_2 attached to E. So, we can take a collection of points $\{x_1, x_2\} \subset \partial H_2 - \partial E$ such that $S \subset E \times [0,1] \cup \partial H_2 \times [0,1] - x_i \times [0,1]$, for $i = 1, 2$. Therefore, in the same way we constructed the π_1-injective embedding of S in H_2 we construct a π_1-injective embedding of S in $H = P_{s_1} \cup_D Q$ and also in $H = P_{s_1} \cup_D P_{s_2}$, for $s_i \in \{0,1\}$, $i = 1, 2$. We will prove the π_1-injectivity in Proposition 4.2. Note that if H is as in Lemma 4.1(a) or (b) we can construct the embedding of S in H following any of the previous sections as H has both an essential separating disk and a non-separating disk; if H is as in Lemma 4.1(c) we can only construct S following Section 3 as H has an essential separating disk but no non-separating disks.

Let $i: S \to H$ be the embedding discussed above and $l: H \to H_2$ the inclusion map. By construction we have the commutativity of the following diagram:

$$
\begin{array}{ccc}
 & H_2 & \\
 {\scriptstyle i}\nearrow & & \uparrow {\scriptstyle l} \\
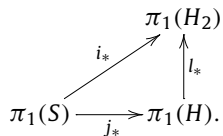S \xrightarrow{\ j\ } & H. &
\end{array}
$$
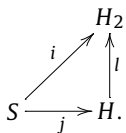
Let C be a compression body as referred above. As in Lemma 4.1 we have the decomposition $C = C^* \cup_J H$, where $C^* \cap H = J$ and J is a collection of disks and annuli where at most two components are annuli. Therefore, $\pi_1(C) = \pi_1(C^*) *_{\pi_1(J)} \pi_1(H)$ is a free product with amalgamation along the group $\pi_1(J)$, which is the trivial group, $\mathbb{Z}$ or $\mathbb{Z} * \mathbb{Z}$.

Consider the inclusion $h: H \to C^* \cup_J H$. Then we have a proper embedding of S in C, induced by j, given by $g = h \circ j$ where the boundary of S lies in $\partial_+ C$.

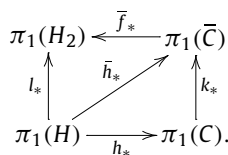**Proposition 4.2.** *The induced homomorphism $g_*: \pi_1(S) \to \pi_1(C)$ is injective.*

Proof. We know that $i = l \circ j$. Therefore, for the induced homomorphisms we have commutativity of the following diagram:
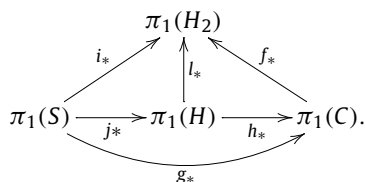
$$\begin{array}{ccc}
 & \pi_1(H_2) & \\
 {\scriptstyle i_*}\nearrow & & \uparrow{\scriptstyle l_*} \\
 \pi_1(S) & \xrightarrow{\ \ j_*\ \ } & \pi_1(H).
\end{array}$$

Let $\bar{C}$ be obtained from C by adding a 2-handle along each core of the annulus components of J. Then $\bar{C} = \overline{C^*} \cup_{\bar{J}} \bar{H}$, where $\overline{C^*}$ and $\bar{H}$ are obtained from C^* and H by the addition of the 2-handles along annulus components of J, and $\bar{J}$ is one or two separating disks in $\bar{C}$. Consider $k : C \to \bar{C}$ the inclusion map. The induced homomorphism $k_* : \pi_1(C) \to \pi_1(\bar{C})$ is onto. Let $\bar{l} : \bar{H} \to H_2$ also be the inclusion map. So the inclusion map l of H in H_2 can be written as the composition of $\bar{l}$ and the inclusion of H in $\bar{H}$. Then, as $\pi_1(\bar{C}) = \pi_1(\overline{C^*}) * \pi_1(\bar{H})$ the homomorphism $\bar{f}_* : \pi_1(\bar{C}) \to \pi_1(H_2)$ defined by

$$\bar{f}_*(a) = \begin{cases} \bar{l}_*(a) & \text{if } a \in \pi_1(\bar{H}), \\ e & \text{if } a \in \pi_1(\overline{C^*}), \end{cases}$$

where e is the identity element of $\pi_1(H_2)$, is onto. Let $f_* = \bar{f}_* \circ k_*$. Consider also the inclusion maps $h : H \to C$ and $\bar{h} = k \circ h$. Therefore, the following diagram is commutative:

$$\begin{array}{ccc}
 \pi_1(H_2) & \xleftarrow{\ \bar{f}_*\ } & \pi_1(\bar{C}) \\
 {\scriptstyle l_*}\uparrow \quad {\scriptstyle \bar{h}_*}\nearrow & & \uparrow{\scriptstyle k_*} \\
 \pi_1(H) & \xrightarrow{\ \ h_*\ \ } & \pi_1(C).
\end{array}$$

From the commutativity of these diagrams we have the commutativity of the following diagram:

$$\begin{array}{ccccc}
 & & \pi_1(H_2) & & \\
 {\scriptstyle i_*}\nearrow & & \uparrow{\scriptstyle l_*} & & \nwarrow{\scriptstyle f_*} \\
 \pi_1(S) & \xrightarrow{\ j_*\ } & \pi_1(H) & \xrightarrow{\ h_*\ } & \pi_1(C). \\
 & & \xrightarrow[g_*]{} & &
\end{array}$$

As f_* is surjective and i_* is injective we have that $g_* = h_* \circ j_*$ is injective. This gives us the conclusion of the proposition. $\square$

Consider the inclusion map $q : C \to M$ and also the embedding $p : S \to M$, where $p = q \circ g$.

