Computers and Mathematics with Applications 62 (2011) 1055–1067

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A numerical technique for solving fractional optimal control problems

A. Lotfi [a], Mehdi Dehghan [b,*], S.A. Yousefi [a]

[a] *Department of Mathematics, Shahid Beheshti University, G.C. Tehran, Iran*

[b] *Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran*

ARTICLE INFO

Keywords:
Fractional optimal control problem
Caputo fractional derivative
Legendre polynomial basis
Operational matrix
Lagrange multiplier method

ABSTRACT

This paper presents a numerical method for solving a class of fractional optimal control problems (FOCPs). The fractional derivative in these problems is in the Caputo sense. The method is based upon the Legendre orthonormal polynomial basis. The operational matrices of fractional Riemann–Liouville integration and multiplication, along with the Lagrange multiplier method for the constrained extremum are considered. By this method, the given optimization problem reduces to the problem of solving a system of algebraic equations. By solving this system, we achieve the solution of the FOCP. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

1. Introduction

Fractional order dynamics appear in several problems in science and engineering such as viscoelasticity [1,2], bioengineering [3], dynamics of interfaces between nanoparticles and substrates [4], etc. It is also shown that the materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction in fractal porous media can be modeled by fractional order models better than integer models [5].

Although the optimal control theory is an area in mathematics which has been under development for years but the fractional optimal control theory is a very new area in mathematics. An FOCP can be defined with respect to different definitions of fractional derivatives. But the most important types of fractional derivatives are the Riemann–Liouville and the Caputo. General necessary conditions of optimality have been developed for fractional optimal control problems. For instance, in [6,7] the authors have achieved the necessary conditions of optimization for FOCPs with the Riemann–Liouville derivative and also have solved the problem numerically by solving the necessary conditions. There also exist other numerical simulations for FOCPs with Riemann–Liouville fractional derivatives such as [8]. In [9], the necessary conditions of optimization are achieved for FOCPs with the Caputo fractional derivative. There exist numerical simulations for such problems such as [9,10], where the author has solved the problem by solving the necessary conditions approximately. The interested reader can see [11–16] for some recent advances on the fractional differential equations.

In the current paper, we focus on optimal control problems with the quadratic performance index and the dynamic system with the Caputo fractional derivative. We solve the problem directly without using Hamiltonian formulas. Our tools for this aim are the Legendre orthonormal basis and the operational matrix of fractional integration. Our problem formulation is as follows:

$$J = \frac{1}{2}\int_{t_0}^{t_1} [q(t)x^2(t) + r(t)u^2(t)]\mathrm{d}t, \tag{1}$$

$${}_{t_0}^{C}D_t^{\alpha}x(t) = a(t)x(t) + b(t)u(t), \tag{2}$$

* Corresponding author.
E-mail addresses: mdehghan@aut.ac.ir, mdehghan.aut@gmail.com, mehdi_dehghan@yahoo.com (M. Dehghan), s-yousefi@sbu.ac.ir (S.A. Yousefi).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.03.044

$$x(t_0) = x_0,$$

where $q(t) \geq 0, r(t) > 0, b(t) \neq 0$, and the fractional derivative is defined in the Caputo sense,

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} x(\tau)\mathrm{d}\tau, & 0 < \alpha < 1, \\ \dot{x}(t), & \alpha = 1. \end{cases} \tag{3}$$

The method we use here consists of reducing the given optimal control problem to a set of algebraic equations. We expand the fractional state rate ${}_{t_0}^{C}D_t^{\alpha}x(t)$ and control variable $u(t)$ with the Legendre orthonormal polynomial basis with unknown coefficients. Then the operational matrices of the Riemann–Liouville fractional integration and multiplication are utilized to achieve a linear system of algebraic equations, instead of performance index (1) and dynamical system (2) in terms of the unknown coefficients. Finally, the method of constrained extremum is applied which consists of adjoining the constraint equations derived from the given dynamical system to the performance index by a set of undetermined Lagrange multipliers. As a result, the necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of ${}_{t_0}^{C}D_t^{\alpha}x(t)$ and $u(t)$ and the Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed. Also illustrative examples are included to demonstrate the applicability of the new approach. In [10], the problem is solved by a discrete iterative method. The main advantage of the new method is that with the use of only few number of Legendre basis we achieve satisfactory results. Also we refer the interested reader to [3,17–21] for more research works in the subject.

This paper is organized as follows. In Section 2, we present some preliminaries in fractional calculus. In Section 3, we describe the basic formulation of the Legendre basis required for our subsequent development. Section 4 is devoted to the formulation of the fractional optimal control problems. In Section 5, we discuss on the convergence of the method. Finally in Section 6, we report our numerical findings and demonstrate the accuracy of the new numerical scheme by considering two test examples. Section 7 consists of a brief summary.

2. Some preliminaries in fractional calculus

Without loss of generality, in Eqs. (1)–(2) consider $t_0 = 0, t_1 = 1$ and $t \in [0, 1]$. It is obvious that with a linear transformation we can transform each closed interval into another.

