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On the connected component of compact composition operators on the Hardy space ☆

Eva A. Gallardo-Gutiérrez [a,*], María J. González [b], Pekka J. Nieminen [c], Eero Saksman [c]

[a] *Departamento de Matemáticas, Universidad de Zaragoza, and IUMA, Plaza San Francisco s/n, 50009 Zaragoza, Spain*

[b] *Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain*

[c] *Department of Mathematics and Statistics, University of Helsinki, PO Box 68, FI-00014 Helsinki, Finland*

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Abstract

We show that there exist non-compact composition operators in the connected component of the compact ones on the classical Hardy space $\mathcal{H}^2$. This answers a question posed by Shapiro and Sundberg in 1990. We also establish an improved version of a theorem of MacCluer, giving a lower bound for the essential norm of a difference of composition operators in terms of the angular derivatives of their symbols. As a main tool we use Aleksandrov–Clark measures.

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* Corresponding author.

E-mail addresses: eva@unizar.es (E.A. Gallardo-Gutiérrez), majose.gonzalez@uca.es (M.J. González), pjniemin@cc.helsinki.fi (P.J. Nieminen), eero.saksman@helsinki.fi (E. Saksman).

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1. Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane and $\mathcal{H}^2$ the classical Hardy space, that is, the space of analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_2 = \left(\sup_{0 \leqslant r < 1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} \right)^{1/2}$$

is finite. It is a well-known consequence of the Littlewood subordination principle that if φ is an analytic map which takes $\mathbb{D}$ into itself, then the composition operator induced by φ,

$$C_\varphi f = f \circ \varphi,$$

is a bounded linear operator on $\mathcal{H}^2$. We let $\mathrm{Comp}(\mathcal{H}^2)$ denote the set of all composition operators acting on $\mathcal{H}^2$.

The properties of composition operators on $\mathcal{H}^2$ and many other function spaces have been studied extensively during the past few decades (see [6,20] for an overview as of the early 1990s). Presently some of the most long-standing open questions in this field are related to the topological structure of the set $\mathrm{Comp}(\mathcal{H}^2)$ endowed with the operator norm metric. Apart from its operator-theoretic significance, this area of study gains interest from the fact that the map $\varphi \mapsto C_\varphi$ provides a remarkable embedding of analytic self-maps of $\mathbb{D}$ into the space of bounded operators on $\mathcal{H}^2$, therefore inducing a natural topology on the unit ball of $\mathcal{H}^\infty$.

The investigation of the topological structure of $\mathrm{Comp}(\mathcal{H}^2)$ was initiated by Berkson [2] and continued in the important papers of MacCluer [9] and Shapiro and Sundberg [22]. Central problems considered in these papers were determining the isolated elements of $\mathrm{Comp}(\mathcal{H}^2)$ and also relating the structure of $\mathrm{Comp}(\mathcal{H}^2)$ to the compactness properties of its members. In particular, it was observed in [9,22] that the collection of all compact composition operators on $\mathcal{H}^2$ is arcwise connected. On the other hand, the authors gave various examples of non-compact composition operators that cannot be connected to the compacts; in fact, in [2,22] it was shown that certain highly non-compact composition operators can be even isolated in $\mathrm{Comp}(\mathcal{H}^2)$.

Towards the end of their paper, Shapiro and Sundberg [22] raised the following fundamental question and conjectured that it had a positive answer:

($*$) Do the compact composition operators form a connected component of the set $\mathrm{Comp}(\mathcal{H}^2)$?

As noted in [22], substantial evidence in favour of the positive answer comes from MacCluer's work [9]. Namely, her results show that for the standard scale of weighted Bergman spaces $\mathcal{A}^2_\beta$, $\beta > -1$, the compact composition operators on $\mathcal{A}^2_\beta$ do form a component of $\mathrm{Comp}(\mathcal{A}^2_\beta)$. So it seemed natural to conjecture that the same phenomenon persists for the limiting ($\beta = -1$) case of $\mathcal{H}^2$. Later it was found out that the answer is positive also in the setting of $\mathcal{H}^\infty$ [10].

The main result of the present paper is a negative solution to question ($*$), stated here in a slightly greater generality:

Main Theorem. *For $0 \leqslant t \leqslant 1$ there are analytic maps $\varphi_t : \mathbb{D} \to \mathbb{D}$ such that C_{φ_0} is compact and C_{φ_1} is non-compact on $\mathcal{H}^p$, and $t \mapsto C_{\varphi_t}$ is continuous from* $[0,1]$ *into* $\mathrm{Comp}(\mathcal{H}^p)$*, where $1 \leqslant p < \infty$.*

As a matter of fact, Shapiro and Sundberg suggested a more general conjecture, according to which operators C_φ and C_ψ would belong to the same component of $\mathrm{Comp}(\mathcal{H}^2)$ if and only if the difference $C_\varphi - C_\psi$ is compact. This has recently been disproved by Moorhouse and Toews [13] and Bourdon [3], who have provided fairly simple and concrete examples of symbols φ and ψ such that the operators C_φ and C_ψ lie in the same component while having a non-compact difference. In these examples, however, both C_φ and C_ψ are non-compact, so they leave question $(*)$ unanswered.

The map φ_1 of Main Theorem is necessarily of fairly complicated function-theoretic nature. In order to illustrate this we recall a result of MacCluer [9] which states that whenever two composition operators belong to the same component of $\mathrm{Comp}(\mathcal{H}^2)$, their symbols must have the same angular derivative (possibly infinite) at each point of the unit circle $\mathbb{T} = \partial\mathbb{D}$. As a consequence, φ_1 cannot have a finite angular derivative at any point of $\mathbb{T}$. In particular, since C_{φ_1} is non-compact, this implies that the valence of φ_1 has to be infinite.

As a main tool in the proof of Main Theorem we will utilize Aleksandrov–Clark measures. These measures, associated to each analytic self-map φ of the unit disc, have lately found several applications in the study of composition operators (see Section 2). Moreover, they are intimately connected to the boundary behaviour of φ; for instance, their mass points correspond to the angular derivatives of φ. The essence of our argument comprises a construction of a certain family of continuously singular measures on $\mathbb{T}$, which are then used to define the maps φ_t in terms of their Aleksandrov–Clark measures.

The rest of the paper is organized as follows. In Section 2, we collect some preliminaries on Aleksandrov–Clark measures and composition operators. In Section 3, we revisit the theorem of MacCluer cited above and establish a slight quantitative strengthening of it. This result will provide some insight into the proof of Main Theorem, which occupies Section 4. Let us however note that the construction of Section 4 itself is completely independent of the results in Section 3. Finally, in Section 5 we pose some additional questions and observations related to Main Theorem.

2. Aleksandrov–Clark measures

In this section we collect some preliminaries and background on Aleksandrov–Clark measures and their relation to composition operators. For more information on these measures and their applications in other areas of analysis, we refer the reader to the lecture notes [16], the book [5] and the surveys [12,15].

