Almost Split Sequences in Subcategories

M. AUSSLANDER* AND SVERRE O. SMALØ†

Brandeis University, Waltham, Massachusetts, 02154 and
University of Trondheim, NLHT, Trondheim, Norway

Communicated by Walter Feit
Received June 9, 1980

INTRODUCTION

Let \( R \) be a commutative artin ring and let \( A \) be an \( R \)-algebra which is a finitely generated \( R \)-module. In [5] Auslander and Reiten introduced the notion of an almost split sequence in \( \text{mod} \, A \), the category of finitely generated modules over \( A \), and the existence and uniqueness of such short exact sequences were established there. The theory of almost split sequences developed further, and similar results for certain subcategories of \( \text{mod} \, A \) were proved by Bautista and Martinez [8] and Roggenkamp [9].

The main purpose of this paper is to develop a more general theory for subcategories of \( \text{mod} \, A \) having almost split sequences, which has the previous examples as special cases.

Notions closely related to that of almost split sequences are those of minimal left and right almost split morphisms. These notions were introduced in [6] for \( \text{mod} \, A \) and further developed in [7] for subcategories of \( \text{mod} \, A \). This paper is based upon [7] and we therefore use the same notations and conventions as used there.

We now give some definitions and then proceed to describe the content of the paper section by section. Let \( C \) be a subcategory of \( \text{mod} \, A \). Then a morphism \( g: B \rightarrow C \) in \( C \) is said to be a right almost split morphism in \( C \) if (i) \( g \) is not a splittable epimorphism and (ii) whenever there is a nonsplittable epimorphism \( h: C' \rightarrow C \) in \( C \), there exists an \( h': C' \rightarrow B \) such that \( gh' = h \). Dually, a morphism \( f: A \rightarrow B \) in \( C \) is said to be left almost split if (i) \( f \) is not a splittable monomorphism and (ii) whenever there is a nonsplittable monomorphism \( h: A \rightarrow A' \) there exists an \( h': B \rightarrow A' \) such that \( h'f = h \). We say that \( C \) has right almost split morphisms if for all indecomposable objects

* Written while a Guggenheim Fellow with the partial support of NSF MCS 77 04 951.
† Supported by the Norwegian Research Council.
ALMOST SPLIT SEQUENCES IN SUBCATEGORIES

C in C there is a B in C and an f: B → C which is right almost split in C. C having left almost split morphisms is defined dually. Finally we say that C has almost split morphisms if C has both left and right almost split morphisms. Subcategories C of mod A having right almost split morphisms, left almost split morphisms and almost split morphisms were studied in [7] and some sufficient conditions for this to happen were given there.

Now assume C is closed with respect to extensions; i.e., if 0 → A → B → C → 0 is exact in mod A with A and C in C, then B is also in C. A module C in C is said to be Ext-projective in C if Ext^1_A(C, X) = 0 for all X in C and a module B in C is said to be Ext-injective in C if Ext^1_A(X, B) = 0 for all X in C. We say that C has almost split sequences if

(i) C has almost split morphisms.

(ii) For each indecomposable non-Ext-projective C in C there exists an exact sequence 0 → A →^f B →^g C → 0 in C with f a left almost split morphism in C and g a right almost split morphism in C.

(iii) For each indecomposable non-Ext-injective module A in C there exists an exact sequence 0 → A →^f B →^g C → 0 in C with f a left almost split morphism in C and g a right almost split morphism in C.

In Section I the basic existence and uniqueness theorem for almost split sequences in subcategories C of mod A which are closed under extensions is given. The criterion developed in Section I for the existence of almost split sequences is not easy to verify, so in Section 2, by restricting the class of subcategories considered, we get an equivalent condition, which is easier to verify. Section 3 is devoted to subcategories C of mod A which are closed either under submodules or under factormodules. In the case C is closed under submodules, the Ext-projective modules are determined and dually, if C is closed under factor modules the Ext-injective modules are determined. The study of the Ext-projective and Ext-injective modules continues in Section 4. Before we state the main result of that section we need one definition. A subcategory C of mod A is said to have a finite cocover if there exists a module C in C such that all modules X in C can be embedded in a direct sum of copies of C. We are now able to state the theorem. If C is closed under extensions and submodules and has a finite cocover, then there are only a finite number of indecomposable nonisomorphic Ext-projective modules and Ext-injective modules in C. We also describe these modules explicitly. The dual results are also stated.

So far we have assumed that C is closed under extensions. Let M be a module in mod A. Denote by Sub M the subcategory of mod A consisting of all objects which are submodules of finite direct sums of copies of M. The subcategory Fac M of mod A is defined dually. In Section 5 amongst other things we have the following result giving sufficient conditions for Sub M
and Fac \( M \) for an \( M \) in mod \( A \) to be closed under extensions. Let \( M \) and \( N \) be in mod \( A \). Then (i) Sub \( M \) is closed under extensions if \( \text{Hom}(\text{Tr} DM, M) = 0 \). (ii) Fac \( N \) is closed under extensions if \( \text{Hom}(N, D \text{ Tr} N) = 0 \). (iii) Sub \( M \cap \text{Fac} N \) is closed under extensions if \( \text{Hom}(\text{Tr} DM, M) = 0 = \text{Hom}(N, D \text{ Tr} N) \).

In Section 6 we apply the theory developed in Sections 1 through 5 of subcategories of mod \( A \) having almost split sequences. For example, we show that if \( S \) is a simple \( A \)-module such that \( \text{Ext}^1(S, S) = 0 \), then Sub(\( D \text{ Tr} S \)) and Fac(\( \text{Tr} DS \)) has almost split sequences. Further, if \( I \) is an injective in mod \( A \) and \( P \) is a projective in mod \( A \) then it is proven that Sub \( I \) and Fac \( P \) have almost split sequences. In these cases also the Ext-injective and Ext-projective modules are determined in fully.

In Section 7 we show that the full subcategory of mod \( A \) consisting of the objects \( C \) such that \( \text{Ext}^1(C, A) = 0 \) has almost split sequences.

1. BASIC EXISTENCE THEOREM

We assume throughout this paper that \( A \) is an artin \( R \)-algebra with \( R \) a commutative artin ring. Following the definitions and notation of [7], a subcategory \( C \) of mod \( A \), the category of finitely generated \( A \)-modules, is always a full subcategory of mod \( A \) closed under isomorphisms and nonzero summands. By an exact sequence in \( C \) we mean an exact sequence of \( A \)-modules \( \cdots \rightarrow C_{i-1} \rightarrow C_i \rightarrow C_{i+1} \) with the nonzero \( C_i \) in \( C \). An object \( C \) in \( C \) is said to be Ext-projective if each exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( C \) splits. Dually an object \( A \) in \( C \) is said to be Ext-injective if each exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( C \) splits.

Next we recall some of the definitions and results given in [7] concerning right and left almost split morphisms in \( C \) as well as almost split sequences in \( C \).

A morphism \( f: B \rightarrow C \) is right almost split in \( C \) if \( f \) is not a splittable epimorphism and any morphism \( g: X \rightarrow C \) in \( C \) which is not a splittable epimorphism can be lifted to \( B \); i.e., there is an \( h: X \rightarrow B \) such that \( hg = f \). We recall that if \( f \) is right almost split then \( C \) is indecomposable. Dually, a morphism \( g: A \rightarrow B \) in \( C \) is said to be left almost split in \( C \) if \( g \) is not a splittable monomorphism and any morphism \( h: A \rightarrow Y \) in \( C \) which is not a splittable monomorphism can be extended to \( B \); i.e., there is a \( j: B \rightarrow Y \) such that \( h = jg \). Finally, an almost split sequence in \( C \) is an exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( C \) such that \( g \) is a left almost split morphism and \( f \) is a right almost split morphism. We recall the uniqueness properties of almost split sequences in \( C \). The following statements are equivalent for two almost split sequences \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) and \( 0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0 \) in \( C \): (a) The sequences are isomorphic, (b) \( A \approx A' \) and (c) \( C \approx C' \).
We say that a subcategory $C$ of $\text{mod } A$ has almost split sequences if it satisfies the following conditions:

(a) If $C$ in $C$ is indecomposable, then there is a right almost split morphism $f: B \to C$ in $C$ and there is a left almost split morphism $g: C \to B$ in $C$.

(b) If $A$ is an indecomposable non-$\text{Ext}$-injective in $C$, then there is an almost split sequence $0 \to A \to B \to C \to 0$ in $C$.

(c) If $C$ is an indecomposable non-$\text{Ext}$-projective in $C$, then there is an almost split sequence $0 \to A \to B \to C \to 0$ in $C$.

Our main purpose in this section is to give a sufficient condition for a subcategory $C$ of $\text{mod } A$ to have almost split sequences. This result serves as our basic existence theorem in the rest of the paper. However, before stating and proving this result, we recall some facts about dualizing $R$-varieties, a notion introduced in [4].