Definition 1. A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}t, \quad \alpha > 0, t > 0, \tag{4}$$

$$ {}_0I_t^{0} f(t) = f(t).$$

As a property for the left Riemann–Liouville fractional integration we have

$$ {}_0I_t^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{\alpha+k}, \quad k \in \mathbb{N} \bigcup \{0\}, t > 0. \tag{5}$$

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as follows

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} f(\tau)\mathrm{d}\tau,$$

$$ n-1 < \alpha < n, \quad n \in \mathbb{N}, f \in C_{-1}^m. \tag{6}$$

On $L_p(0, 1)$ $(1 < p < \infty)$ we have the semigroup property for the left Riemann–Liouville fractional integration almost every where [22]

$$ {}_0I_t^{\alpha}\, {}_0I_t^{\beta} = {}_0I_t^{\alpha+\beta}, \quad (\alpha, \beta > 0), \tag{7}$$

where $L_p(0, 1)$ is the space of Lebesgue measurable functions f on $[0, 1]$ such that

$$ \|f\|_p = \left[\int_0^1 |f(t)|^p \mathrm{d}t \right]^{\frac{1}{p}} < \infty.$$

With the aid of semigroup property (7) we have the following properties [22]:

(1) For real values of $\alpha > 0$, the Caputo fractional derivative provides operation inverse to the Riemann–Liouville integration from the left

$$ {}_0^C D_t^{\alpha} {}_0 I_t^{\alpha} f(t) = f(t), \quad \alpha > 0, f(t) \in C[0, 1]. \tag{8}$$

(2) If $f(t) \in C^n[0, 1]$, then

$$ {}_0 I_t^{\alpha} {}_0^C D_t^{\alpha} f(t) = f(t) - \sum_{j=0}^{n-1} \frac{1}{j!} t^j (D_t^j f)(0), \quad n-1 < \alpha \le n,$$

where $C^n[0, 1]$ is the space of functions, which are n times continuously differentiable on $[0, 1]$ and D_t^j is j times differentiation of function f. In particular, if $0 < \alpha \le 1$ and $f(t) \in C^1[0, 1]$ then

$$ {}_0 I_t^{\alpha} {}_0^C D_t^{\alpha} f(t) = f(t) - f(0). \tag{9}$$

3. Properties of the Legendre basis

3.1. The Legendre polynomials

The Legendre polynomials are orthogonal polynomials on the interval $[-1, 1]$ and can be determined with the following recurrence formula:

$$L_{i+1}(y) = \frac{2i+1}{i+1} y L_i(y) - \frac{i}{i+1} L_{i-1}(y), \quad i = 1, 2, \dots$$

where $L_0(y) = 1$ and $L_1(y) = y$. By the change of variable $y = 2t - 1$ we will have the well-known shifted Legendre polynomials. Here, $p'_m(t)$s are the shifted Legendre polynomials of order m which are defined on the interval $[0, 1]$ and can be determined with the following recurrence formula:

$$p'_{m+1}(t) = \frac{2m+1}{m+1}(2t-1)p'_m(t) - \frac{m}{m+1} p'_{m-1}(t), \quad m = 1, 2, 3, \dots$$

$$p'_0(t) = 1, \quad p'_1(t) = 2t - 1. \tag{10}$$

Now we define $p_m(t) = \sqrt{2m+1} p'_m(t)$. For polynomials $p_m(t)$, $m = 0, 1, \dots$ we have

$$\int_0^1 p_i(t) p_j(t) \mathrm{d}x = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \tag{11}$$

The analytical form of the shifted Legendre polynomial of degree i, $p_i(t)$ is as follows

$$p_i(t) = \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)! t^k}{(i-k)!(k!)^2}. \tag{12}$$

3.2. The function approximation

Suppose that $H = L^2[0, 1]$ and

$$\{p_0, p_1, \dots, p_m\} \subset H, \quad m \in \mathbb{N} \bigcup \{0\},$$

be the set of Legendre polynomials and

$$Y = \mathrm{Span}\{p_0, p_1, \dots, p_m\},$$

and f be an arbitrary element in H. Since Y is a finite-dimensional vector space, f has the unique best approximation out of Y such as $y_0 \in Y$, that is

$$\exists y_0 \in Y; \quad \forall y \in Y \ \|f - y_0\|_2 \le \|f - y\|_2,$$

where $\|f\|_2 = \sqrt{<f, f>}$.

Since $y_0 \in Y$, there exist unique coefficients c_j, $j = 0, \dots, m$ such that

$$f(t) \simeq y_0 = \sum_{j=0}^{m} c_j p_j = C^T \Psi, \tag{13}$$

where c_j can be calculated as follows [24]

$$c_j = < f(t), p_j(t) >, \tag{14}$$

and we have

$$C^T = [c_0, \ldots, c_m],$$
$$\Psi^T = [p_0, \ldots, p_m].$$

The fractional integration of the vector Ψ can be approximated as

$$_0I_t^\alpha \Psi \simeq I^\alpha \Psi, \tag{15}$$

where I^α is the $(m+1)\times(m+1)$ Riemann–Liouville fractional operational matrix of integration. We construct I^α as follows, Consider (4). With the aid of Eqs. (5), (12) we calculate $_0I_t^\alpha p_i(t)$, $i = 0, 1, \ldots, m$ as follows

$$_0I_t^\alpha p_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} p_i(\tau)\mathrm{d}\tau = \sqrt{2i+1}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!k!\Gamma(k+1+\alpha)}t^{k+\alpha}. \tag{16}$$

Now we approximate $t^{k+\alpha}$ by $m+1$ terms of the Legendre basis

$$t^{k+\alpha} \simeq \sum_{j=0}^{m} b_j p_j, \tag{17}$$

where

$$b_j = \int_0^1 t^{k+\alpha} p_j(t)\mathrm{d}t = \sqrt{2j+1}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(l+k+\alpha+1)}. \tag{18}$$