Let φ be an analytic self-map of $\mathbb{D}$. For any $\alpha \in \mathbb{T}$, the real part of the function $(\alpha+\varphi)/(\alpha-\varphi)$ is positive and harmonic in $\mathbb{D}$, so it may be expressed as the Poisson integral of a positive Borel measure $\tau_{\varphi,\alpha}$ supported on $\mathbb{T}$. That is,

$$\mathrm{Re}\,\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)} = \frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \int_{\mathbb{T}} P_z\, d\tau_{\varphi,\alpha},$$

where $P_z(\zeta) = (1-|z|^2)/|\zeta - z|^2$ is the Poisson kernel for $z \in \mathbb{D}$. The family of measures $\{\tau_{\varphi,\alpha}\colon \alpha \in \mathbb{T}\}$ are called the *Aleksandrov–Clark measures* associated to φ. Alternatively, one can invoke the Herglotz formula to write

$$\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)} = H\tau_{\varphi,\alpha}(z) + ic_\alpha, \tag{2.1}$$

where

$$H\tau_{\varphi,\alpha}(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\tau_{\varphi,\alpha}(\zeta)$$

is the Herglotz integral and c_α is the imaginary part of $(\alpha + \varphi(0))/(\alpha - \varphi(0))$. Note that if α is given and τ is any positive and finite Borel measure on $\mathbb{T}$, one can proceed in the reverse direction and use the Herglotz formula to construct a map φ whose Aleksandrov–Clark measure at α equals τ.

For any Borel measure τ on $\mathbb{T}$, we write $\tau = \tau^a\, dm + \tau^s$ for the Lebesgue decomposition of τ, where τ^a is the density of the absolutely continuous part, m is the normalized Lebesgue measure on $\mathbb{T}$ and τ^s is singular. It follows from the basic properties of Poisson integrals that $\tau^s_{\varphi,\alpha}$ is carried by the set where $\varphi(\zeta) = \alpha$ and

$$\tau^a_{\varphi,\alpha}(\zeta) = \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2}.$$

In particular, $\tau_{\varphi,\alpha}$ is singular if and only if φ is an inner function.

A nice feature of the Aleksandrov–Clark measures is that their discrete parts (i.e. mass points, or atoms) have a perfect correspondence with the finite angular derivatives of φ. Let us recall that if the quotient $(\varphi(z) - \eta)/(z - \zeta)$ has a finite non-tangential limit at $\zeta \in \mathbb{T}$ for some $\eta \in \mathbb{T}$, then this limit is called the *angular derivative* of φ at ζ and denoted by $\varphi'(\zeta)$. It satisfies $\varphi'(\zeta) = |\varphi'(\zeta)|\bar{\zeta}\eta$ with $\eta = \varphi(\zeta)$. Now the following holds:

- *The map φ has a finite angular derivative at $\zeta \in \mathbb{T}$ if and only if there is $\alpha \in \mathbb{T}$ such that $\tau_{\varphi,\alpha}(\{\zeta\}) > 0$. In that case $\varphi(\zeta) = \alpha$ and $|\varphi'(\zeta)| = \tau_{\varphi,\alpha}(\{\zeta\})^{-1}$.*

For the proof of this result convenient references are [5,16], where it is established in conjunction with the classical Julia–Carathéodory theorem.

To bring Aleksandrov–Clark measures into the theory of composition operators, we follow Sarason's [17] idea of describing composition operators as integral operators acting on the unit circle. Let us denote by $\mathcal{M}$ the space of all complex Borel measures on $\mathbb{T}$ endowed with the total variation norm. Then, if $\mu \in \mathcal{M}$ is given, the Poisson integral $u(z) = \int_{\mathbb{T}} P_z\, d\mu$ defines a harmonic function on $\mathbb{D}$. Consequently the function $v = u \circ \varphi$ is also harmonic, and it follows easily that v is the Poisson integral of a unique measure $\nu \in \mathcal{M}$. Thus it makes sense to define $C_\varphi \mu = \nu$. One can show that $C_\varphi : \mathcal{M} \to \mathcal{M}$ is bounded and, furthermore, that C_φ restricts to a bounded operator $L_p \to L_p$, where $L_p = L_p(\mathbb{T}, m)$ for $1 \leqslant p \leqslant \infty$. Moreover, viewing the Hardy space $\mathcal{H}^p$ as a subspace of L^p (through the non-tangential boundary values of $\mathcal{H}^p$ functions), we see that the restriction of C_φ to $\mathcal{H}^p$ coincides with the standard definition of C_φ.

By definition we have $\tau_{\varphi,\alpha} = C_\varphi \delta_\alpha$, where δ_α is the Dirac measure at α. More generally, the correspondence $C_\varphi \mu = \nu$ can be written as

$$\int_{\mathbb{T}} f\, d\nu = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f\, d\tau_{\varphi,\alpha} \right) d\mu(\alpha) \tag{2.2}$$

for a suitable class of functions f. Indeed, if f is a Poisson kernel P_z, this follows directly from the definitions. The case of continuous f is then obtained by approximating with linear combinations of Poisson kernels. Finally one may invoke a further approximation argument (e.g. a monotone class theorem; cf. [5, Sec. 9.4]) to establish (2.2) for all bounded Borel functions f on $\mathbb{T}$.

In [17] Sarason characterized those composition operators C_φ that are compact on $\mathcal{M}$ and L^1 by a condition which says that $\tau^s_{\varphi,\alpha} = 0$ for all $\alpha \in \mathbb{T}$; that is, the Aleksandrov–Clark measures of φ are required to be absolutely continuous. Later Shapiro and Sundberg [21] observed that Sarason's criterion is equivalent to Shapiro's [19] characterization of compact composition operators on $\mathcal{H}^p$, $1 \leqslant p < \infty$, involving the Nevanlinna counting function. Moreover, Cima and Matheson [4] have shown that the essential norm (i.e. distance, in the operator norm, from the compact operators) of any C_φ acting on $\mathcal{H}^2$ equals $\sup_\alpha \|\tau^s_{\varphi,\alpha}\|^{1/2}$. Thus, a necessary condition for the compactness of C_φ on all the spaces mentioned is that the symbol φ has no finite angular derivative at any point of $\mathbb{T}$. This condition, however, is not sufficient unless φ is of finite valence (see e.g. [20]).

Aleksandrov–Clark measures have also been used to study differences and more general linear combinations of composition operators in [8,14,18]. In particular, a characterization for compact differences of composition operators on $\mathcal{M}$ and L^1 was found in [14].

3. Extension of MacCluer's theorem

In 1989 Barbara MacCluer obtained the following result concerning differences of composition operators on $\mathcal{H}^2$.