Let $I = I(R/r)$, the injective envelope over $R$ of $R/r$ where $r$ is the radical of $R$. Suppose $F: \text{mod } A \to \text{Ab}$ is an additive functor. Then for each $C$ in $\text{mod } A$ the abelian group $F(C)$ has a natural $R$-module structure. Define $DF: (\text{mod } A)^{\text{op}} \to \text{Ab}$ by $(DF)(C) = \text{Hom}_R(F(C), I)$. We then obtain the contravariant functor $D: (\text{mod } A, Ab) \to ((\text{mod } A)^{\text{op}}, Ab)$ where $(\text{mod } A, Ab)$ and $((\text{mod } A)^{\text{op}}, Ab)$ are the categories of covariant and contravariant additive functors from $\text{mod } A$ to $\text{Ab}$, the category of abelian group. Similarly we have the contravariant functor $D: ((\text{mod } A)^{\text{op}}, Ab) \to (\text{mod } A, Ab)$.

Next we recall that a functor $F: \text{mod } A \to \text{Ab}$ is said to be finitely presented if there is an exact sequence of functors $(C_1, \to (C_2, \to F \to 0$ where $(C_i, ) = \text{Hom}(C_i, )$ in $\text{mod } A$ for $i = 1, 2$. Similarly a functor $G: (\text{mod } A)^{\text{op}} \to \text{Ab}$ is said to be finitely presented if there is an exact sequence of functors $(, C_1, ) \to (, C_2, ) \to G \to 0$. The full subcategories $\text{f.p.}(\text{mod } A, Ab)$ and $\text{f.p.}((\text{mod } A)^{\text{op}}, Ab)$ of $(\text{mod } A, Ab)$ and $((\text{mod } A)^{\text{op}}, Ab)$ respectively consisting of the finitely presented functors have the following basic properties:

(a) If $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ is an exact sequence with $F_2$ and $F_3$ finitely presented, then $F_1$ and $F_4$ are also finitely presented.

(b) If $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of functors with $F_1$ and $F_3$ finitely presented, then $F_2$ is finitely presented.

(c) A functor $F$ is finitely presented if and only if $DF$ is finitely presented.

(d) The induced contravariant functors $D: \text{f.p.}(\text{mod } A, Ab) \to \text{f.p.}((\text{mod } A)^{\text{op}}, Ab)$ and $D: \text{f.p.}((\text{mod } A)^{\text{op}}, Ab) \to \text{f.p.}(\text{mod } A, Ab)$ are dualities which are dual inverses.
Suppose now that \( C \) is an additive subcategory of \( \text{mod} \; A \); i.e., \( C = \text{add} C \), the subcategory of \( \text{mod} \; A \) consisting of all finite sums of objects in \( C \). Then the contravariant functors \( D: (\text{mod} \; A, Ab) \to ((\text{mod} \; A)^{\text{op}}, Ab) \) and \( D: ((\text{mod} \; A)^{\text{op}}, Ab) \to (\text{mod} \; A, Ab) \) induce contravariant functors \( D: (C, Ab) \to (C^{\text{op}}, Ab) \) and \( D: (C^{\text{op}}, Ab) \to (C, Ab) \) in an obvious way. We say that \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \; A \) if \( F: C \to Ab \) is finitely presented in \( (C, Ab) \) if and only if \( DF: C^{\text{op}} \to Ab \) is finitely presented in \( (C^{\text{op}}, Ab) \) and \( G: C^{\text{op}} \to Ab \) is finitely presented in \( (C^{\text{op}}, Ab) \) if and only in \( DG: C \to Ab \) is finitely presented. If \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \; A \), then \( D: \text{f.p.}(C, Ab) \to \text{f.p.}(C^{\text{op}}, Ab) \) and \( D: \text{f.p.}(C^{\text{op}}, Ab) \to \text{f.p.}(C, Ab) \) are dualities which are dual inverses. In other words, \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \; A \) if and only if \( C \) is a dualizing \( R \)-variety in the sense of [4]. The reader is referred to [4] for details concerning the basic properties of dualizing \( R \)-varieties we use in this paper.

We now state and prove our main existence theorem.

**Theorem 1.1.** If \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \; A \) closed under extensions, then \( C \) has almost split sequences.

**Proof.** Since \( C \) is a dualizing \( R \)-variety, then we know by [4, Proposition 3.2], that all simple functors in \( (C, Ab) \) and \( (C^{\text{op}}, Ab) \) are finitely presented. In other words, if \( C \) is an indecomposable object in \( C \), then there is a right almost split morphism \( B \to C \) in \( C \) and a left almost split morphism \( C \to B' \) in \( C \).

Suppose \( A \) in \( C \) is an indecomposable non-Ext-injective object. Then there is a nonsplit exact sequence \( 0 \to A \to B' \to C' \to 0 \) in \( C \) which induces the exact sequence of functors \( 0 \to (, A) \to (, B') \to (, C') \to F \to 0 \) when \( F = \text{Coker}(, v) \). Since the exact sequence \( 0 \to A \to B' \to C' \to 0 \) does not split \( F \) is not zero. Therefore we know by [4, p. 324] that \( C \) being a dualizing \( R \)-variety implies that \( F \) contains a simple subfunctor \( S \).

Let \( C \) be the uniquely determined indecomposable in \( C \) such that \( S(C) \neq 0 \). Therefore there is a nonzero morphism \( (, C) \to S \) which is an epimorphism since \( S \) is simple. Because \( (, C) \) is projective in \( (C^{\text{op}}, Ab) \) we know there is a morphism \( h: C \to C' \) such that the exact diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
(, C) \xrightarrow{f} S \\
\downarrow (, h) \\
(, C') \rightarrow F \rightarrow 0
\end{array}
\]
commutes where \( S \to F \) is the inclusion of \( S \) into \( F \). Let

\[
\begin{array}{c}
0 \to A \xrightarrow{g} B \xrightarrow{\omega} C \to 0 \\
\downarrow \quad \downarrow \\
0 \to A \xrightarrow{u} B' \xrightarrow{\nu} C' \to 0
\end{array}
\]

be the pullback diagram in \( \text{mod } A \). Since \( C \) is closed under extensions, \( B \) is in \( C \). Furthermore it is straightforward to check that (*) induces the exact commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to (\cdot, A) \to (\cdot, B) \to (\cdot, C) \to S \to 0 \\
\downarrow \\
0 \to (\cdot, A) \to (\cdot, B') \to (\cdot, C') \to F \to 0
\end{array}
\]

Since \( S \) is simple and \( \text{End } A \) is a local ring, the fact that \( C \) is closed under extensions implies that the exact sequence \( 0 \to A \to B \to C \to 0 \) in \( C \) is almost split in \( C \) [2, Chap II, Proposition 4.4].

Consequently we have shown that there is an almost split sequence \( 0 \to A \to B \to C \to 0 \) in \( C \) if \( A \) is an indecomposable non-Ext-injective module in \( C \). A similar argument shows that there is an almost split sequence \( 0 \to A \to B \to C \to 0 \) in \( C \) if \( C \) is an indecomposable non-Ext-projective module in \( C \). Therefore we have established our desired result that \( C \) has almost split sequence if \( C \) is a dualizing \( R \)-subvariety of \( \text{mod } A \) closed under extensions.

2. MAIN EXISTENCE THEOREM

In view of our main existence theorem for a subcategory \( C \) of \( \text{mod } A \) to have almost split sequences, it is important to know when \( C \) is a dualizing \( R \)-subvariety of \( \text{mod } A \). Unfortunately it does not seem to be easy in practice to check whether or not a subcategory \( C \) of \( \text{mod } A \) satisfies the general description of dualizing \( R \)-varieties given in [4]. For this reason this section is devoted to exploring the connection between \( C \) being a dualizing \( R \)-subvariety of \( \text{mod } A \) and the more easily verified condition of \( \text{mod } A \) being functorially finite over \( C \).

The notion of \( \text{mod } A \) being functorially finite over \( C \), or \( C \) being functorially finite in \( \text{mod } A \), was first introduced in [7], which serves as the foundation for much of this paper. Recall that \( C \) is said to be contravariantly
finite in \( \text{mod } A \) if for each \( X \) in \( \text{mod } A \) the restriction \( (\cdot, X)|_C \) to \( C \) of the representable functor \( (\cdot, X) \) is a finitely generated functor on \( C \); i.e., there is an epimorphism \( \text{Hom}_C(\cdot, C) \to (\cdot, X)|_C \) for some \( C \) in \( C \). Equivalently, \( C \) is contravariantly finite in \( \text{mod } A \) if for each \( X \) in \( \text{mod } A \), there is a morphism \( C \to X \) with \( C \) in \( \text{add } C \) such that \( (C', C) \to (C', X) \to 0 \) is exact for all \( C' \) in \( C \). Dually \( C \) is covariantly finite in \( \text{mod } A \) if for each \( X \) in \( \text{mod } A \) the restriction \( (X, \cdot)|_C \) of the representable functor \( (X, \cdot) \) is a finitely generated functor on \( C \). Equivalently \( C \) is covariantly finite in \( \text{mod } A \) if for each \( X \) in \( \text{mod } A \) there is a morphism \( X \to C \) with \( C \) in \( \text{add } C \) such that \( (C', C) \to (X, C') \to 0 \) is exact for all \( C' \) in \( C \). Finally \( C \) is said to be functorially finite in \( \text{mod } A \) if it is both covariantly and contravariantly finite in \( \text{mod } A \).