So we have

$$_0I_t^\alpha p_i(t) \simeq \sum_{j=0}^{m} B_{ij} p_j(t),$$

where

$$B_{ij} = \sqrt{(2i+1)(2j+1)}\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!}{(i-k)!k!\Gamma(k+1+\alpha)(j-l)!(l!)^2(l+k+\alpha+1)}.$$

So we achieve the left Riemann–Liouville operational matrix I^α as follows

$$I^\alpha = \begin{bmatrix} I_{11} & I_{12} & \cdots & I_{1(m+1)} \\ \vdots & & & \\ I_{(m+1)1} & I_{(m+1)2} & \cdots & I_{(m+1)(m+1)} \end{bmatrix}, \tag{19}$$

where

$$I_{ij} = B_{i-1j-1}, \quad 1 \le i, j \le m+1.$$

In [14], the operational matrix of the fractional Caputo derivative is constructed where for our basis Ψ is as follows

$$D^\alpha = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1(m+1)} \\ \vdots & & & \\ D_{(m+1)1} & D_{(m+1)2} & \cdots & D_{(m+1)(m+1)} \end{bmatrix}, \tag{20}$$

where

$$D_{ij} = \hat{B}_{i-1j-1}, \quad 1 \le i, j \le m+1,$$
$$\hat{B}_{ij} = \sqrt{(2i+1)(2j+1)}\sum_{k=\lceil\alpha\rceil}^{i}\sum_{l=0}^{j}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!}{(i-k)!k!\Gamma(k+1-\alpha)(j-l)!(l!)^2(l+k-\alpha+1)}.$$

Although we do not use the operational matrix of the fractional derivative for solving problem (1), we will need it in Section 5 when we discuss on the convergence of the method.

4. Solving fractional optimal control problems

Consider the following fractional optimal control problem

$$J=\frac{1}{2}\int_0^1[q(t)x^2(t)+r(t)u^2(t)]\mathrm{d}t, \tag{21}$$

$$ {}_0^C D_t^\alpha x(t)=a(t)x(t)+b(t)u(t), \tag{22}$$
$$x(0)=x_0.$$

We expand the fractional derivative of the state variable by the Legendre basis Ψ

$${}_0^C D_t^\alpha x(t)\simeq C^T\Psi(t), \tag{23}$$

$$U(t)\simeq U^T\Psi(t), \tag{24}$$

where

$$C^T=[c_0,\ldots,c_m], \tag{25}$$

$$U^T=[u_0,\ldots,u_m], \tag{26}$$

are unknown. Using (9) and (19), $x(t)$ can be represented as

$$x(t)={}_0I_t^\alpha\,{}_0^C D_t^\alpha x(t)+x(0)\simeq(C^TI^\alpha+d^T)\Psi, \tag{27}$$

where I^α is the fractional operational matrix of integration of order α and

$$d^T=[x_0,0,\ldots,0].$$

Also using (13) and (14) we approximate functions $a(t)$, $b(t)$, $q(t)$, $r(t)$ by the Legendre basis as:

$$a(t)\simeq A^T\Psi,\qquad b(t)\simeq B^T\Psi, \tag{28}$$

$$q(t)\simeq Q^T\Psi,\qquad r(t)\simeq R^T\Psi, \tag{29}$$

where

$$A^T=[a_0,\ldots,a_m],\qquad B^T=[b_0,\ldots,b_m], \tag{30}$$

$$Q^T=[q_0,\ldots,q_m],\qquad R^T=[r_0,\ldots,r_m], \tag{31}$$

and we have

$$a_j=\int_0^1 a(t)p_j(t)\mathrm{d}t,\qquad b_j=\int_0^1 b(t)p_j(t)\mathrm{d}t,$$
$$q_j=\int_0^1 q(t)p_j(t)\mathrm{d}t,\qquad r_j=\int_0^1 r(t)p_j(t)\mathrm{d}t,$$
$$j=0,1,\ldots,m.$$

Using Eqs. (24), (27) and (29), the performance index J can be approximated as

$$J\simeq J[C,U]=\frac{1}{2}\int_0^1[(Q^T\Psi(t))((C^TI^\alpha+d^T)\Psi(t)\Psi(t)^T(C^TI^\alpha+d^T)^T)+(R^T\Psi(t))(U^T\Psi(t)\Psi^T(t)U)]\mathrm{d}t. \tag{32}$$

Using Eqs. (23), (24), (27) and (28), the dynamical system (22) can also be approximated as

$$C^T\Psi-A^T\Psi\Psi^T(C^TI^\alpha+d^T)^T-B^T\Psi\Psi^TU=0. \tag{33}$$

Consider $A^T\Psi\Psi^T$ and $B^T\Psi\Psi^T$ given in the following:

$$A^T\Psi\Psi^T=[v_1(t),\ldots,v_{m+1}(t)],$$
$$B^T\Psi\Psi^T=[w_1(t),\ldots,w_{m+1}(t)].$$