Theorem 3.1. *(See MacCluer [9].) Assume that $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ are analytic maps and φ has a finite angular derivative at $\zeta \in \mathbb{T}$. Then, unless $\psi(\zeta) = \varphi(\zeta)$ and $\psi'(\zeta) = \varphi'(\zeta)$, one has*

$$\|C_\varphi - C_\psi\|_e^2 \geqslant \frac{1}{|\varphi'(\zeta)|},$$

where $\| \ \|_e$ denotes the essential norm of an operator on $\mathcal{H}^2$.

The relationship between angular derivatives and the atoms of the Aleksandrov–Clark measures (see Section 2) allows us to restate Theorem 3.1 as follows:

- *Assume that $\tau_{\varphi,\alpha}(\{\zeta\}) > 0$ for some $\alpha \in \mathbb{T}$. Then, unless $\tau_{\psi,\alpha}(\{\zeta\}) = \tau_{\varphi,\alpha}(\{\zeta\})$, one has $\|C_\varphi - C_\psi\|_e^2 \geqslant \tau_{\varphi,\alpha}(\{\zeta\})$.*

Theorem 3.1 implies that, for each non-zero complex number d and point $\zeta \in \mathbb{T}$, the set of all C_φ with $\varphi'(\zeta) = d$ is both open and closed in $\mathrm{Comp}(\mathcal{H}^2)$, even in the topology induced by the essential norm. Hence a necessary condition for two composition operators to lie in the same component or essential component of $\mathrm{Comp}(\mathcal{H}^2)$ is that the angular derivatives of their symbols coincide. In particular, it follows that if C_φ belongs to the component containing all compact composition operators, then φ has no finite angular derivative at any point of $\mathbb{T}$—or, equivalently, the Aleksandrov–Clark measure $\tau_{\varphi,\alpha}$ has no atoms for any $\alpha \in \mathbb{T}$.

Remark 3.2. MacCluer's work was actually carried out in a general context of weighted Dirichlet (or Bergman) spaces $\mathcal{D}_\beta$, $\beta \geqslant 1$, which includes as special cases the Hardy space $\mathcal{H}^2$ ($\beta = 1$) as well as the standard Bergman space $\mathcal{A}^2$ ($\beta = 2$). For $\beta > 1$ it is known that the non-existence of finite angular derivatives is both necessary and sufficient for the compactness of a composition operator on $\mathcal{D}_\beta$ (see [11] or [6]). So, in these spaces, MacCluer's theorem implies (e.g. by the argument of the preceding paragraph) that the compacts indeed form a component of $\mathrm{Comp}(\mathcal{D}_\beta)$.

In another direction, Kriete and Moorhouse [8] have recently obtained various interesting refinements of MacCluer's results. In particular, they establish a version of Theorem 3.1 for higher-order boundary data of the symbols.

In this section we provide a slight improvement of Theorem 3.1. Our lower bound involves the whole discrete part of the Aleksandrov–Clark measure at α. This result will provide some heuristics for the proof of our Main Theorem in the next section (see the discussion at the end of that section).

Theorem 3.3. *Let $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic maps and $\alpha \in \mathbb{T}$. Write*

$$Z_\alpha = \bigl\{\zeta \in \mathbb{T}\colon\ 0 < \tau_{\varphi,\alpha}\bigl(\{\zeta\}\bigr) \neq \tau_{\psi,\alpha}\bigl(\{\zeta\}\bigr)\bigr\}.$$

Then

$$\|C_\varphi - C_\psi\|_e^2 \geqslant \tau_{\varphi,\alpha}(Z_\alpha).$$

In the proof of Theorem 3.3 we will use as test functions the normalized reproducing kernels

$$f_w(z) = \frac{\sqrt{1-|w|^2}}{1-\overline{w}z}.$$

They have the property that $\|f_w\|_2 = 1$ for all $w \in \mathbb{D}$ and $f_w \to 0$ weakly as $|w| \to 1$, whence

$$\|C_\varphi - C_\psi\|_e \geqslant \limsup_{|w|\to 1} \bigl\|(C_\varphi - C_\psi)f_w\bigr\|_2.$$

Apart from being standard test functions, the functions f_w are very useful in connection with the Aleksandrov–Clark measures. Indeed, a result of J.E. Shapiro [18] already shows that $|C_\varphi f_w|^2 \to \tau^s_{\varphi,\alpha}$ weak* on $\mathbb{T}$ as $w \to \alpha$ non-tangentially. In the present context, a slightly more careful analysis of the local limiting behaviour of $C_\varphi f_w$ is needed and the crucial estimates are recorded in Lemma 3.4 below. These estimates depend on the "non-tangential conformality" of φ at boundary points with a finite angular derivative. Indeed, if $\varphi(\zeta) = \alpha$ for some $\zeta, \alpha \in \mathbb{T}$ and $\varphi'(\zeta)$ exists, then φ admits an expansion $\varphi(z) - \alpha = \varphi'(\zeta)(z-\zeta) + o(z-\zeta)$ in any non-tangential approach region for the point ζ. In particular, this implies that φ maps any curve in $\mathbb{D}$ terminating at ζ and making an angle $-\pi/2 < \theta < \pi/2$ with the radius to ζ onto a curve terminating at α and making the same angle θ with the radius to α.

In order to give some heuristics for Lemma 3.4 and its application, let us assume for a moment that $\tau_{\varphi,1}(\{1\}) = a > 0$. This means that $\varphi(1) = 1$ and $\varphi'(1) = 1/a$. It is easy to see that for large $|w|$, the support of the function f_w gets concentrated around the radius that goes through w. For w close to 1, the non-tangential conformality of φ now implies that the composition $C_\varphi f_w$

attains large values (approximately) around the radius through the point $1 - a(1 - w)$, whose argument is $\approx a \cdot \arg w$. Thus the location of the local mass of $C_\varphi f_w$ depends on the value of a, and by making w approach 1 almost tangentially we can use this phenomenon to distinguish between different values of a. To obtain the result of Theorem 3.3 in its sharpest possible form (without any constant factors), it seems necessary to introduce several parameters and limiting processes in the course of the argument.

Let us also note here that the idea of almost tangential approach was already used by MacCluer [9] in her proof for Theorem 3.1. However, instead of analysing the action of C_φ and C_ψ on the kernel functions f_w, she worked with the corresponding adjoint operators.

Lemma 3.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and fix $a > 0$. For $\delta, \kappa, \lambda, r > 0$, write*

$$I(\delta, \kappa, \lambda, r) = \frac{1}{2\pi} \int_{\kappa ra - \lambda r}^{\kappa ra + \lambda r} \left| C_\varphi f_{(1-r)e^{i\kappa r}}\bigl((1 - \delta r)e^{it}\bigr)\right|^2 dt.$$

(1) *If $\tau_{\varphi,1}(\{1\}) = a$, then*

$$\lim_{r \to 0} I(\delta, \kappa, \lambda, r) = \frac{a \cdot c(a; \delta, \lambda)}{1 + \delta/a},$$

where $0 < c(a; \delta, \lambda) < 1$ and $\lim_{\lambda \to \infty} c(a; \delta, \lambda) = 1$ for all $\delta > 0$.