Next we recall that a subcategory \( C' \) of \( \text{mod } A \) consisting of indecomposable objects in \( C \) is said to be a finite cover for \( C \) if (a) there is only a finite number of nonisomorphic objects in \( C' \) and (b) for each \( C \) in \( C \) there is a surjection \( C' \to C \) with \( C' \) in \( \text{add } C' \). It was shown in [7, Proposition 3.7] that a subcategory \( C' \) of \( \text{mod } A \) consisting of only a finite number of nonisomorphic objects in \( C \) is a finite cover for \( C \) if and only if there is a morphism \( A \to C' \) with \( C' \) in \( \text{add } C' \) such that \( (C', C) \to (A, C) \to 0 \) is exact for all \( C \) in \( C \). Thus if \( C \) is covariantly finite in \( \text{mod } A \), then \( C \) has a finite cover.

Dually a subcategory \( C'' \) of \( \text{mod } A \) consisting of indecomposable objects in \( C \) is said to be a finite cocover for \( C \) if (a) \( C'' \) has only a finite number of nonisomorphic objects and (b) for each \( C \) in \( C \) there is an injection \( C \to C'' \) with \( C'' \) in \( \text{add } C'' \). It was shown in [7, Proposition 3.6] that a subcategory \( C'' \) of \( \text{mod } A \) consisting of only a finite number of nonisomorphic indecomposable objects in \( C \) is a cocover for \( C \) if and only if there is a morphism \( C'' \to D(A) = \text{Hom}_A(A, I) \) with \( C'' \) in \( \text{add } C'' \) such that \( (C, C'') \to (C, D(A)) \to 0 \) is exact for all \( C \) in \( C \). Thus if \( C \) is contravariantly finite in \( \text{mod } A \), then \( C \) has a finite cocover.

Before starting to prove the main result of this section we recall one more bit of terminology.

Let \( C \) be an additive subcategory of \( \text{mod } A \); i.e., \( C = \text{add } C \). And let \( C_1 \to f_1; C_2 \to f_2; C_3 \) be morphisms in \( C \). Then \( f_1 \) is said to be a pseudokernel of \( f_2 \) in \( C \) if \( (C, C_1) \to (C, C_2) \to (C, C_3) \) is exact for all \( C \) in \( C \) and \( f_2 \) is said to be a pseudocokernel of \( f_1 \) if \( (C_3, C) \to (C_2, C) \to (C_1, C) \) is exact for all \( C \) in \( C \). We say that \( C \) has pseudokernels if each morphism in \( C \) has a pseudokernel. Similarly, we say that \( C \) has pseudocokernels if each morphism in \( C \) has a pseudocokernel.

In connection with the above definitions we point out the following which will be useful in establishing the main result of this section.

**Proposition 2.1.** Let \( C \) be an additive subcategory of \( \text{mod } A \) which is contravariantly finite in \( \text{mod } A \). Then
(a) \( C \) has pseudokernels.

(b) If \( F: (\text{mod} \ A)^{\text{op}} \to Ab \) is a finitely presented functor, then \( F \mid C \) is finitely presented over \( C \).

Proof. (a) Let \( 0 \to K \to C_1 \to C_2 \) be exact with \( C_1 \) and \( C_2 \) in \( C \). Since \( C \) is an additive subcategory of \( \text{mod} \ A \) which is contravariantly finite in \( \text{mod} \ A \), there is a morphism \( C_0 \to K \) with \( C_0 \) in \( C \) such that \( (C, C_0) \to (C, K) \to 0 \) is exact for all \( C \) in \( C \). Therefore the induced morphism \( C_0 \to C_1 \) is a pseudokernel for \( C_1 \to C_2 \). Hence \( C \) has pseudokernels.

(b) We first show that \( (,X) \mid C \) is finitely presented for all \( X \) in \( \text{mod} \ A \). Since \( C \) is contravariantly finite in \( \text{mod} \ A \), there is a morphism \( C_0 \to X \) with \( C_0 \) in \( C \) such that \( (C, C_0) \to (C, X) \to 0 \) is exact for all \( C \) in \( C \). Applying the same argument to \( \text{Ker}(C_0 \to X) \) we obtain that there is a sequence \( C_1 \to C_0 \to X \) such that \( (C, C_1) \to (C, C_0) \to (C, X) \to 0 \) is exact for all \( C \) in \( C \). Thus \( (,X) \mid C \) is finitely presented.

Suppose \( F: (\text{mod} \ A)^{\text{op}} \to Ab \) is finitely presented and \( (,X_i) \to (,X_0) \to F \to 0 \) is exact. Then \( (,X_i) \mid C \to (,X_0) \mid C \to F \mid C \to 0 \) is exact. By our previous remarks above we know that the \( (,X_i) \mid C \) are finitely presented. Since \( C \) has pseudokernels, f.p. \( (C^{\text{op}}, Ab) \) is closed under cokernels [4, p. 315], so \( F \mid C \) is finitely presented.

For the sake of completeness we state the dual of this result.

Proposition 2.2. Let \( C \) be an additive subcategory of \( \text{mod} \ A \) which is covariantly finite in \( \text{mod} \ A \). Then

(a) \( C \) has pseudocokernels.

(b) If \( F: \text{mod} \ A \to Ab \) is finitely presented, then \( F \mid C \) is finitely presented.

We now turn our attention to the main result of this section.

Theorem 2.3. The following are equivalent for an additive subcategory \( C \) of \( \text{mod} \ A \).

(a) \( C \) is functorially finite in \( \text{mod} \ A \).

(b) \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \ A \) with a finite cover and a finite cocover.

(c) The functors \( (,D(A)) \mid C \) and \( (A , ,) \mid C \) are finitely presented and \( C \) has pseudocokernels and pseudokernels.

Proof. (a) implies (b). We first show that \( C \) is a dualizing \( R \)-subvariety of \( \text{mod} \ A \). Let \( F: C^{\text{op}} \to Ab \) be a finitely presented functor. Then there is a morphism \( f: C_1 \to C_0 \) in \( C \) such that \( \text{Hom}_C( , C_1) \to (\cdot f) \text{Hom}_C( , C_0) \to F \to 0 \) is exact. The morphism \( f: C_1 \to C_0 \) also gives us an exact sequence of
functors in \(((\text{mod} \ A)^{\text{op}}, \text{Ab})\) \(\text{Hom}_A(\cdot, C_1) \to(\cdot, J)\text{Hom}_A(\cdot, C_0) \to G \to 0\) which has the property that the exact sequence

\[
\text{Hom}_A(\cdot, C_1) | C \to \text{Hom}_A(\cdot, C_0) | C \to G | C \to 0
\]

is isomorphic to \(\text{Hom}_C(\cdot, C_1) \to(\cdot, J)\text{Hom}_C(\cdot, C_0) \to F \to 0\). Since \(\text{mod} \ A\) is a dualizing \(R\)-variety we know that \(DG\) is a finitely presented functor. Hence by Proposition 2.1 we have that \((DG) | C\) is a finitely presented functor. But \((DG) | C = D(G | C) = D(F)\). Therefore if \(F: C^{\text{op}} \to \text{Ab}\) is finitely presented, then \(DF\) is finitely presented.

The fact that if \(F: C \to \text{Ab}\) is finitely presented, then \(DF\) is finitely presented follows by a similar, actually dual, argument. Hence we have shown that \(C\) is a dualizing \(R\)-subvariety of \(\text{mod} \ A\). We have already remarked that \(C\) being functorially finite in \(\text{mod} \ A\) implies that \(C\) has a finite cocover and a finite cover.

(b) implies (c). As was remarked earlier, the fact that \(C\) has finite cover and cocover means that the functors \((A, \cdot) | C\) and \((\cdot, D(A)) | C\) are finitely generated. But \((A, X) = A \otimes_X X\) and so \(D(A, X) = \text{Hom}_R(A \otimes_X X, I) \approx \text{Hom}_A(X, D(A))\). Hence \(D((A, \cdot) | C) = (\cdot, D(A)) | C\). Since \(C\) is a dualizing \(R\)-variety, we know that a functor \(F: C \to \text{Ab}\) is finitely presented if and only if \(F\) and \(DF\) are finitely generated \([4, \text{Proposition 3.1}]\). Therefore \((A, \cdot) | C\) and \((\cdot, D(A)) | C\) are finitely presented functors. The fact that \(C\) has pseudokernels and pseudocokernels is also a consequence of \(C\) being a dualizing \(R\)-variety \([4, \text{Theorem 2.4}]\).

(c) implies (a). We first show that \(C\) is covariantly finite in \(\text{mod} \ A\) by showing that if \(X\) is in \(\text{mod} \ A\), then \((X, \cdot) | C\) is finitely generated. Let \(P_1 \to P_0 \to X \to 0\) be in \(\text{mod} \ A\) with the \(P_i\) projective. Then the exact sequence \(0 \to (X, \cdot) \to (P_0, \cdot) \to (P_1, \cdot)\) gives rise to the exact sequence \(0 \to (X, \cdot) | C \to (P_0, \cdot) | C \to (P_1, \cdot) | C\). Since \((A, \cdot) | C\) is finitely presented, the \((P_i, \cdot) | C\) are finitely presented. But \(C\) having pseudocokernels implies that the kernel of a morphism between finitely presented functors is also finitely presented. Therefore \((X, \cdot) | C\) is finitely generated since it is finitely presented.