Now we approximate $A^T\Psi\Psi^T$ and $B^T\Psi\Psi^T$ by Ψ as:

$$v_i(t)\simeq\tilde{v}_{i1}p_0+\cdots+\tilde{v}_{i(m+1)}p_m,$$
$$w_i(t)\simeq\tilde{w}_{i1}p_0+\cdots+\tilde{w}_{i(m+1)}p_m,$$

where

$$\tilde{v}_{ij} = \int_0^1 v_i(t)p_{j-1}\mathrm{d}t,$$
$$\tilde{w}_{ij} = \int_0^1 w_i(t)p_{j-1}\mathrm{d}t, \quad 1 \le i, j \le m+1,$$

and we achieve the operational matrices of multiplication as:

$$\tilde{V} = [\tilde{v}_{ij}]_{1\le i,j\le m+1}, \qquad \tilde{W} = [\tilde{w}_{ij}]_{1\le i,j\le m+1},$$

and we have

$$A^T\Psi\Psi^T \simeq \Psi^T\tilde{V}^T, \tag{34}$$
$$B^T\Psi\Psi^T \simeq \Psi^T\tilde{W}^T. \tag{35}$$

Now using Eqs. (34) and (35) in (33) we obtain:

$$C^T\Psi - \Psi^T\tilde{V}^T(C^TI^\alpha + d^T)^T - \Psi^T\tilde{W}^TU = 0,$$

or

$$(C^T - (C^TI^\alpha + d^T)\tilde{V} - U^T\tilde{W})\Psi = 0, \tag{36}$$

and finally using (36) we convert the dynamical system (22) to the following linear system of algebraic equations:

$$(C^T - (C^TI^\alpha + d^T)\tilde{V} - U^T\tilde{W}) = 0. \tag{37}$$

Let

$$J^*[C, U, \lambda] = J[C, U] + [C^T - (C^TI^\alpha + d^T)\tilde{V} - U^T\tilde{W}]\lambda,$$

where

$$\lambda = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}, \tag{38}$$

is the unknown Lagrange multiplier. Now the necessary conditions for the extremum are

$$\frac{\partial J^*}{\partial C} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{39}$$

(Of course $\frac{\partial J^*}{\partial C} = 0$, is the system $\frac{\partial J^*}{\partial c_j} = 0\ j = 0, \dots, m$.) The above equations can be solved for C, U, λ using the Newton iterative method. By determining C, U, we can determine the approximate values of $u(t)$ and $x(t)$ from (24) and (27), respectively. The method we presented here is based on Rietz direct method for solving variational problems. Also we refer the interested reader to [23].

5. On the convergence of the method

In this section, we discuss on the convergence of the method presented in Section 4. First we will find an error upper bound for the operational matrix of the fractional integration I^α and derivative D^α introduced in Section 3.2. To this aim, we restate a theorem from [24].

Theorem 1. *Suppose that H is a Hilbert space and Y is a closed subspace of H such that* $\dim Y < \infty$ *and $\{y_1, y_2, \dots, y_n\}$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then*

$$\|x - y_0\|_2^2 = \frac{G(x, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)},$$

where

$$G(x, y_1, y_2, \dots, y_n) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \cdots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, x\rangle & \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{vmatrix}.$$

Consider the following statement:

$$e_I^\alpha := I^\alpha.\Psi - {}_0I_t^\alpha\Psi,$$

$$e_I^\alpha = \begin{bmatrix} e_{I\,0}^\alpha \\ e_{I\,1}^\alpha \\ \vdots \\ e_{I\,m}^\alpha \end{bmatrix}.$$

We call e_I^α error vector of the operational matrix. In (17) when we approximated $t^{k+\alpha}$ we had

$$t^{k+\alpha} \simeq \sum_{j=0}^m b_j p_j,$$

where b_js obtained by the best approximation. Hence according to Theorem 1 we have

$$\left\| t^{k+\alpha} - \sum_{j=0}^m b_j p_j \right\|_2 = \left(\frac{G(t^{k+\alpha}, p_0, \ldots, p_m)}{G(p_0, \ldots, p_m)} \right)^{\frac{1}{2}}.$$

So according to (16) and (17) we have

$$\|e_{I\,i}^\alpha\|_2 = \left\| {}_0I_t^\alpha p_i(t) - \sum_{j=0}^m B_{ij} p_j(t) \right\|_2$$
$$\leq \sqrt{2i+1} \sum_{k=0}^{i} \left| \frac{(i+k)!}{(i-k)!k!\Gamma(k+1+\alpha)} \right| \left(\frac{G(t^{k+\alpha}, p_0, \ldots, p_m)}{G(p_0, \ldots, p_m)} \right)^{\frac{1}{2}}, \quad 0 \leq i \leq m. \tag{40}$$

In the same way we find an error upper bound for D^α

$$\|e_{Di}^\alpha\|_2 \leq \sqrt{2i+1} \sum_{k=\lceil\alpha\rceil}^{i} \left| \frac{(i+k)!}{(i-k)!k!\Gamma(k+1-\alpha)} \right| \left(\frac{G(t^{k-\alpha}, p_0, \ldots, p_m)}{G(p_0, \ldots, p_m)} \right)^{\frac{1}{2}}, \tag{41}$$

$$0 \leq i \leq m,$$

where

$$e_D^\alpha := D^\alpha.\Psi - {}_0^C D_t^\alpha \Psi,$$

$$e_D^\alpha = \begin{bmatrix} e_{D0}^\alpha \\ e_{D1}^\alpha \\ \vdots \\ e_{Dm}^\alpha \end{bmatrix}.$$

In the above discussion with the aid of Theorem 1 we presented the error upper bounds for the operational matrices of integration and derivative in terms of Gram determinant G. Now we show that with increase in the number of Legendre polynomials, the error vectors e_D^α and e_I^α tend to zero. To this aim, we state the following fact from [25].