(2) *If $\tau_{\varphi,1}(\{1\}) \neq a$, then*

$$\lim_{r \to 0} I(\delta, \kappa, \lambda, r) = \varepsilon(a; \delta, \kappa, \lambda),$$

where $\lim_{\kappa \to \infty} \varepsilon(a; \delta, \kappa, \lambda) = 0$ for all $\delta, \lambda > 0$.

Proof. Let us fix $\delta, \kappa, \lambda > 0$, and write $w_r = (1 - r)e^{i\kappa r}$ and $z_r(t) = (1 - \delta r)e^{it}$. Then

$$I(\delta, \kappa, \lambda, r) = \frac{2r - r^2}{2\pi} \int_{\kappa ra - \lambda r}^{\kappa ra + \lambda r} \frac{dt}{|1 - \overline{w_r}\varphi(z_r(t))|^2}. \tag{3.1}$$

We first consider the case when $\tau_{\varphi,1}(\{1\}) = b$ for some $b > 0$. That is, $\varphi(1) = 1$ and φ has a finite angular derivative equal to $1/b$ at 1. Note that the points $z_r(t)$ for $\kappa ra - \lambda r < t < \kappa ra + \lambda r$ and $0 < r < 1$ all lie in a non-tangential approach region for the point 1 (whose opening angle depends on δ, κ, a and λ). Therefore, for these $z_r(t)$, the non-tangential conformality of φ at 1 yields an expansion

$$1 - \varphi\bigl(z_r(t)\bigr) = b^{-1}\bigl(1 - z_r(t)\bigr) + r\varepsilon_r(t),$$

uniformly in t. Here and elsewhere in this proof we use ε_r (with or without additional parameters) as a generic symbol for a quantity which tends to zero as $r \to 0$. With this notation, we also have $1 - \overline{w_r} = r + i\kappa r + r\varepsilon_r$ and $1 - z_r(t) = \delta r - it + r\varepsilon_r(t)$. Consequently,

$$\begin{aligned}1-\overline{w_r}\varphi\big(z_r(t)\big) &= (1-\overline{w_r})+\big\{1-\varphi\big(z_r(t)\big)\big\}+r\varepsilon_r(t)\\ &= r(1+\delta/b)+i(\kappa r-t/b)+r\varepsilon_r(t).\end{aligned}$$

Now substitute this expression into the integrand in (3.1) and perform the change of variables $u=t/ra-\kappa$ to get

$$\begin{aligned}I(\delta,\kappa,\lambda,r) &= \frac{(2r-r^2)ra}{2\pi}\int_{-\lambda/a}^{+\lambda/a}\frac{du}{|r(1+\delta/b)+i(\kappa r-\kappa ra/b-rau/b)+r\varepsilon_r(u)|^2}\\ &= \frac{(2-r)a}{2\pi}\int_{-\lambda/a}^{+\lambda/a}\frac{du}{|(1+\delta/b)+i((1-a/b)\kappa-au/b)+\varepsilon_r(u)|^2}.\end{aligned}$$

Hence

$$\lim_{r\to 0} I(\delta,\kappa,\lambda,r)=\frac{a}{\pi}\int_{-\lambda/a}^{+\lambda/a}\frac{du}{(1+\delta/b)^2+((1-a/b)\kappa-au/b)^2}. \tag{3.2}$$

If $b=a$, this limit equals

$$\frac{a}{\pi}\int_{-\lambda/a}^{+\lambda/a}\frac{du}{(1+\delta/a)^2+u^2},$$

which is of the desired form $ac(a;\delta,\lambda)/(1+\delta/a)$. On the other hand, if $b\neq a$, then the integrand in (3.2) tends to zero as $\kappa\to\infty$, uniformly in u. So, in this case (3.2) goes to zero as $\kappa\to\infty$.

Finally assume that $\tau_{\varphi,1}(\{1\})=0$, so φ has no finite angular derivative at 1 or $\varphi(1)\neq 1$. By the Julia–Carathéodory theorem, we now have $(1-\varphi(z))/(1-z)\to\infty$ as $z\to 1$ non-tangentially. By considerations similar to those in the first part of the proof, this implies that $\{1-\overline{w_r}\varphi(z_r(t))\}/r\to\infty$ as $r\to 0$, uniformly in t, and hence $I(\delta,\kappa,\lambda,r)\to 0$ as $r\to 0$. We leave the details to the reader. □

Proof of Theorem 3.3. Without loss of generality, we may take $\alpha=1$. We first treat the case of a single mass point and then indicate the general argument. Let us assume that $\tau_{\varphi,1}(\{1\})=a\neq\tau_{\psi,1}(\{1\})$ for some $a>0$. Then, for $\delta,\kappa,\lambda>0$ and small enough $r>0$, we have

$$\begin{aligned}\big\|(C_\varphi-C_\psi)f_{(1-r)e^{i\kappa r}}\big\|_2 &\geqslant \left(\frac{1}{2\pi}\int_{\kappa ra-\lambda r}^{\kappa ra+\lambda r}\big|(C_\varphi-C_\psi)f_{(1-r)e^{i\kappa r}}\big((1-\delta r)e^{it}\big)\big|^2\,dt\right)^{1/2}\\ &\geqslant I_\varphi(\delta,\kappa,\lambda,r)^{1/2}-I_\psi(\delta,\kappa,\lambda,r)^{1/2},\end{aligned}$$

where I_φ and I_ψ refer to the integrals of Lemma 3.4 corresponding to φ and ψ, respectively. Passing to the limit as $r\to 0$, we then get the following type of lower bound for the essential norm of $C_\varphi-C_\psi$:

$$\|C_\varphi - C_\psi\|_e \geqslant \left(\frac{a \cdot c(a;\delta,\lambda)}{1+\delta/a}\right)^{1/2} - \varepsilon(a;\delta,\kappa,\lambda)^{1/2}.$$

Letting $\kappa \to \infty$, $\lambda \to \infty$ and $\delta \to 0$ now yields $\|C_\varphi - C_\psi\|_e \geqslant a^{1/2}$ as desired.