The rest of the implication (c) implies (a) follows by a similar, actually dual, argument. Thus the theorem is established.

As an immediate consequence of Theorems 1.1 and 2.3 we have the following.

**Theorem 2.4.** Let \(C\) be a functorially finite subcategory of \(\text{mod} \ A\) which is closed under extensions. Then

(a) \(C\) has a finite cover and a finite cocover.

(b) \(C\) has almost split sequences.
Thus we see that subcategories $C$ of $\text{mod } A$ which are closed under extensions and are functorially finite in $\text{mod } A$ are similar to categories of finitely generated modules over artin algebras in quite a few respects. These similarities suggest the following problems about such subcategories $C$ of $\text{mod } A$.

(a) Describe the Ext-projective and Ext-injective objects in $C$. In particular, need there be only a finite number of isomorphism classes of indecomposable objects which are Ext-projective or Ext-injective? Also, need these numbers be the same when finite?

(b) Suppose $0 \to A \to B \to C \to 0$ is an almost split sequence in $C$. Does there exist a method of constructing $A$ from $C$ and vice versa similar to the construction given by the functors $D$ and $\text{Tr } D$?

Although we cannot answer these problems in general, we do have information along these lines in some special situations as we now proceed to show.

3. Subcategories Closed under Submodules

Throughout this section we assume that $C$ is a subcategory of $\text{mod } A$ closed under extensions. Clearly an indecomposable $C$ in $C$ is Ext-projective (Ext-injective) if and only if $\text{Ext}^1_A(C, X) = 0$ ($\text{Ext}^1_A(X, C) = 0$) for all $X$ in $C$. Suppose $C$ has a minimal finite cover $P_0(C)$; i.e., $P_0(C)$ is a finite cover of $C$ which is contained in all other covers for $C$. Then $C$ is in $P_0(C)$ if and only if $C$ is a splitting projective in $C$; i.e., if $f: Y \to C$ is a surjection in $C$, then $f$ is a splittable surjection (see [7, pp. 14–17]). Hence the objects in $P_0(C)$ are Ext-projective in $C$. However, in general, there may be Ext-projective objects in $C$ which are not in $P_0(C)$. Dually, suppose $C$ has a minimal finite cocover $I_0(C)$; i.e., $I_0(C)$ is a finite cocover of $C$ which is contained in all cocovers of $C$. Then $C$ is in $I_0(C)$ if and only if $C$ is a splitting injective in $C$; i.e., if $g: C \to Y$ is an injection in $C$, then $g$ is a splittable injection (see [7, pp. 14–17]). Hence the objects of $I_0(C)$ are Ext-injective in $C$. However, in general, there may be Ext-injective objects in $C$ which are not in $I_0(C)$.

Suppose now that $C$ is closed under submodules; i.e., if $C'$ is a submodule of $C$ in $C$, then $C'$ is in $C$. Then it is easily seen (see [7, Proposition 4.8.]) that for each $X$ in $\text{mod } A$ there is a unique submodule $t_cX$ minimal with respect to the property $X/t_cX$ is in $C$. Moreover the surjection $X \to X/t_cX$ induces an isomorphism $(X/t_cX, C) \approx (X, C)$ for all $C$ in $C$. From this it follows that if $P_1, \ldots, P_n$ is a complete set of nonisomorphic indecomposable projective $A$-modules then $C$ has a minimal finite cover $P_0(C)$ which consists of the modules isomorphic to the nonzero $P_i/t_cP_i$ for $i = 1, \ldots, n$. Finally, if
an indecomposable $C$ in $\mathcal{C}$ is Ext-projective, then $C$ is in $P_0(\mathcal{C})$. For we have an exact sequence $0 \to A \to B \to C \to 0$ with $B$ in $\text{add} \ P_0(\mathcal{C})$. Since $\mathcal{C}$ is closed under submodules, $A$ is in $\mathcal{C}$. Therefore if $C$ is Ext-projective, the sequence $0 \to A \to B \to C \to 0$ splits, so $C$ is in $P_0(\mathcal{C})$. Thus we have shown the following.

**Proposition 3.1.** Suppose $\mathcal{C}$ is closed under submodules. Then

(a) $\mathcal{C}$ has a minimal finite cover $P_0(\mathcal{C})$.

(b) $C$ in $\mathcal{C}$ is in $P_0(\mathcal{C})$ if and only if $C \simeq P/t_C P$ for some indecomposable projective $A$-module $P$ such that $P/t_C P \neq 0$.

(c) An indecomposable $C$ in $\mathcal{C}$ is Ext-projective if and only if $C$ is in $P_0(\mathcal{C})$.

Dually, suppose $\mathcal{C}$ is closed under factor modules; i.e., if $C$ is in $\mathcal{C}$, $C/C'$ is in $\mathcal{C}$ for all submodules $C'$ of $C$. Then it is easily seen (see [7, Proposition 4.8]) that for each $X$ in mod $A$, there is a unique submodule $\tau_C(X)$ of $X$ maximal with respect to being in $\mathcal{C}$. Moreover the injection $\tau_C(X) \to X$ induces an isomorphism $(C, \tau_C(X)) \to (C, X)$ for all $C$ in $\mathcal{C}$. We also have the following dual version of Proposition 3.1.

**Proposition 3.2.** Suppose $\mathcal{C}$ is closed under factor modules. Then

(a) $\mathcal{C}$ has a minimal finite cocover $I_0(\mathcal{C})$.

(b) $C$ is in $I_0(\mathcal{C})$ if and only if $C \simeq \tau_C(I)$ for some indecomposable injective $A$-module $I$ such that $\tau_C(I) \neq 0$.

(c) An indecomposable $C$ in $\mathcal{C}$ is Ext-injective if and only if $C$ is in $I_0(\mathcal{C})$.

Suppose once again that $\mathcal{C}$ is closed under submodules. Having described the indecomposable Ext-projectives in $\mathcal{C}$, we now turn our attention to characterizing the indecomposable Ext-injective modules in $\mathcal{C}$.

**Proposition 3.3.** Suppose $\mathcal{C}$ is closed under submodules. Then the following statements are equivalent for an indecomposable object $A$ in $\mathcal{C}$.

(a) $A$ is not Ext-injective in $\mathcal{C}$.

(b) $t_C(\text{Tr} \ DA) \neq \text{Tr} \ DA$.

(c) The almost split sequence $0 \to A \to B \to \text{Tr} \ DA \to 0$ in mod $A$ has the property that the induced sequence in $\mathcal{C}$, $0 \to A \to B/t_C B \to \text{Tr} \ DA/t_C(\text{Tr} \ DA) \to 0$, is exact and therefore nonsplit.
Proof. (a) implies (b). Since \( A \) is not \( \text{Ext} \)-injective in \( \mathbf{C} \), there is a nonzero element \( x \) in \( \text{Ext}_A^1(X, A) \) for some \( X \) in \( \mathbf{C} \). Then there is a morphism \( f: \text{Tr} DA \to X \) such that \( \text{Ext}_A^1(f, A)(x) \) in \( \text{Ext}_A^1(\text{Tr} DA, A) \) is the almost split sequence. Hence \( f: \text{Tr} DA \to X \) is not zero, so \( t_C \text{Tr} DA \neq \text{Tr} DA \).

(b) implies (c). Since \( 0 \to A \to B \to \text{Tr} DA \to 0 \) is an almost split sequence and \( t_C(\text{Tr} DA) \neq \text{Tr} DA \), there is a morphism \( h: t_C(\text{Tr} DA) \to B \) such that \( (g|\text{Im} h)h = \text{id}_{t_C(\text{Tr} DA)} \). Letting \( B' = \text{Im} h \), we have the exact sequence \( 0 \to A \to B/B' \to \text{Tr} DA/t_C(\text{Tr} DA) \to 0 \). Since \( \mathbf{C} \) is closed under extensions \( B/B' \) is in \( \mathbf{C} \). Therefore \( B' \supseteq t_C B \) and so \( t_C(\text{Tr} DA) = g(B') \supseteq g(t_C B) \). So we have the exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B/t_C B & \to & \text{Tr} DA/g(t_C B) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A & \to & B/B' & \to & \text{Tr} DA/t_C(\text{Tr} DA) & \to & 0 \\
\end{array}
\]

Since \( B/t_C B \) is in \( \mathbf{C} \), we have that \( B'/t_C B \) is in \( \mathbf{C} \) and hence that \( \text{Tr} DA/g(t_C B) \) is in \( \mathbf{C} \) (remember that \( \mathbf{C} \) is closed under submodules and extensions).

Hence \( g(t_C B) \supseteq t_C(\text{Tr} DA) \) and so \( g(t_C B) = t_C(\text{Tr} DA) \). Therefore \( B'/t_C B = 0 \), which shows that \( 0 \to A \to B/t_C B \to \text{Tr} DA/t_C \text{Tr} DA \to 0 \) is exact. The fact that this sequences does not split follows from the fact that \( 0 \to A \to B \to \text{Tr} DA \to 0 \) does not split.

(c) implies (a). Trivial.

As an immediate consequence of Proposition 3.3, we have the following characterization of the indecomposable \( \text{Ext} \)-injective objects in \( \mathbf{C} \).