Theorem 2. *Suppose that, function $f \in L^2[0, 1]$ is approximated by $q_n(t)$ as follows*

$$q_n(t) = \lambda_0 p_0(t) + \cdots + \lambda_n p_n(t),$$

where

$$\lambda_i = \int_0^1 p_i(t) f(t) \mathrm{d}t, \quad i = 0, 1, \ldots, n.$$

Consider

$$s_n(f) = \int_0^1 [f(t) - q_n(t)]^2 \mathrm{d}t,$$

then we have

$$\lim_{n\to\infty} s_n(f) = 0.$$

Now by considering Theorems 1 and 2 and Eqs. (40) and (41) we can easily observe that with increase in the number of the Legendre basis, the error vectors e_D^α and e_I^α tend to zero.

Consider problem (1), of course when $t_0 = 0$ and $t_1 = 1$. By considering (9) it is easy to see that problem (1) is equivalent to the following problem:

$$J[x] = \frac{1}{2}\int_0^1 \left[q(t)({}_0I_t^\alpha {}_0^C D_t^\alpha x(t) + x_0)^2 + r(t)\left(\frac{1}{b(t)} {}_0^C D_t^\alpha x(t) - \frac{a(t)}{b(t)} x(t)\right)^2 \right] dt.$$

Obviously the set of Legendre polynomials form a basis for the space

$$D_1[0, 1] = \{f(t)|f \text{ is continuously differentiable on interval } [0, 1]\},$$

with uniform norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Consider M_n as n-dimensional subspace of $D_1[0, 1]$ generated by $\{p_i|i = 0, \dots, n\}$. So each element of M_n is in the form

$$\alpha_0 p_0 + \cdots + \alpha_n p_n,$$

where $\alpha_0, \dots, \alpha_n$ are arbitrary real numbers and on each set M_n the functional J leads to a function $J[\alpha_0 p_0 + \cdots + \alpha_n p_n]$ of n variables $\alpha_0, \dots, \alpha_n$. We choose $\alpha_0, \dots, \alpha_n$ in such a way that minimize J, denoting the minimum by μ_n and the element of M_n which yields the minimum by x_n. Clearly

$$M_n \subset M_{n+1},$$

so we have

$$\mu_n \geq \mu_{n+1}.$$

Theorem 3. *Consider the functional J, then*

$$\lim_{n\to\infty} \mu_n = \mu,$$

where

$$\mu = \inf_{x \in D_1[0,1]} J[x].$$

Proof. Given any $\epsilon > 0$ let x^* be such that

$$J[x^*] < \mu + \epsilon,$$

such an x^* exist for any $\epsilon > 0$ by the definition of μ. Since $J[x]$ is continuous on $D_1[0, 1]$

$$|J[x] - J[x^*]| < \epsilon, \tag{42}$$

provided that $\|x - x^*\| < \delta(\epsilon)$. Let η_n be an element of M_n s.t.

$$\|\eta_n - x^*\| < \delta.$$

Obviously such an η_n exists for sufficiently large n. Then using (42), we find that

$$\mu \leq \mu_n \leq J[\eta_n] < \mu + 2\epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} \mu_n = \mu. \quad \square$$

As mentioned above, μ_m is the minimum of the functional J on the set M_m. Also for each element $x(t)$ of M_m we have

$$x(t) = x_0 p_0 + \cdots + x_m p_m = X^T \Psi,$$

so we obtain

$${}_0^C D_t^\alpha x(t) = X^T {}_0^C D_t^\alpha \Psi(t) = X^T.D^\alpha.\Psi(t) + X^T.e_D^\alpha.$$

We consider $X^T.D^\alpha$ as C^T and we also have

$$x(t) = C^T.I^\alpha.\Psi(t) + C^T e_I^\alpha + X_0^T I_t^\alpha e_D^\alpha + d^T\Psi.$$

Now we demonstrate problem (1) on M_m as follows

$$J = \frac{1}{2}\int_0^1 [(Q^T\Psi(t) + e_q)(C^T.I^\alpha.\Psi(t) + C^T e_I^\alpha + X_0^T I_t^\alpha e_D^\alpha + d^T\Psi)^2 + (R^T\Psi(t) + e_r)(U^T\Psi(t))^2]dt, \tag{43}$$

$$C^T.\Psi(t) + X^T.e_D^\alpha = (A^T\psi(t) + e_a)(C^T.I^\alpha.\Psi(t) + C^T e_I^\alpha + X_0^T I_t^\alpha e_D^\alpha + d^T\Psi) + (B^T\Psi(t) + e_b)(U^T\Psi(t)), \tag{44}$$

where Q, R, A, B achieved in (28)–(31) by the least squares approximation, and e_q, e_r, e_a, e_b are remainders of approximations. In Section 4, by considering (23) and (27) we achieved (32) and (33). Then we calculated the operational matrices of multiplication $\tilde{V}$ and $\tilde{W}$ for $A^T\Psi\Psi^T$ and $B^T\Psi\Psi^T$ in (34) and (35). We show the error of the operational matrices of multiplication as:

$$A^T\Psi\Psi^T = \Psi^T\tilde{V}^T + e_v,$$
$$B^T\Psi\Psi^T = \Psi^T\tilde{W}^T + e_w,$$

where

$$e_v{}^T = \begin{bmatrix} e_{v_1} \\ e_{v_2} \\ \vdots \\ e_{v_{m+1}} \end{bmatrix}, \qquad e_w{}^T = \begin{bmatrix} e_{w_1} \\ e_{w_2} \\ \vdots \\ e_{w_{m+1}} \end{bmatrix},$$
$$v_i(t) = \tilde{v}_{i1}p_0 + \cdots + \tilde{v}_{i(m+1)}p_m + e_{v_i},$$
$$w_i(t) = \tilde{w}_{i1}p_0 + \cdots + \tilde{w}_{i(m+1)}p_m + e_{w_i}.$$