To prove the theorem in full (assuming still $\alpha = 1$), we observe that the above reasoning is local in the sense that the interval $[\kappa ra - \lambda r,\ \kappa ra + \lambda r]$ shrinks to 0 as $r \to 0$. Let $Z' = \{\zeta_1, \dots, \zeta_n\}$ be any finite subset of the (possibly infinite) set Z_1, where $\zeta_k \neq \zeta_l$ for $k \neq l$. Write $t_k = \arg \zeta_k$ and $a_k = \tau_{\varphi,1}(\{\zeta_k\})$. We proceed as above, now integrating over the union of the intervals $[t_k + \kappa r a_k - \lambda r,\ t_k + \kappa r a_k + \lambda r]$, $k = 1, \dots, n$. Since these are disjoint for small r, we get, after passing to the appropriate limits as above,

$$\|C_\varphi - C_\psi\|_e \geqslant \left(\sum_{k=1}^n a_k\right)^{1/2} = \tau_{\varphi,1}(Z')^{1/2}.$$

Finally, if Z_1 is infinite, we take the supremum over all finite subsets $Z' \subset Z_1$ to complete the proof of the theorem. $\square$

4. Proof of Main Theorem: non-compact composition operators in the component of compacts

In this section we establish our Main Theorem, giving a negative answer to question $(*)$ stated in Section 1. The same construction turns out to work for a variety of spaces in addition to $\mathcal{H}^2$.

Main Theorem. *For $0 \leqslant t \leqslant 1$ there are analytic maps $\varphi_t : \mathbb{D} \to \mathbb{D}$ such that C_{φ_0} is compact and C_{φ_1} is non-compact on X, and $t \mapsto C_{\varphi_t}$ is continuous from $[0,1]$ into $\mathrm{Comp}(X)$, where X is any of the spaces $\mathcal{M}$, L^p or $\mathcal{H}^p$ with $1 \leqslant p < \infty$.*

We begin with some preliminary observations. First of all, we note that it is enough to deal with the case $X = \mathcal{M}$. Indeed, as we pointed out in Section 2, the compactness of composition operators is equivalent in any two of the spaces mentioned. Furthermore, we may apply interpolation between L^1 (a subspace of $\mathcal{M}$) and L^∞ to conclude that for any $1 \leqslant p < \infty$ and $s, t \in [0,1]$,

$$\begin{aligned}\left\|C_{\varphi_s} - C_{\varphi_t} : L^p \to L^p\right\| &\leqslant \left\|C_{\varphi_s} - C_{\varphi_t} : L^1 \to L^1\right\|^{1/p} \left\|C_{\varphi_s} - C_{\varphi_t} : L^\infty \to L^\infty\right\|^{1-1/p} \\ &\leqslant 2^{1-1/p} \|C_{\varphi_s} - C_{\varphi_t} : \mathcal{M} \to \mathcal{M}\|^{1/p}.\end{aligned}$$

(See e.g. [1, Sec. 4.1] for the classical Riesz–Thorin interpolation theorem which is applicable here.)

Throughout the proof we will utilize Sarason's way of viewing composition operators as acting on the unit circle (cf. Section 2). If φ is an analytic self-map of $\mathbb{D}$ and $E \subset \mathbb{T}$ is a Borel set, we let $\chi_E C_\varphi$ denote the restriction of C_φ to E. More precisely, if $\mu \in \mathcal{M}$ and $C_\varphi \mu = \nu$, then $\chi_E C_\varphi \mu$ refers to the Borel measure $B \mapsto \nu(E \cap B)$ on $\mathbb{T}$. For functions $f \in L^1$, this simply means that $\chi_E C_\varphi f(\zeta) = \chi_E(\zeta) C_\varphi f(\zeta)$ for m-a.e. $\zeta \in \mathbb{T}$. In this context, Eq. (2.2) yields that

$$\|\chi_E C_\varphi : \mathcal{M} \to \mathcal{M}\| = \sup\{\tau_{\varphi,\alpha}(E) \colon \alpha \in \mathbb{T}\} \tag{4.1}$$

(just replace f in (2.2) by $f\chi_E$ and take supremum over $\|f\|_\infty \leqslant 1$). Similarly, for differences we have

$$\big\|\chi_E(C_\varphi - C_\psi) : \mathcal{M} \to \mathcal{M}\big\| = \sup\big\{|\tau_{\varphi,\alpha} - \tau_{\psi,\alpha}|(E) \colon \alpha \in \mathbb{T}\big\}, \tag{4.2}$$

where $|\tau_{\varphi,\alpha} - \tau_{\psi,\alpha}|$ denotes the total variation measure.

We use ρ to denote the hyperbolic distance in $\mathbb{D}$; it is the conformally invariant metric induced by the arc length element $2|dz|/(1-|z|^2)$ (see e.g. [7, Sec. I.1]). When working with hyperbolic distances, it is often convenient to shift to the right half-plane $\mathbb{H} = \{z' \colon \operatorname{Re} z' > 0\}$, where the hyperbolic metric $\rho_{\mathbb{H}}$ is induced by the arc length element $|dz'|/\operatorname{Re} z'$. For any $\alpha \in \mathbb{T}$, this is accomplished through the Möbius transformation $z' = (\alpha + z)/(\alpha - z)$, which takes $\mathbb{D}$ onto $\mathbb{H}$ isometrically relative to ρ and $\rho_{\mathbb{H}}$. Note that this transformation also occurs in the definition of Aleksandrov–Clark measures.

Lemma 4.1. *Let $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic, and let $E \subset \mathbb{T}$ be a Borel set such that $\tau_{\varphi,\alpha}(\partial E) = \tau_{\psi,\alpha}(\partial E) = 0$ for all $\alpha \in \mathbb{T}$. Also let $0 < \varepsilon < 1$. Suppose that for m-a.e. $\zeta \in E$ the following holds: if one of $\varphi(\zeta)$ and $\psi(\zeta)$ is unimodular, then $\varphi(\zeta) = \psi(\zeta)$, and otherwise $\rho(\varphi(\zeta), \psi(\zeta)) \leqslant \varepsilon$. Then*

$$\big\|\chi_E(C_\varphi - C_\psi) : \mathcal{M} \to \mathcal{M}\big\| \leqslant \frac{C\varepsilon}{1 - |\varphi(0)|},$$

where $C > 0$ is a universal constant.

Proof. We first note that the Poisson kernel functions P_z satisfy the following estimate: for all $z, w \in \mathbb{D}$ with $\rho(z, w) \leqslant 1$ and $\alpha \in \mathbb{T}$,

$$\big|P_z(\alpha) - P_w(\alpha)\big| \leqslant C\rho(z, w) P_z(\alpha), \tag{4.3}$$

where $C > 0$ is a universal constant. In fact, one may use the transformation $z' = (\alpha + z)/(\alpha - z)$ to pass to the right half-plane where (4.3) becomes

$$\big|\operatorname{Re}(z' - w')\big| \leqslant C\rho_{\mathbb{H}}(z', w') \operatorname{Re} z',$$

which is easy to verify by geometric reasoning.