**Corollary 3.4.** Suppose \( \mathbf{C} \) is closed under submodules. Then an indecomposable \( A \) in \( \mathbf{C} \) is \( \text{Ext} \)-injective in \( \mathbf{C} \) if and only if \( t_C(\text{Tr} DA) = \text{Tr} DA \).

As another consequence of Proposition 3.3 we have the following result concerning the existence and structure of almost split sequence in \( \mathbf{C} \).

**Corollary 3.5.** Suppose \( \mathbf{C} \) is closed under submodules. Let \( A \) be an indecomposable object in \( \mathbf{C} \) which is not \( \text{Ext} \)-injective in \( \mathbf{C} \) and let
0 \rightarrow A \rightarrow B \rightarrow \text{Tr} DA \rightarrow 0 be an almost split sequence in \text{mod} A. Then there is a commutative exact diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
C & \rightarrow & C \\
\downarrow & & \downarrow \\
0 & \rightarrow & A \rightarrow B/\text{Tr} DA \rightarrow 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & A/\text{Tr} DA \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

satisfying:

(a) \(0 \rightarrow A \rightarrow B' \rightarrow A' \rightarrow 0\) is an almost split sequence in \(C\);

(b) \(C \rightarrow B/\text{Tr} DA\) and \(C \rightarrow \text{Tr} DA\) are splittable monomorphisms.

**Proof:** The fact that \(0 \rightarrow A \rightarrow B/\text{Tr} DA \rightarrow 0\) is exact was established in Proposition 3.3. The fact that \(0 \rightarrow A \rightarrow B\) is left almost split in \(\text{mod} A\) implies that \(0 \rightarrow A \rightarrow B/\text{Tr} DA\) is left almost split in \(C\). Now the exact sequence \(0 \rightarrow A \rightarrow B/\text{Tr} DA \rightarrow 0\) can be written as a sum of exact sequences

\[
\begin{array}{ccc}
0 & \rightarrow & A \rightarrow B' \rightarrow A' \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & C
\end{array}
\]

where \(f'\) is left minimal [7, Proposition 1.2]). Hence the exact sequence \(0 \rightarrow A \rightarrow f' B' \rightarrow g' A' \rightarrow 0\) in \(C\) has the property that \(f'\) is minimal left almost split in \(C\), which by [2, Chap. II, Proposition 4.4] implies that it is almost split in \(C\).

For the sake of completeness we state without proof the duals of these last results if \(C\) has factor modules instead of submodules.

**Proposition 3.6.** Suppose \(C\) is closed under factor modules. Then the following statements are equivalent for an indecomposable object \(C\) in \(C\).

(a) \(C\) is not Ext-projective.

(b) \(\tau_c(D \text{Tr} C) \neq 0\).

(c) The almost split sequence \(0 \rightarrow D \rightarrow C \rightarrow B \rightarrow 0\) in \(\text{mod} A\) has the property that the induced sequence in \(C\) \(0 \rightarrow \tau_c(D \text{Tr} C) \rightarrow \tau_c(B) \rightarrow C \rightarrow 0\) is exact and therefore not split.
Corollary 3.7. Suppose C is closed under factor modules. Then an indecomposable C in C is Ext-projective if and only if $\tau_C(D \operatorname{Tr} C) = 0$.

Corollary 3.8. Suppose C is closed under factor modules. Let C be an indecomposable in C which is not Ext-projective and let $0 \to D \operatorname{Tr} C \to B \to C \to 0$ an almost split sequence in mod $\Lambda$. Then there is a commutative exact diagram

\[
\begin{array}{ccccccccc}
0 & \to & C' & \to & B' & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tau_C(D \operatorname{Tr} C) & \to & \tau_C(B) & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & = & A & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & & 
\end{array}
\]

satisfying:

(a) $0 \to C' \to B' \to C \to 0$ is an almost split sequence in C;
(b) $0 \to C' \to \tau_C(D \operatorname{Tr} C)$ and $0 \to B' \to \tau_C(B)$ are splittable monomorphisms.

4. The Ext-Projective and Ext-Injective Modules in Sub $M$

We assume throughout this section that C is a subcategory of mod $\Lambda$ which is closed under extensions. In the previous section we examined the Ext-injective and Ext-projective objects in C under the additional hypothesis that C was closed under either submodules or factor modules. In this section we obtain somewhat sharper results by adding the hypothesis that if C is closed under submodules, then it also has a finite cover and that if C is closed under factor modules, then it also has a finite cover. Amongst other things we show that under either of these additional hypothesis C has only a finite number of nonisomorphic, indecomposable Ext-projective and Ext-injective modules.

Suppose C is a subcategory of mod $\Lambda$ closed under submodules. Then C has a finite cocover if and only if there is a $C$ in C such that $C = \text{Sub } C$ where $\text{Sub } C$ is the subcategory of mod $\Lambda$ consisting of all modules isomorphic to submodules of finite sums of copies of C. We now assume that $C = \text{Sub } C$ for some $C$ in C. Then by [7, Proposition 4.7] we know that C is functorially finite in mod $\Lambda$. Therefore C has almost split sequences by
Theorem 2.4 since we have the blanket assumption that \( C \) is closed under extensions. Before describing the indecomposable Ext-injective modules in \( C \) we introduce some notation.

Since \( C \) is functorially finite in \( \text{mod} \ A \) we know by [7, Proposition 3.9] that for each \( X \) in \( \text{mod} \ A \) there is a unique (up to isomorphism) right minimal morphism \( f: C \rightarrow X \) with \( C \) in \( C \) such that \( \langle C', C \rangle \rightarrow \langle C', X \rangle \rightarrow 0 \) is exact for all \( C' \) in \( C \). For each \( X \) we denote by \( f_X: X_C \rightarrow X \) one fixed such morphism. The fact that \( C \) is closed under submodules implies \( \text{Ker} \ f_X \) is contained in \( C \) and we denote \( \text{Ker} \ f_X \) by \( A^X_C \). Finally, we recall that if \( D \) is a subcategory of \( \text{mod} \ A \), we denote by \( \text{Ind} \ D \) the subcategory of \( \text{mod} \ A \) consisting of the indecomposable modules in \( D \) and if \( X \) is a module in \( \text{mod} \ A \) we denote by \( \text{Ind} \ X \) the subcategory of \( \text{mod} \ A \) consisting of the indecomposable modules isomorphic to summands of \( X \).

**Theorem 4.1.** Suppose \( C = \text{Sub} \ C \) for some \( C \) in \( C \). Denoting \( I_0(\text{mod} \ A) \) by \( I_0 \) we have

(a) \( I_0(\mathcal{C}) = \bigcup_{I \in I_0} \text{Ind} \ I_C \),

(b) \( \bigcup_{I \in I_0} \text{Ind} \ A^I_C \cap I_0(\mathcal{C}) = \emptyset \),

(c) \( I_0(\mathcal{C}) \cup \bigcup_{I \in I_0} \text{Ind} \ A^I_C \) is the subcategory of \( \text{Ind}(\text{mod} \ A) \) consisting of the Ext-injective modules in \( \text{Ind} \ C \).

**Proof.** (a) See [7, Lemma 3.5].

(b) Suppose \( X \) is in \( I_0(\mathcal{C}) \cap \bigcup_{I \in I_0} \text{Ind} \ A^I_C \). Then there is a \( I \) in \( I_0 \) such that \( X \) is a summand of \( A^I_C \). Thus the composition of monomorphisms \( X \rightarrow A^I_C \rightarrow I_C \) is a splittable monomorphism. But then \( f_I : I_C \rightarrow I \) is not right minimal, which is a contradiction. Hence \( I_0(\mathcal{C}) \cap \bigcup_{I \in I_0} \text{Ind} \ A^I_C = \emptyset \).

(c) We first show that if \( X \) is in \( I_0(\mathcal{C}) \cup \bigcup_{I \in I_0} \text{Ind} \ A^I_C \), then \( X \) is Ext-injective in \( C \), or what is the same thing, \( \text{Ext}^1(\mathcal{C}, X) = 0 \) for all \( C \) in \( C \). We have already seen at the beginning of Section 3 that \( \text{Ext}^1(\mathcal{C}, X) = 0 \) for all \( C \) in \( C \) if \( X \) is in \( I_0(\mathcal{C}) \). Let \( I \) be in \( \text{Ind} \ I_0 \) and let \( T = \text{Im}(I_C \rightarrow I) \). Since \( (C, f_I) : (C, I_C) \rightarrow (C, I) \rightarrow 0 \) is exact for all \( C \) in \( C \), we have that \( (C, I_C) \rightarrow (C, T) \rightarrow 0 \) is exact for all \( C \) in \( C \). Hence the exact sequence \( 0 \rightarrow A^I_C \rightarrow I_C \rightarrow T \rightarrow 0 \) gives the exact sequence \( 0 \rightarrow \text{Ext}^1(\mathcal{C}, A^I_C) \rightarrow \text{Ext}^1(\mathcal{C}, I_C) \). Therefore \( \text{Ext}^1(\mathcal{C}, A^I_C) = 0 \) for all \( C \) in \( C \) since \( \text{Ext}^1(\mathcal{C}, I_C) = 0 \) for all \( C \) in \( C \). Therefore we have shown that \( I_0(\mathcal{C}) \cup \bigcup_{I \in I_0} \text{Ind} \ A^I_C \) consists of Ext-injective objects in \( C \).