Finally, after using the operational matrices of multiplication we achieved the approximated form of problem (1) on M_m

$$J \simeq J[C, U] = \frac{1}{2}\int_0^1 [(Q^T\Psi(t))((C^TI^\alpha + d^T)\Psi(t)\Psi(t)^T(C^TI^\alpha + d^T)^T) + (R^T\Psi(t))(U^T\Psi(t)\Psi^T(t)U)]\mathrm{d}t. \tag{45}$$

$$(C^T - (C^TI^\alpha + d^T)\tilde{V} - U^T\tilde{W})\Psi(t) = 0. \tag{46}$$

In (39) using the Lagrange multiplier method we achieved the minimum of the functional J of problem (45)–(46) on M_m, call it $\hat{\mu}_m$. Theorem 2 ensures that as $m \to \infty$, then e_I^α, e_D^α, e_r, e_q, e_a, e_b, e_v, e_w tend to zero. So we can observe that as m increases Eq. (46) gets close to Eq. (33) and Eq. (33) also gets close to Eq. (44) continuously. Also the same is true for functional J. As m increases the functional J in (45) gets close to functional J in (43). So we can deduce that for large enough values of m, $\hat{\mu}_m$ and μ_m will be close to each other. On the other hand, from Theorem 3 we had

$$\lim_{m\to\infty} \mu_m = \mu,$$

so we can conclude that

$$\lim_{m\to\infty} \hat{\mu}_m = \mu.$$

6. Illustrative test problems

In this section we apply the method presented in Section 4 to solve the following two test examples.

6.1. Example 1

Consider the following time invariant problem

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\mathrm{d}t,$$

subject to the system dynamics

$${}_0^CD_t^\alpha x(t) = -x(t) + u(t),$$

with initial condition:

$$x(0) = 1.$$

Our aim is to find $u(t)$ which minimizes the performance index J. For this problem we have the exact solution in the case of $\alpha = 1$ as follows

$$x(t) = \cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t),$$
$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t),$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} \simeq -0.98.$$

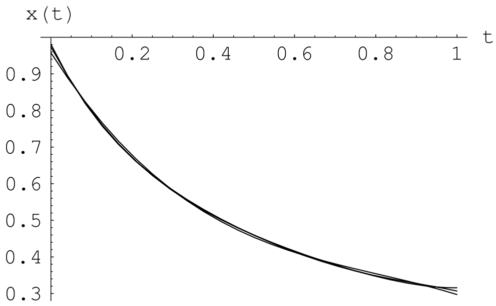

Fig. 1. Approximate solutions for $x(t)$ with $\alpha = 0.8$ and $m = 3, 4, 5$.

As (23) and (24) we approximate ${}_0^C D_t^{\alpha} x(t)$ and $u(t)$. By means of (19) we calculate the operational matrices of fractional integration. These matrices for when $m = 3$ and $\alpha = 0.8, 0.9, 0.99, 1$ are given in the following

$$I^{0.8} = \begin{bmatrix} 0.5965 & 0.2952 & -0.0200 & 0.0059 \\ -0.2952 & 0.0942 & 0.1650 & -0.0156 \\ -0.0200 & -0.1650 & 0.0595 & 0.1186 \\ -0.0059 & -0.0156 & -0.1186 & 0.0449 \end{bmatrix},$$

$$I^{0.9} = \begin{bmatrix} 0.5472 & 0.2941 & -0.0097 & 0.0026 \\ 0.2941 & 0.04210 & 0.1477 & -0.0071 \\ -0.0097 & -0.1477 & 0.0250 & 0.1013 \\ -0.0026 & -0.0071 & -0.1013 & 0.0181 \end{bmatrix},$$

$$I^{0.99} = \begin{bmatrix} 0.5046 & 0.2894 & -0.0009 & 0.0002 \\ -0.2894 & 0.0038 & 0.1310 & -0.0006 \\ -0.0009 & -0.1310 & 0.0021 & 0.0861 \\ -0.0002 & -0.0006 & -0.0861 & 0.0015 \end{bmatrix},$$

$$I^{1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 \\ 0 & -\frac{1}{2\sqrt{15}} & 0 & \frac{1}{2\sqrt{35}} \\ 0 & 0 & -\frac{1}{2\sqrt{35}} & 0 \end{bmatrix}.$$

So by (27) we approximate $x(t)$ while $d^T = [1, 0, 0, 0]$. Also, according to (28)–(31) we have

$$A^T = [-1, 0, 0, 0], \qquad B^T = Q^T = R^T = [1, 0, 0, 0].$$

According to (34) and (35) we also achieve, $\tilde{V} = -I_{4\times 4}$ and $\tilde{W} = I_{4\times 4}$, where $I_{4\times 4}$ is the identity matrix. Finally by solving (39) we achieve vectors C and U for $\alpha = 0.8, 0.9, 0.99, 1$. We show them by C_α and U_α as

$$C_{0.8}^T = [-0.6889, 0.2666, -0.0586, 0.0208],$$
$$U_{0.8}^T = [-0.1773, 0.0978, -0.0068, 0.0066],$$
$$C_{0.9}^T = [-0.7025, 0.2898, -0.0560, 0.0135],$$
$$U_{0.9}^T = [-0.1717, 0.1035, -0.0091, 0.0042],$$
$$C_{0.99}^T = [-0.7164, 0.3062, -0.0517, 0.0073],$$
$$U_{0.99}^T = [-0.1667, 0.1075, -0.0115, 0.0025],$$
$$C_1^T = [-0.7180, 0.3085, -0.0511, 0.0066],$$
$$U_1^T = [-0.1661, 0.1078, -0.0118, 0.0023].$$

After substituting C_α and U_α in (24), (27) we achieve $u(t)$ and $x(t)$ for different values of α, respectively.