Now fix $\alpha \in \mathbb{T}$ and $0 < r < 1$. Since $\rho(r\varphi(\zeta), r\psi(\zeta)) \leqslant \varepsilon$ for m-a.e. $\zeta \in E$, we get by (4.3) that

$$\int_E \left|\frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2} - \frac{1 - |r\psi|^2}{|\alpha - r\psi|^2}\right| dm \leqslant C\varepsilon \int_E \frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2}\, dm \leqslant C\varepsilon \frac{1 - |r\varphi(0)|^2}{|\alpha - r\varphi(0)|^2}.$$

The last inequality was obtained by extending the integral over the whole circle $\mathbb{T}$ and using the harmonicity of the integrand. The definition of the Aleksandrov–Clark measures implies that the absolutely continuous measure $(1 - |r\varphi|^2)/|\alpha - r\varphi|^2\, dm$ converges to $\tau_{\varphi,\alpha}$ weak* as $r \to 1$. Similarly $(1 - |r\psi|^2)/|\alpha - r\psi|^2\, dm$ converges to $\tau_{\psi,\alpha}$. Therefore, the preceding chain of inequalities yields, as $r \to 1$,

$$|\tau_{\varphi,\alpha} - \tau_{\psi,\alpha}|(E) \leqslant C\varepsilon \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2} \leqslant \frac{2C\varepsilon}{1 - |\varphi(0)|}.$$

(Here we used the assumption that $\tau_{\varphi,\alpha}$ and $\tau_{\psi,\alpha}$ both assign measure zero to the boundary of E.) The lemma now follows from (4.2). □

We are now in a position to define the maps φ_t. Recall from Section 2 that a composition operator C_φ is non-compact on any of the spaces mentioned in Main Theorem if and only if at least one of the Aleksandrov–Clark measures $\tau_{\varphi,\alpha}$ fails to be absolutely continuous. In the other direction, if C_φ is required to belong to the component of compact composition operators, MacCluer's theorem (Theorem 3.1) implies that none of $\tau_{\varphi,\alpha}$ may have atoms. That is why we have to consider Aleksandrov–Clark measures with continuous singularity.

Let λ be any non-trivial, positive and finite continuously singular Borel measure on the unit circle $\mathbb{T}$. For $0 \leqslant t \leqslant 1$, let

$$\tau_{t,1} = m + \chi_{I(t)}\lambda, \tag{4.4}$$

where $I(t) \subset \mathbb{T}$ is the closed arc connecting the point 1 to $e^{2\pi i t}$ in the positive direction and, as before, m denotes the normalized Lebesgue measure on $\mathbb{T}$. We define φ_t in terms of the Herglotz integral

$$\frac{1+\varphi_t}{1-\varphi_t} = H\tau_{t,1} \tag{4.5}$$

(see (2.1)). Then $\tau_{t,1}$ becomes the Aleksandrov–Clark measure of φ_t at 1. Moreover, since $\operatorname{Re} H\tau_{t,1}(z) = \int_{\mathbb{T}} P_z\, d\tau_{t,1} \geqslant 1$, it follows that φ_t either takes $\mathbb{D}$ into the open disc $\{w\colon |w - \frac{1}{2}| < \frac{1}{2}\}$ or is constant 0 (for small t). In general, we let $\tau_{t,\alpha}$ denote the Aleksandrov–Clark measure of φ_t at $\alpha \in \mathbb{T}$.

The compactness statements of Main Theorem are now immediate. Since $\tau_{1,1} = m + \lambda$ is not absolutely continuous, the operator C_{φ_1} is non-compact. On the other hand, $\varphi_0 \equiv 0$, so C_{φ_0} is clearly compact.

The hard part of the proof consists of showing that the map $t \mapsto C_{\varphi_t}$ is indeed continuous. This will be based on the following two lemmas, in conjunction with Lemma 4.1.

Lemma 4.2. *Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $I \subset \mathbb{T}$ is an arc with $m(I) \leqslant \delta$, then the Aleksandrov–Clark measures of the maps φ_t satisfy $\tau_{t,\alpha}(I) \leqslant \varepsilon$ for all $t \in [0,1]$ and $\alpha \in \mathbb{T}$. In particular, none of $\tau_{t,\alpha}$ have atoms.*

Proof. We first argue that all the measures $\tau_{t,\alpha}$ are indeed continuous, i.e. have no atoms. For $\alpha = 1$ this is clear from (4.4). For $\alpha \neq 1$ we need to note that since the closure of the image of φ_t does not contain α, the harmonic function

$$\operatorname{Re} \frac{\alpha + \varphi_t(z)}{\alpha - \varphi_t(z)} = \int_{\mathbb{T}} P_z\, d\tau_{t,\alpha} \tag{4.6}$$

is bounded and hence $\tau_{t,\alpha}$ is absolutely continuous.

Next, using (4.4) and (4.5) one can easily show that for each fixed z the left-hand side of (4.6) is continuous as a function of the pair (t,α) in $[0,1] \times \mathbb{T}$. Since linear combinations of Poisson kernels are dense among the continuous functions on $\mathbb{T}$, it follows that the map $(t,\alpha) \mapsto \tau_{t,\alpha}$ is continuous in the weak* sense.

Now assume that the claim of the lemma fails. Thus, there are arcs $I_n \subset \mathbb{T}$ and points $t_n \in [0, 1]$ and $\alpha_n \in \mathbb{T}$ such that $\tau_{t_n,\alpha_n}(I_n) > \varepsilon$ for all $n \geqslant 1$ while $m(I_n) \to 0$. By passing to a subsequence we may further assume that the intervals I_n (i.e. their endpoints) converge to a point $\zeta_0 \in \mathbb{T}$ and also that $t_n \to t_0$ and $\alpha_n \to \alpha_0$. Then, for each closed arc $I \subset \mathbb{T}$ whose midpoint is ζ_0, we have $\tau_{t_n,\alpha_n}(I) > \varepsilon$ whenever n is large enough. Since the map $(t, \alpha) \mapsto \tau_{t,\alpha}$ is weak* continuous, this implies that $\tau_{t_0,\alpha_0}(I) \geqslant \varepsilon$ for every such I, and hence $\tau_{t_0,\alpha_0}(\{\zeta_0\}) \geqslant \varepsilon$. This is a contradiction since we observed that τ_{t_0,α_0} cannot have atoms. $\square$

Lemma 4.3. *Fix $t_0 \in [0, 1]$ and let $I_0 \subset \mathbb{T}$ be an arc whose midpoint is $e^{2\pi i t_0}$. If $\varepsilon > 0$ is given, there exists $\delta > 0$ such that*

$$\rho\big(\varphi_{t_0}(\zeta), \varphi_t(\zeta)\big) \leqslant \varepsilon \quad \textit{for } \zeta \in \mathbb{T} \setminus I_0$$

whenever $|t_0 - t| \leqslant \delta$.