Suppose now that \( X \) in \( C \) is an indecomposable Ext-injective module which is not in \( I_0(C) \). We want to show that \( X \) is in \( \bigcup_{I \in I_0} \text{Ind} \ A^I_C \). Since \( I_0(\mathcal{C}) \) is a cocover for \( C \), we know there is an exact sequence \( 0 \rightarrow X \rightarrow^h \rightarrow Y \rightarrow U \rightarrow 0 \) with \( Y \) in \( \text{Ind} \ I_0(\mathcal{C}) \) such that \( h \) is left minimal. Since \( \text{Ext}^1(\mathcal{C}, X) = 0 \) for all \( C \) in \( C \), it follows that \( (C, Y) \rightarrow (C, U) \rightarrow 0 \) is exact for all \( C \) in \( C \). Hence \( 0 \rightarrow (\mathcal{C}, X) \rightarrow (\mathcal{C}, Y) \rightarrow (\mathcal{C}, U) \rightarrow (\mathcal{C}, 0) \) is a minimal projective presentation of \( (\mathcal{C}, U) \) in \( (\mathcal{C}^{op}, Ab) \).
Let $U \to I$ be an injective envelope of $U$ in $\text{mod } A$. Then we have the exact commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to A'_c \to f^{-1}(U) \to U \to 0 \\
\| \\
0 \to A'_c \to I_c \to I
\end{array}
\tag{\ast}
\]

which implies $f^{-1}(U) \to U$ has the property $(f^{-1}(U))|\mathcal{C} \to (U)|\mathcal{C} \to 0$ is exact. Since $\mathcal{C}$ is closed under submodules, $f^{-1}(U)$ is in $\mathcal{C}$, so we have that $(f^{-1}(U))|\mathcal{C}$ is projective. Hence the fact that $(Y)|\mathcal{C} \to (U)|\mathcal{C} \to 0$ is a projective cover means that there is an exact commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to X \to Y \to U \to 0 \\
\downarrow \\
0 \to A'_c \to f^{-1}(U) \to U \to 0
\end{array}
\tag{\ast\ast}
\]

with $Y \to f^{-1}(U)$ a splittable monomorphism.

Combining $\ast$ and $\ast\ast$ we obtain the exact commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to X \to Y \to U \to 0 \\
\downarrow \\
0 \to A'_c \to I_c \to I
\end{array}
\]

Since $Y$ is in $I_0(\mathcal{C})$, we know that $Y \to I_c$ is a splittable monomorphism. So we obtain the exact sequence of functors on $\mathcal{C}^{\text{op}}$

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to (X) \to (Y) \to (U)|\mathcal{C} \to 0 \\
\downarrow \\
0 \to (A'_c) \to (I_c) \to (I)|\mathcal{C} \to 0 \\
\downarrow \\
0 \to G \to (I_c/Y)|\mathcal{C} \to F \to 0 \\
\downarrow \\
0 \\
0
\end{array}
\]

where $G = \text{Coker}((X) \to (A'_c))$ and $F = \text{Coker}((U)|\mathcal{C} \to (I)|\mathcal{C})$. Since $Y \to I_c$ is a splittable monomorphism, $I_c/Y$ is in $\mathcal{C}$ so $(I_c/Y)|\mathcal{C} = (I_c/Y)$ is projective in $(\mathcal{C}^{\text{op}}, Ab)$. 
Next we observe that $F$ is a subfunctor of $(\cdot, I/U) | C$. Therefore $pd F \leq 1$ where $pd F$ means projective dimension of $F$ in $(C^{op}, Ab)$. To see this we first observe that if $M$ is in mod $A$, then the exact sequence $0 \rightarrow A_C^M \rightarrow M_C \rightarrow M$ induces the exact sequence of functors $0 \rightarrow (\cdot, A_C^M) \rightarrow (\cdot, M_C) \rightarrow (\cdot, M) | C \rightarrow 0$ with the $(\cdot, A_C^M)$ and $(\cdot, M_C)$ projective in $(C^{op}, Ab)$. Thus $pd(\cdot, M) | C \leq 1$ for all $M$ in mod $A$. In particular, $pd(\cdot, I/U) | C \leq 1$. But the fact that $C$ is closed under submodules implies $gl \ dim \ f.p(C^{op}, Ab) \leq 2$. Hence the fact that $F \subset (\cdot, I/U) | C$ and $pd(\cdot, I/U) | C \leq 1$, implies $pd F \leq 1$.

Consequently, it follows from the exact sequence $0 \rightarrow G \rightarrow (\cdot, I_C/Y) \rightarrow F \rightarrow 0$ that $G$ is projective since $(\cdot, I_C/Y)$ is projective. Hence the exact sequence $0 \rightarrow (\cdot, X) \rightarrow (\cdot, A_C^I) \rightarrow G \rightarrow 0$ splits, which means that $X \rightarrow A_C^I$ is a splittable monomorphism. Therefore $X$ is in Ind $A_C^I$, our desired result. This completes the proof of the theorem.

As an immediate consequence of this theorem we have the following.

**Corollary 4.2.** Suppose $C = \text{Sub} C$ for some $C$ in $C$. Then there are a finite number of nonisomorphic indecomposable objects in $C$ which are Ext-injective as well as only a finite number which are Ext-projective.

We end this section with the statements of the duals of Theorem 4.1 and Corollary 4.2.

Suppose $C$ is a subcategory of mod $A$ which is closed under factor modules. Then $C$ has a finite cover if and only if $C = \text{Fac} C$ for some $C$ in $C$, where $\text{Fac} C$ is the subcategory of mod $A$ consisting of all modules isomorphic to factor modules of finite sums of $C$. Assume $C = \text{Fac} C$. Then by [7, Proposition 4.6] we know that $C$ is functorially finite in mod $A$. Hence for each $X$ in mod $A$, there is a unique, up to isomorphism, left minimal morphism $g: X \rightarrow C$ with $C$ in $C$ such that $(C, C') \rightarrow (X, C') \rightarrow 0$ is exact for all $C'$ in $C$. For each $X$ in mod $A$ we denote by $g^X: X \rightarrow X^C$ one fixed such morphism. The fact that $C$ is closed under factor modules implies $\text{Coker} g^X$ is in $C$ and we denote $\text{Coker} g^X$ by $A_C^X$.

We now give the dual of Theorem 4.1.

**Theorem 4.3.** Suppose $C = \text{Fac} C$ for some $C$ in $C$. Denoting $\text{P}_0(\text{mod} A)$ by $\text{P}_0$ we have

(a) $\text{P}_0(C) = \bigcup_{P \in \text{P}_0} \text{Ind} P^C$.

(b) $\bigcup_{P \in \text{P}_0} \text{Ind} A_C^P \cap \text{P}_0(C) = \emptyset$.

(c) $\text{P}_0(C) \cup \bigcup_{P \in \text{P}_0} \text{Ind} A_C^P$ is the subcategory of Ind(\text{mod} A) consisting of the Ext-projective modules in Ind $C$.

Finally, we have the following dual of Corollary 4.2.

**Corollary 4.4.** Suppose $C = \text{Fac} C$ for some $C$ in $C$. Then $C$ has only
a finite number of nonisomorphic indecomposable modules which are Ext-projective as well as only a finite number which are Ext-injective.

5. When Is Sub M Closed under Extensions?

In view of Theorems 4.1 and 4.3 just established it is of interest to know when for a module \( M \) in \( \text{mod}\ A \) the subcategory \( \text{Sub} M \) of \( \text{mod}\ A \) is closed under extensions as well as when the subcategory \( \text{Fac} M \) is closed under extensions. We begin our discussion of these problems with some observations concerning the subcategories of \( \text{mod}\ A \) of the forms \( \text{Sub} M \) and \( \text{Fac} M \) for some \( M \) in \( \text{mod}\ A \).

Let \( M \) be in \( \text{mod}\ A \). Then add \( M \) has a unique minimal finite cocover \( I_0(\text{add} M) \). Let \( \bar{M} \) be a sum of a complete set of nonisomorphic modules in \( I_0(\text{add} M) \). It is now easy to see that \( \bar{M} \) has the following properties:

(i) \( \bar{M} \) is a sum of nonisomorphic indecomposable modules;
(ii) \( \text{ind} \bar{M} = I_0(\text{add} M) \);
(iii) \( \text{Sub} \bar{M} = \text{sub} M \);
(iv) The above properties of \( \bar{M} \) uniquely determine \( \bar{M} \) up to isomorphism.

In view of these remarks, we see that in discussing the subcategories of \( \text{mod}\ A \) of the form \( \text{Sub} M \) for some \( M \) we might as well assume that \( M = \bar{M} \). Unless stated to the contrary, we make the convention that when we write \( \text{Sub} M \) we are automatically assuming that \( M = \bar{M} \).