In Table 1, the absolute error of $x(t)$ for when $\alpha = 1$ is demonstrated. In Figs. 1 and 2, the state variable $x(t)$ and the control variable $u(t)$ are plotted for $\alpha = 0.8$ and different values of m. It is obvious that with increase in the number of the Legendre basis, the approximate values of $x(t)$ and $u(t)$ converge to the exact solutions. Figs. 3 and 4 demonstrate the approximation of $x(t)$ and $u(t)$ for different values of α together with the exact solution for $\alpha = 1$.

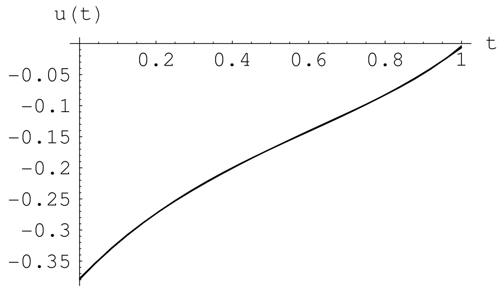

Fig. 2. Approximate solutions for $u(t)$ with $\alpha = 0.8$ and $m = 3, 4, 5$.

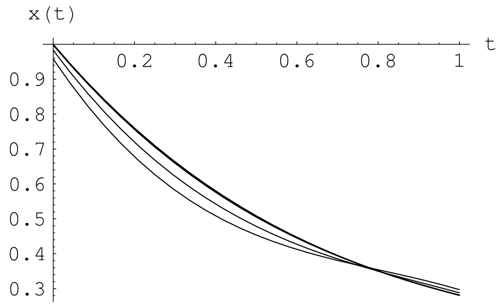

Fig. 3. Approximate solutions of $x(t)$ for $m = 3$ and $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution for $\alpha = 1$.

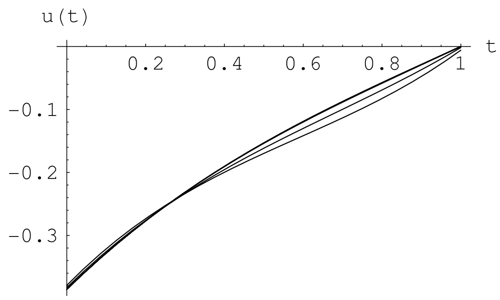

Fig. 4. Approximate solutions of $u(t)$ for $m = 3$ and $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution for $\alpha = 1$.

Table 1
Absolute error of $x(t)$ in Example 1 when $\alpha = 1$.

t	$M = 3$	$M = 4$	$M = 5$
0	−0.00123	−0.0000899	−0.00000625
0.1	0.000341	0.0000477	0.0000134
0.2	0.000508	0.0000325	0.0000212
0.3	0.000112	0.00000774	0.0000324
0.4	−0.000287	0.0000213	0.0000473
0.5	−0.000397	0.0000643	0.0000620
0.6	−0.000150	0.000103	0.0000749
0.7	0.000293	0.000112	0.0000888
0.8	0.000629	0.0000914	0.000107
0.9	0.000371	0.0000941	0.000131

6.2. Example 2

This example considers a time varying fractional optimal control problem. Find the control $u(t)$ which minimizes the performance index J given in Example 1 subject to the following dynamical system

$$
\begin{aligned}
&{}_0^C D_t^{\alpha} x(t) = t x(t) + u(t),\\
&x(0) = 1.
\end{aligned}
$$

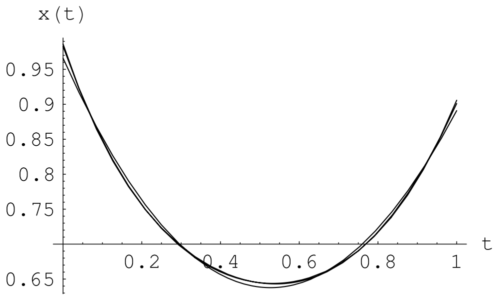

Fig. 5. Approximate solutions for $x(t)$ with $\alpha = 0.8$ and $m = 3, 4, 5$.

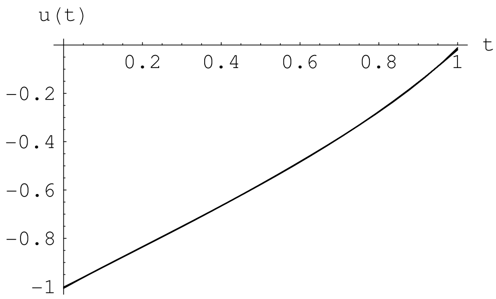

Fig. 6. Approximate solutions for $u(t)$ with $\alpha = 0.8$ and $m = 3, 4, 5$.