Proof. Assume that $|t_0 - t|$ is so small that the distance of the point $e^{2\pi i t}$ to the set $\mathbb{T} \setminus I_0$ is greater than a positive constant c. Then $H\tau_{t,1} = H\tau_{t_0,1} \pm H(\chi_{J(t)}\lambda)$, where $J(t)$ is the arc connecting the points $e^{2\pi i t_0}$ and $e^{2\pi i t}$ in I_0. Moreover, for $\zeta \in \mathbb{T} \setminus I_0$ we have

$$\big|H(\chi_{J(t)}\lambda)(\zeta)\big| = \left| \int\limits_{J(t)} \frac{\xi + \zeta}{\xi - \zeta}\, d\lambda(\xi) \right| \leqslant \frac{2}{c}\lambda\big(J(t)\big).$$

Since this upper bound tends to zero as $t \to t_0$ and $\operatorname{Re} H\tau_{t_0,1} \geqslant 1$, we see that the distance between $H\tau_{t,1}(\zeta)$ and $H\tau_{t_0,1}(\zeta)$ in the hyperbolic metric of the right half-plane tends to zero as $t \to t_0$, uniformly for $\zeta \in \mathbb{T} \setminus I_0$. In view of (4.5) and the conformal invariance of the hyperbolic metric, the same conclusion holds true for the distance of $\varphi_t(\zeta)$ and $\varphi_{t_0}(\zeta)$ in the metric ρ. $\square$

We are now ready to finish the proof of Main Theorem.

Proof of Main Theorem (final part). It remains to show that the map $t \mapsto C_{\varphi_t}$ is continuous with respect to the operator norm on $\mathcal{M}$. Let $0 < \varepsilon < 1$. By Lemma 4.2 we can find $\delta > 0$ such that $\tau_{t,\alpha}(I) \leqslant \varepsilon$ for all $t \in [0, 1]$ and $\alpha \in \mathbb{T}$ whenever $I \subset \mathbb{T}$ is an arc with $m(I) \leqslant \delta$. For all such I, Eq. (4.1) yields the estimate

$$\|\chi_I C_{\varphi_t}\| \leqslant \varepsilon. \tag{4.7}$$

(Here and throughout the rest of the proof $\|\ \|$ refers to the operator norm on $\mathcal{M}$.)

Now fix $t_0 \in [0, 1]$ and pick an arc $I_0 \subset \mathbb{T}$ with $m(I_0) \leqslant \delta$ whose midpoint is $e^{2\pi i t_0}$. By Lemma 4.3 there exists $\eta > 0$ such that if $|t_0 - t| \leqslant \eta$, then $\rho(\varphi_{t_0}(\zeta), \varphi_t(\zeta)) \leqslant \varepsilon$ for all $\zeta \in \mathbb{T} \setminus I_0$. Hence Lemma 4.1 shows that

$$\big\|\chi_{\mathbb{T}\setminus I_0}(C_{\varphi_{t_0}} - C_{\varphi_t})\big\| \leqslant C\varepsilon/\big(1 - \big|\varphi_{t_0}(0)\big|\big) \tag{4.8}$$

whenever $|t_0 - t| \leqslant \eta$. To finish the argument we just write

$$C_{\varphi_{t_0}} - C_{\varphi_t} = \chi_{I_0} C_{\varphi_{t_0}} - \chi_{I_0} C_{\varphi_t} + \chi_{\mathbb{T}\setminus I_0}(C_{\varphi_{t_0}} - C_{\varphi_t})$$

and, when $|t_0 - t| \leqslant \eta$, invoke estimates (4.7) and (4.8) to conclude that

$$\|C_{\varphi_{t_0}} - C_{\varphi_t}\| \leqslant \varepsilon + \varepsilon + C\varepsilon/\big(1 - |\varphi_{t_0}(0)|\big).$$

Since $\varepsilon > 0$ was arbitrary, this clearly shows that the norm of $C_{\varphi_{t_0}} - C_{\varphi_t}$ on $\mathcal{M}$ tends to zero as $t \to t_0$.

This completes the proof of Main Theorem. □

We close this section by discussing some heuristics behind the above construction. First of all, one can easily show that if a continuous path (C_{φ_t}) yielding the desired example exists, then one may assume that the image of each map φ_t is contained in the disc $\{w\colon |w - \frac{1}{2}| \leqslant \frac{1}{2}\}$. Then $\tau_{1,1}$ is necessarily of the form $g\,dm + \lambda$ where $g \geqslant 1$ and λ is non-trivial and continuously singular. One may also assume that $\varphi_0 \equiv 0$. The central issue now is to find the intermediate maps φ_t for $0 < t < 1$.

A seemingly natural choice might be $\varphi_t = (1-t)\varphi_0 + t\varphi_1$, but this obviously fails to work since each corresponding operator is compact. On the other hand, in certain spectral-theoretic applications one considers the maps corresponding to the measures $\tau_{t,1} = (1-t)\tau_{0,1} + t\tau_{1,1}$. However, Theorem 3.3 suggests that this approach might not work either. Namely, in the case of a discrete singular part, Theorem 3.3 shows that if one makes a simultaneous change—no matter how small—to all the mass points of an Aleksandrov–Clark measure, this induces a big difference in the corresponding composition operator. In fact, if we restrict our attention to the special case when the absolutely continuous part of $\tau_{t,1}$ is a constant function (as in the construction of the present section), it is rather easy to verify directly that the corresponding path is discontinuous; see Lemma 4.4 below. These observations motivated our actual choice (4.4), where the singularity λ is continuously "wiped off" in such a way that the change in $\tau_{t,1}$ is strictly local at every instant t.

Lemma 4.4. *Let λ be any singular Borel probability measure on $\mathbb{T}$. For $0 \leqslant t \leqslant 1$ define analytic maps $\psi_t : \mathbb{D} \to \mathbb{D}$ by the Herglotz integral*

$$\frac{1 + \psi_t}{1 - \psi_t} = H(m + t\lambda),$$

so that $m + t\lambda$ becomes the Aleksandrov–Clark measure of ψ_t at 1. Then the essential norm of $C_{\psi_s} - C_{\psi_t}$ on $\mathcal{H}^2$ satisfies $\|C_{\psi_s} - C_{\psi_t}\|_e^2 \geqslant s$ whenever $s \neq t$.

Proof. Define an analytic map σ by $(1+\sigma)/(1-\sigma) = H\lambda$. Then σ is an inner function which vanishes at the origin and whose Aleksandrov–Clark measure at 1 equals λ. Also, for $0 \leqslant t \leqslant 1$ let φ_t be an analytic map with $m + t\delta_1$ as its Aleksandrov–Clark measure at 1. Now we have $C_{\varphi_t \circ \sigma}\delta_1 = C_\sigma C_{\varphi_t}\delta_1 = C_\sigma(m + t\delta_1) = m + t\lambda$. Thus the Aleksandrov–Clark measures of ψ_t and $\varphi_t \circ \sigma$ at the point 1 coincide. Since σ fixes the origin, it is easy to deduce that actually $\psi_t = \varphi_t \circ \sigma$.