Similarly, add \( M \) has a unique finite minimal cover \( P_0(\text{add} M) \). Let \( M \) be a sum of a complete set of nonisomorphic modules in \( P_0(\text{add} M) \). Then we have

(i) \( M \) is a sum of nonisomorphic indecomposable modules;
(ii) \( \text{Ind} M = P_0(\text{add} M) \)
(iii) \( \text{Fac} M = \text{Fac} M \)
(iv) The above properties of \( M \) uniquely determine \( M \) up to isomorphism.

Unless stated to the contrary, we will assume \( M = \bar{M} \) when we write \( \text{Fac} M \).

One final notation. For a subcategory \( C \) of \( \text{mod}\ A \) we denote by \( \text{Ext}(C) \) the subcategory of \( \text{mod}\ A \) consisting of summands of modules \( M \) which have a sequence of submodules \( 0 = M_0 \subset M \subset \cdots \subset M_n = M \) such that \( M_{i+1}/M_i \) is in \( C \) for \( i = 0, \ldots, n - 1 \).

With these conventions and notations in mind, we have the following result which is basic to our entire discussion in this section.
PROPOSITION 5.1. Suppose $C$ is a subcategory of $\text{Sub} M$ for some $M$ in $\text{mod} \mathcal{A}$.

(a) If $\text{Ext}^1_{\mathcal{A}}(C, M) = 0$ for all $C$ in $C$, then $\text{Ext}(C)$ is contained in $\text{Sub} M$.

(b) Moreover if $\text{Ind} M$ is contained in $C$, then $\text{Ext}(C)$ being contained in $\text{Sub} M$ implies that $\text{Ext}^1_{\mathcal{A}}(C, M) = 0$ for all $C$ in $C$.

Proof: (a) Suppose $\text{Ext}^1_{\mathcal{A}}(C, M) = 0$ for all $C$ in $C$. Let $X$ be in $\text{Ext}(C)$ and suppose $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a chain of submodules of $X$ such that $X_{i+1}/X_i$ is in $C$. The proof goes by induction on the minimal length of these chains. If $n = 1$, then $X$ is in $C$ and hence in $\text{Sub} M$. Assume now that the claim is proved for all modules in $\text{Ext}(C)$ with chains of length $n < k$ and suppose $X$ in $\text{Ext}(C)$ has a chain of length $n = k$. Then there is an exact sequence $0 \rightarrow X_{n-1} \rightarrow X \rightarrow X/X_{n-1} \rightarrow 0$ with $X_{n-1}$ and $X/X_{n-1}$ in $\text{Sub} M$. Then we have that there is an exact commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
0 & \rightarrow & X \\
\downarrow & & \downarrow \\
0 & \rightarrow & X/X_{n-1} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
$$

with $M'$ in $\text{add} M$ since $\text{Ind} M$ is a cocover for $\text{Sub} M$. Since $X/X_{n-1}$ is in $C$, we have that $\text{Ext}^1(X/X_{n-1}, M') = 0$, so $E = M' \oplus X/X_{n-1}$. But $X/X_{n-1} \subset M''$ in $\text{add} M$ so $E \subset M' \oplus M''$. Hence $X$ is contained in $M' \oplus M''$ and is therefore in $\text{Sub} M$. So part (a) is proven.

(b) This is a trivial consequence of the following.

LEMMA 5.2. Let $M$ be a module such that $\text{Ind} M = \text{I}_0(\text{add} M)$. If $0 \rightarrow M \rightarrow E \rightarrow A \rightarrow 0$ is a nonsplit exact sequence in $\text{mod} \mathcal{A}$, then $E$ is not in $\text{Sub} M$.

Proof. Suppose $E$ is in $\text{Sub} M$. Then there is an injection $E \rightarrow M'$ with $M'$ in $\text{add} M$. Thus the composition $M \rightarrow E \rightarrow M'$ is an injection which is a splittable monomorphism since $\text{Ind} M = \text{I}_0(\text{add} M)$. Therefore $0 \rightarrow M \rightarrow E$ is a splittable monomorphism. This contradiction shows that $E$ is not in $\text{Sub} M$.

As a consequence of Proposition 5.1 we have the following criterion for $\text{Sub} M$ closed under extensions.

PROPOSITION 5.3. The following are equivalent for $C = \text{Sub} M$.
(a) \( C \) is closed under extensions.
(b) \( \text{Ext}^1_C(M, C) = 0 \) for all \( C \) in \( C \).
(c) \( \text{Ext}^1_C(M', M) \) for all submodules \( M' \) of \( M \).

**Proof.** (a) equivalent to (b). Trivial consequence of Proposition 5.1.

(b) implies (c). Trivial.

(c) implies (b). Suppose \( C \subseteq nM \). If \( n = 1 \), there is nothing to prove. Assume true for \( n < k \). Then the natural projection \( kM \rightarrow (k-1)M \) induces an exact sequence \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) with \( C' \subseteq M \) and \( C'' \subseteq (k-1)M \).

The exact sequence

\[
\text{Ext}^1_C(C'', M) \rightarrow \text{Ext}^1_C(C, M) \rightarrow \text{Ext}^1_C(C', M)
\]

gives that \( \text{Ext}^1_C(C', M) = 0 \) since \( \text{Ext}^1_C(C'', M) = 0 = \text{Ext}^1_C(C', M) \).

For the sake of completeness we state the duals of Propositions 5.1 and 5.3.

**PROPOSITION 5.4.** Suppose \( C \) is a subcategory of \( \text{Fac} M \) for some \( M \) in \( \text{mod} \ A \).

(a) If \( \text{Ext}^1_C(M, C) = 0 \) for all \( C \) in \( C \), then \( \text{Ext}(C) \) is contained in \( \text{Fac} M \).

(b) Moreover if \( \text{Ind} M \) is contained in \( C \), then \( \text{Ext}(C) \) being contained in \( \text{Fac} M \) implies that \( \text{Ext}^1_C(M, C) = 0 \) for all \( C \) in \( C \).

The dual of Proposition 5.3 is the following.

**PROPOSITION 5.5.** The following are equivalent for \( C = \text{Fac} M \).

(a) \( C \) is closed under extensions.
(b) \( \text{Ext}^1_C(M, C) = 0 \) for all \( C \) in \( C \).
(c) \( \text{Ext}^1_C(M, M'') = 0 \) for all factor modules \( M'' \) of \( M \).

In view of these results it is of interest to know when a module \( M \) has the property \( \text{Ext}^1_C(M', M) = 0 \) for all submodules \( M' \) of \( M \) or the property \( \text{Ext}^1_C(M, M'') = 0 \) for all factor modules \( M'' \) of \( M \). In this connection we have the following results.

**PROPOSITION 5.6.** The following statements are equivalent for a pair of modules \( M, N \) in \( \text{mod} \ A \).

(a) \( \text{Ext}^1_C(N', M) = 0 \) for all submodules \( N' \) of \( N \).
(b) \( \text{Hom}_A(\text{Tr} DM, N') = 0 \) for all submodules \( N' \) of \( N \).
(c) \( \text{Hom}_A(\text{Tr} DM, N) = 0 \).
Proof: (a) equivalent to (b). We know by [2, Chap. I, Proposition 5.4] that $\text{Ext}^1_A(Y, X) \cong D(\text{Hom}(\text{Tr} DX, Y))$ for all $X$ and $Y$ in $\text{mod} \, A$. From this the equivalence of (a) and (b) follows trivially.

(b) implies (c). Suppose $f: \text{Tr} \, DM \to N$ is a nonzero morphism with $N' = \text{Im} \, f$. Then the induced epimorphism $f': \text{Tr} \, DM \to N'$ does not factor through a projective. For let $P \to N'$ be a projective cover of $N'$ and $h: \text{Tr} \, DM \to P$ any morphism. Then since $\text{Tr} \, DM$ has no nontrivial projective summands, $h(\text{Tr} \, DM) \subseteq rP$ and so the composition $\text{Tr} \, DM \to h' \to h \to N'$ is not an epimorphism. Therefore the image of $f'$ in $\text{Hom}_A(\text{Tr} \, DM, N')$ is not zero. Hence if $\text{Hom}_A(\text{Tr} \, DM, N') = 0$ for all submodules $N'$ of $N$, then $\text{Hom}_A(\text{Tr} \, DM, N) = 0$.

(c) implies (b). Trivial.

As an immediate application of this proposition we have the following.

**Corollary 5.7.** The following statements are equivalent for a module $M$ in $\text{mod} \, A$.

(a) $\text{Ext}^1_A(M', M) = 0$ for all submodules $M'$ of $M$.

(b) $\text{Hom}_A(\text{Tr} \, DM, M') = 0$ for all submodules $M'$ of $M$.

(c) $\text{Hom}_A(\text{Tr} \, DM, M) = 0$.

For the sake of completeness we give the duals of Proposition 5.6 and Corollary 5.7.

**Proposition 5.8.** The following statements are equivalent for a pair of modules $M, N$ in $\text{mod} \, A$.

(a) $\text{Ext}^1_A(M, N'') = 0$ for all factor modules $N''$ of $N$.

(b) $\text{Hom}_A(N'', D \, \text{Tr} \, M) = 0$ for all factor modules $N''$ of $N$.

(c) $\text{Hom}_A(N, D \, \text{Tr} \, M) = 0$.