Lemma 4.3. *The induced homomorphism* $q_* : \pi_1(C) \to \pi_1(M)$ *is injective.*

Proof. Assume M is connected and let N be defined by $M = C \cup_S N$. C is connected but N may not be, and we let $N_1, N_2, \ldots, N_n$ be the connected components of N. The components $S_1, S_2, \ldots, S_m$ of S are incompressible boundary components of both C and the N_i. We build M from C and the N_i by gluing along the S_j's. At each gluing, we apply either van Kampen's theorem (if this is the first time we join a particular N_i to C) or perform an HNN extension (if this is not the first time we join a particular N_i to C). In both cases the fundamental group of the component containing C injects into the fundamental group of the resulting manifold after the gluing along S_j (this uses the injectivity of $\pi_1(S_j)$ into both $\pi_1(C)$ and $\pi_1(N_i)$). Thus by induction, $\pi_1(C)$ injects into $\pi_1(M)$. $\square$

Proposition 4.4. *The induced homomorphism* $p_* : \pi_1(S) \to \pi_1(M)$ *is injective.*

Proof. We have from Proposition 4.2 that g_* is injective and from Lemma 4.3 that q_* is injective. These imply that p_* is injective. $\square$

Proof of Theorem 1.3. Let M be a 3-manifold and B a compressible boundary component of genus greater than or equal to two, of type *ns*. Consider a maximal compression body, C, in M obtained from B. As in the proof of Lemma 4.1, we consider a graph G where the vertices correspond to the components of $\partial_- C$, and the edges correspond to the 1-handles. Each compressing disk of B in M can be seen as the co-core of a 1-handle and (by possibly changing G) corresponds to a point in the corresponding edge of G.

Assume B is of type *ns*. So, it has a compressing disk D in M where ∂D is non-separating in B, and so D is necessarily non-separating in M. This implies that the point in G (possibly changed by considering a different 1-handle decomposition

of C) corresponding to D is non-separating. Therefore, G has at least one cycle. Then, from the proof of Lemma 4.1, C has a decomposition as in Lemma 4.1(a) or (b). In both cases, H has an essential disk that is separating, and an essential disk that is non-separating, in both H and M. In fact, from Lemma 4.1(a) or (b) the disk D (as denoted there) is also separating in M and a non-separating disk of Q (as denoted in the lemma) is also non-separating in M. Let D_1 be an essential non-separating disk in H and M. Then we can construct embeddings of surfaces in H, and hence in C and M, following Section 2 and the discussion in this section. Note that we start this construction in H with a non-separating essential disk in H, with the same property in C and M. So, we construct a non-separating embedding in M for each orientable, and non-orientable, compact surface with boundary. Now, let D_2 be an essential separating disk in H and M. Then we can construct embeddings of surfaces in H, and hence in C and M, following Section 3 and the discussion in this section. As the starting disk is separating in M we construct a separating embedding for all orientable, compact surfaces with boundary in M. By Proposition 4.4 these embeddings are π_1-injective. This completes the proof of Theorem 1.3. $\quad\square$

Proof of Theorem 1.4. Consider the compressible boundary components of M with genus greater than or equal to two. If one of these components is of type ns then we have Theorem 1.3 and our result follows. Otherwise no boundary component is of type ns. So all compressing disks D of a boundary component B in M are such that ∂D separates B. Fix a choice of compressing disk D of a compressible boundary component B of M. As before, let C denote the maximal compression body defined by B in M. As B is not of type ns, the compression body C has the decomposition given by Lemma 4.1(c). The disk D is dual to a 1-handle in a 1-handle decomposition of C and D is a compressing disk of $\partial_+ C$ so we can adjust the decomposition such that D is the separating disk in H from Lemma 4.1(c). We follow Section 3 and the discussion in this section to construct the embeddings of orientable surfaces in H. Although the surfaces constructed in this way are separating in H and C they are not necessarily separating in M. We have two cases. If D is (non-)separating in M then the surfaces are (resp., non-)separating in M. By Proposition 4.4 these embeddings are π_1-injective. Since the choice of D was arbitrary, this completes the proof of Theorem 1.4. $\quad\square$

Acknowledgement

References

[1] M. Eudave-Muñoz, Incompressible surfaces in tunnel number one knot complements, Topology Appl. 98 (1999) 167–189.
[2] H.N. Howards, Generating disjoint incompressible surfaces, Topology Appl., in press, doi:10.1016/j.topol.2010.11.008.
[3] W. Jaco, Lectures on Three-Manifold Topology, CBMS Reg. Conf. Ser. Math., vol. 43, Amer. Math. Soc., Providence, RI, 1997, reprint.
[4] R. Qiu, Incompressible surfaces in handlebodies and closed 3-manifolds of Heegaard genus 2, Proc. Amer. Math. Soc. 128 (10) (2000) 3091–3097.
[5] R. Qiu, Incompressible surfaces in ∂-reducible 3-manifolds, unpublished, 13 pp.