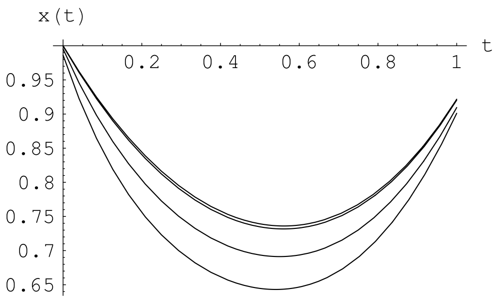

Fig. 7. Approximate solutions of $x(t)$ for $m = 5$ and $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution for $\alpha = 1$.

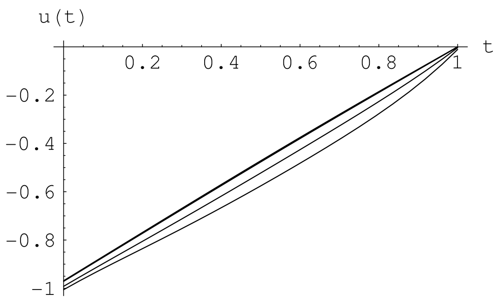

Fig. 8. Approximate solutions of $u(t)$ for $m = 5$ and $\alpha = 0.8, 0.9, 0.99, 1$ and exact solution for $\alpha = 1$.

In Figs. 5 and 6, the state variable $x(t)$ and the control variable $u(t)$ are plotted for $\alpha = 0.8$ and different values of m. It is obvious that with increase in the number of the Legendre basis, the approximate values of $x(t)$ and $u(t)$ converge to the exact solution. Figs. 7 and 8 demonstrate the approximation of $x(t)$ and $u(t)$ for different values of α together with the exact solution for $\alpha = 1$.

Test problems 1 and 2 have been solved in [10] by a different way. Our results, shown in Figs. 1–8 are in good agreement with the results demonstrated in [10]. But we achieved satisfactory numerical results with at last 5 numbers of the Legendre basis while in [10], number of approximations starts in 10 and increases up to 320. So it is worth to point out that we achieved our numerical results with very small order of approximations.

7. Conclusion

In the present work we developed an efficient and accurate method for solving a class of fractional optimal control problems. By utilizing the Legendre basis and the operational matrices of fractional integration and multiplication and the Lagrange multiplier method for constrained optimization we reduced the main problem to the problem of solving a system of algebraic equations. Illustrative examples presented to demonstrate the validity and applicability of the new method.

Acknowledgments

The authors are very grateful to both referees for carefully reading the paper and for their comments and suggestions.

References

[1] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol. 27 (1983) 201–210.
[2] R.L. Bagley, P.J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J. 23 (1985) 918–925.
[3] R.L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng. 32 (2004) 1–104.
[4] T.S. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, Phys. Lett. A 342 (2005) 148–155.
[5] M. Zamani, M. Karimi-Ghartemani, N. Sadati, FOPID controller design for robust performance using particle swarm optimization, J. Fract. Calc. Appl. Anal. 10 (2007) 169–188.
[6] O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynam. 38 (2004) 323–337.
[7] O.P. Agrawal, D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, J. Vib. Control 13 (2007) 1269–1281.
[8] C. Tricaud, Y.Q. Chen, An approximation method for numerically solving fractional order optimal control problems of general form, Comput. Math. Appl. 59 (2010) 1644–1655.
[9] O.P. Agrawal, A formulation and numerical scheme for fractional optimal control problems, J. Vib. Control 14 (2008) 1291–1299.
[10] O.P. Agrawal, A quadratic numerical scheme for fractional optimal control problems, Trans. ASME, J. Dyn. Syst. Meas. Control 130 (1) (2008) 011010-1–011010-6.
[11] M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, Numer. Methods Partial Differential Equations 26 (2010) 448–479.
[12] M. Dehghan, J. Manafian, A. Saadatmandi, The solution of the linear fractional partial differential equations using the homotopy analysis method, Z. Naturforsch. 65a (2010) 935–949.
[13] M. Dehghan, S.A. Yousefi, A. Lotfi, The use of He's variational iteration method for solving the telegraph and fractional telegraph equations, Int. J. Numer. Methods Biomed. Eng. 27 (2011) 219–231.
[14] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, Comput. Math. Appl. 59 (2010) 1326–1336.
[15] A. Saadatmandi, M. Dehghan, A Legendre collocation method for fractional integro-differential equations, J. Vib. Control, in press (doi:10.1177/1077546310395977).
[16] J.R. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Anal. RWA 12 (2011) 262–272.
[17] E.M. Rabei, K.I. Nawafleh, R.S. Hijjawi, S.I. Muslih, D. Baleanu, The Hamilton formalism with fractional derivatives, J. Math. Anal. Appl. 327 (2007) 891–897.
[18] D. Baleanu, About fractional quantization and fractional variational principles, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 2520–2523.
[19] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Application of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach, New York, 1993.
[21] E.M. Rabei, I. Almayteh, S.I. Muslih, D. Baleanu, Hamilton–Jacobi formulation of systems within Caputo's fractional derivative, Phys. Scr. 77 (2008) Article Number 015101.
[22] V.E. Tarasov, Fractional vector calculus and fractional Maxwell's equations, J. Ann. Physics 323 (2008) 2756–2778.
[23] I.M. Gelfand, S.V. Fomin, Calculus of Variation (R.A. Silverman, Trans.), PrenticeHall, 1963.
[24] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and sons, Inc., 1978.
[25] T.J. Rivlin, An Introduction to the Approximation of Functions, Dover Publications, 1981.