We recall that the composition operator induced by an inner function fixing the origin is always an isometry on $\mathcal{H}^2$ (see e.g. [6, Thm. 3.8]). Therefore $\|C_{\psi_s} - C_{\psi_t}\|_e = \|C_\sigma C_{\varphi_s} - C_\sigma C_{\varphi_t}\|_e = \|C_{\varphi_s} - C_{\varphi_t}\|_e$, and the claim follows from Theorem 3.1. □

5. Further remarks

After the work of Section 4 it is natural to search for a larger class of composition operators that could be continuously joined to the compacts. For instance, one might be tempted to expect a positive answer to the following question:

- Assume that φ and $\alpha_0 \in \mathbb{T}$ are such that the measure τ_{φ,α_0} has no atoms and, for all $\alpha \neq \alpha_0$, the measure $\tau_{\varphi,\alpha}$ is absolutely continuous. Does C_φ belong to the same component of $\mathrm{Comp}(\mathcal{H}^2)$ as the compact composition operators?

The answer to this question is, however, negative.

Example 5.1. There is a symbol ψ such that C_ψ is isolated in $\mathrm{Comp}(\mathcal{H}^2)$ and the following properties hold: $\tau_{\psi,1}$ has a continuous non-trivial singular part while all the other measures $\tau_{\psi,\alpha}$ are absolutely continuous. In fact, one may choose $\psi = \varphi \circ \sigma$, where σ is an inner function and φ is a conformal map from $\mathbb{D}$ onto a region $\Omega \subset \mathbb{D}$ with $\overline{\Omega} \cap \mathbb{T} = \{1\}$.

The above example is based on a construction of Shapiro and Sundberg [22]. We first recall some terminology. Shapiro and Sundberg call a continuous and 2π-periodic function $\kappa : \mathbb{R} \to [0, 1)$ a *contact function* if it is increasing and positive on $(0, \pi]$, decreasing and positive on $[-\pi, 0)$ and vanishes at the origin. Such a function determines an approach region

$$\Omega(\kappa) = \{re^{i\theta}\colon 0 \leqslant r < 1 - \kappa(\theta)\},$$

whose boundary is a Jordan curve in $\overline{\mathbb{D}}$ that meets the unit circle only at the point 1. In this setting Shapiro and Sundberg prove the following (see Theorem 4.1 and Remark 5.1 of [22]).

Theorem 5.2. *(See Shapiro and Sundberg [22].) Suppose κ is a C^2 contact function and φ is a conformal map from $\mathbb{D}$ onto $\Omega(\kappa)$. If $\int_0^\pi \log\kappa(\theta)\,d\theta = -\infty$, then C_φ is (essentially) isolated in* $\mathrm{Comp}(\mathcal{H}^2)$.

We observe that this theorem can be extended as follows.

Proposition 5.3. *Let φ be a function given by Theorem* 5.2, *and let σ be an inner function with* $\sigma(0) = 0$. *Put $\psi = \varphi \circ \sigma$. Then C_ψ is (essentially) isolated in* $\mathrm{Comp}(\mathcal{H}^2)$.

In order to produce the symbol ψ needed for Example 5.1, we apply the above proposition and the idea utilized in the proof of Lemma 4.4. Let σ be an inner function such that $\sigma(0) = 0$ and $\tau_{\sigma,1}$ is continuously singular. Also let φ be a conformal map given by Theorem 5.2, with the additional requirement that $\varphi(1) = 1$. Then $\tau^s_{\varphi,1} = c\delta_1$ for some constant c. It is now easy to check that the map $\psi = \varphi \circ \sigma$ has the required properties; in particular, $\tau^s_{\psi,1}$ is the continuously singular measure $c\tau_{\sigma,1}$.

Proof of Proposition 5.3. We start by recalling some ideas from the proof of Theorem 5.2. Write $\Omega = \Omega(\kappa)$ for the image of φ. A crucial part of Shapiro's and Sundberg's argument is the construction of a sequence of test functions $f_n \in \mathcal{H}^2$ which converges to zero weakly in $\mathcal{H}^2$. Their functions satisfy the following properties: $|f_n|^2 \geqslant c/m(J_n)$ on Γ_n, where $\Gamma_n \subset \partial\Omega$ are arcs converging to 1 and $J_n = \varphi^{-1}(\Gamma_n)$; and $|f_n| \leqslant 1$ on $\mathbb{D} \setminus T_n$, where $T_n \subset \mathbb{D}$ is a set containing Γ_n whose diameter is roughly twice the length of Γ_n. Now suppose that $\eta : \mathbb{D} \to \mathbb{D}$ is any analytic map different from φ. Shapiro and Sundberg consider the sets $E_n = \{\zeta \in J_n\colon |\varphi(\zeta) - \eta(\zeta)| \geqslant c_n\}$ where c_n is approximately twice the diameter of T_n. They observe that for $\zeta \in E_n$ one has $\varphi(\zeta) \in \Gamma_n$ and $\eta(\zeta) \in \mathbb{D} \setminus T_n$. Therefore $|f_n \circ \varphi - f_n \circ \eta|^2 \geqslant c/m(J_n)$ on E_n. Since $f_n \to 0$ weakly, this yields the estimate

$$\|C_\varphi - C_\eta\|_e^2 \geqslant c \limsup_{n\to\infty} \frac{m(E_n)}{m(J_n)}.$$

Finally Shapiro and Sundberg show that $\limsup m(E_n)/m(J_n) = 1$, based simply on the fact that $\int_{\mathbb{T}} \log|\varphi - \eta|\, dm > -\infty$.

Our argument is just a minor adaptation of the one explained above. Suppose that $\eta : \mathbb{D} \to \mathbb{D}$ is an analytic map different from ψ, and put $J_n' = \psi^{-1}(\Gamma_n)$ and $E_n' = \{\zeta \in J_n'\colon |\psi(\zeta) - \eta(\zeta)| \geqslant c_n\}$. Then $J_n' = \sigma^{-1}(J_n)$, and since σ is an inner function fixing the origin, we have $m(J_n') = m(J_n)$. Thus, using the test functions f_n as before, we arrive at the estimate

$$\|C_\psi - C_\eta\|_e^2 \geqslant c \limsup_{n\to\infty} \frac{m(E_n')}{m(J_n')}.$$

The proof is now completed by using the same argument as Shapiro and Sundberg to show that the limit superior here equals 1. $\square$

Given the above example, it seems appropriate to close this section with the following general open problem.

Problem 5.4. Determine all the non-compact composition operators that lie in the component of the compact ones in Comp($\mathcal{H}^2$).

This problem might be quite hard. As a first step one could try to describe interesting subsets of the component of the compacts that are larger than those provided by obvious modifications of our construction presented in Section 4. For instance, it would be instructive to know—and probably not difficult to check—if the extremality condition in Theorem 5.2 that was essential for the example provided by Proposition 5.3 can be relaxed.

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