**Corollary 5.9.** The following statements are equivalent for a module $M$ in $\text{mod} \, A$.

(a) $\text{Ext}^1_A(M, M'') = 0$ for all factor modules $M''$ of $M$.

(b) $\text{Hom}_A(M'', D \, \text{Tr} \, M) = 0$ for all factors modules $M''$ of $M$.

(c) $\text{Hom}_A(M, D \, \text{Tr} \, M) = 0$.

Summarizing some of these results we obtain the following existence theorem for subcategories $C$ of $\text{mod} \, A$ which are closed under extensions and are functorially finite in $\text{mod} \, A$. 
THEOREM 5.10. Let $M$ and $N$ be in $\text{mod } A$.

(a) If $\text{Hom}_A(\text{Tr } DM, M) = 0$, then $\text{Sub } M$ is closed under extensions and is functorially finite in $\text{mod } A$.

(b) If $\text{Hom}_A(N, D \text{ Tr } N) = 0$, then $\text{Fac } N$ is closed under extensions and is functorially finite in $\text{mod } A$.

(c) If $\text{Hom}_A(\text{Tr } DM, M) = 0 = \text{Hom}_A(N, D \text{ Tr } N)$, then $C = \text{Sub } M \cap \text{Fac } N$ is closed under extensions and is functorially finite in $\text{mod } A$.

**Proof:** (a) and (b) Already established.

(c) The fact that $C$ is closed under extensions follows from (a) and (b). From the fact that $C = \text{Sub } M \cap \text{Fac } N$, it follows easily that if $f: C_1 \to C_2$ is a morphism in $C$ then $\text{Im } f$ is in $C$. Thus $C$ is closed under images. Therefore we know by [7, Propositions 4.5, 4.10] that $C$ is functorially finite in $\text{mod } A$ if $C$ has a finite cocover and a finite cover. But it is not difficult to see that $M$ has a unique submodule $M'$ maximal with respect to being in $C$ and that $\text{Ind } M'$ is a finite cocover for $C$. Similarly, it is not difficult to see that there is a unique submodule $N'$ of $N$ minimal with respect to $N/N'$ being in $C$ and that $\text{Ind } (N/N')$ is a finite cover for $C$.

6. EXAMPLES

This section is devoted to applying our previous results to obtain examples of subcategories $C$ of $\text{mod } A$ which have almost split sequences because they are closed under extensions and are functorially finite in $\text{mod } A$.

We begin with subcategories of $\text{mod } A$ which are hereditary torsion or cotorsion theories. Our first result was first obtained by Bautista and Martinez [8] in the case of the torsionless modules over 1-Gorenstein artin algebras and by Roggenkamp [9] for certain other cases.

Let $A$ be an arbitrary artin algebra. For each simple $A$-module $S$ we choose fixed projective covers $P(S)$ and injective envelopes $I(S)$. We also assume that we also have a fixed complete set $S_1, \ldots, S_n$ of nonisomorphic simple modules and $P_i = P(S_i)$ and $I_i = I(S_i)$.

Let $J$ be a subset of $\{1, \ldots, n\}$ and let $P_J$ denote $\bigcap_{i \in J} P_i$ and let $I_J$ denote $\bigcap_{i \in J} I_i$. $\text{Sub } I_J$ consists of all modules $M$ with the property that each simple submodule of $M$ is isomorphic to $S_j$ for some $j$ in $J$ and $\text{Fac } P_J$ consists of all modules $M$ with the property that each simple submodule of $M/\text{r } M$ is isomorphic to $S_j$ for some $j$ in $J$. Hence the subcategories of $\text{mod } A$ which are of the form $\text{Sub } I_J$ for some subset $J$ of $\{1, \ldots, n\}$ are precisely the hereditary torsion theories of $\text{mod } A$ and the subcategories $\text{Fac } P_J$ are precisely the hereditary cotorsion theories of $\text{mod } A$. 


**Proposition 6.1.** For each subset $J$ of $\{1, \ldots, n\}$ the subcategories $\text{Sub} I_J$ and $\text{Fac} P_J$ have almost split sequences since they are functorially finite in mod $A$ and are closed under extensions.

**Proof:** Since $\text{Tr} DI_J = 0$, we have $(\text{Tr} DI_J, I_J) = 0$, so by Theorem 5.10 $\text{Sub} I_J$ is closed under extensions and is functorially finite in mod $A$. Similarly, since $D \text{Tr} P_J = 0$, we have $(P_J, D \text{Tr} P_J) = 0$, so again by Theorem 5.10 $\text{Fac} P_J$ is closed under extensions and is functorially finite in mod $A$.

We next want to determine the Ext-projective and Ext-injective objects in the subcategories $\text{Sub} I_J$ and $\text{Fac} P_J$. This result is based on the following.

**Lemma 6.2.** Let $M$ be a $A$-module.

(a) Suppose $\text{Sub} M$ is closed under extensions. Then $X$ in $\text{Sub} M$ is Ext-injective in $\text{Sub} M$ if and only if $(\text{Tr} DX, M) = 0$.

(b) Suppose $\text{Fac} M$ is closed under extensions. Then $Y$ in $\text{Fac} M$ is Ext-projective if and only if $(M, D \text{Tr} Y) = 0$.

**Proof:** (a) By Proposition 5.6, we know that $\text{Ext}^1_A(M', X) = 0$ for all submodules $M'$ of $M$ if and only if $(\text{Tr} DX, M) = 0$. Hence if $X$ is Ext-injective in $\text{Sub} M$, then $(\text{Tr} DX, M) = 0$. On the other hand it is not hard to show that $\text{Ext}^1_A(C, X) = 0$ for all $C$ in $\text{Sub} M$ if $\text{Ext}^1_A(M', X) = 0$ for all submodules $M'$ of $M$ using the same argument as that given in proving Proposition 5.3. Hence if $(\text{Tr} DX, M) = 0$, then $X$ is Ext-injective in $\text{Sub} M$.

(b) Dual of (a).

**Proposition 6.3.** Let $J$ be a subset of $\{1, \ldots, n\}$, let $K$ be the subset of $\{1, \ldots, n\}$ consisting of all $i$ in $\{1, \ldots, n\}$ such that $(P_i, I_J) \neq 0$ and let $C = \text{Sub} I_J$. Then

(a) $K \supset J$.

(b) The set of all $P_k/t_{C} P_k$ with $k$ in $K$ is a complete set of nonisomorphic indecomposable Ext-projectives in $\text{Sub} I_J = C$.

(c) The set $\{I_J\}_{i \in J} \cup \{D \text{Tr}(P_i/\tau_{P_i}(P_i))\}_{i \in K-J}$ is a complete set of nonisomorphic indecomposable Ext-injective modules in $\text{Sub} I_J$.

(d) The number of isomorphism classes of indecomposable Ext-projective is the same as the number of isomorphism classes of indecomposable Ext-injective modules.

**Proof:** (a) Trivial.

(b) Follows from the general description of the Ext-projective objects in subcategories of the form $\text{Sub} M$. 

(c) Clearly each $I_j$ with $j \in J$ is Ext-injective in $\text{Sub } I_j$. Suppose $X$ is an indecomposable noninjective module. By Lemma 6.2 we know that an indecomposable $X$ is Ext-injective in $\text{Sub } I_j$ if and only if $(\text{Tr } DX, I_j) = 0$. Hence $X$ is Ext-injective if and only if $\text{Tr } DX$ has no simple composition factors $S_i$ with $i \in J$. Therefore we want to determine the indecomposable nonprojective modules $Y$ in $\text{mod } A$ which satisfy: (i) $Y$ has no composition factors in $\{S_j\}_{j \in J}$ and (ii) $D \text{Tr } Y$ is in $\text{Sub } I_j$ or equivalently, such that the composition factors of $\text{Soc } D \text{Tr } Y$ are in $\{S_j\}_{j \in J}$.

Suppose $P_i \rightarrow \mathcal{P}_0 \rightarrow Y \rightarrow 0$ is a minimal projective presentation of $Y$. Now it is easily seen that $\text{Soc } D \text{Tr } Y \simeq \bigsqcup_{j \in J} n_j S_j$ if and only if $P_i \simeq \bigsqcup_{j \in J} n_j P_j$. Also, $Y$ has no composition factors in $\{S_j\}_{j \in J}$ if and only if $\text{Im } g \supseteq \tau_{P_j}(\mathcal{P}_0)$. Therefore $Y$ satisfies (i) and (ii) above if and only if $\text{Im } g = \tau_{P_j}(\mathcal{P}_0)$.

Suppose $Y$ satisfies $\text{Im } g = \tau_{P_j}(\mathcal{P}_0)$. Since $\tau_{P_j}$ is an additive functor, $Y$ is indecomposable if and only if $\mathcal{P}_0$ is indecomposable, i.e., if and only if $Y \simeq P_i/\tau_{P_j}(P_i)$ with $i$ not in $J$. But $\tau_{P_j}(P_i) \neq 0$ if and only if $(P_i, I_j) \neq 0$. Therefore $Y$ is an indecomposable nonprojective module satisfying (i) and (ii) if and only if $Y \simeq P_i/\tau_{P_j}(P_i)$ with $i$ in $K - J$. Hence a noninjective indecomposable $X$ in $\mathcal{C}$ is Ext-injective in $\mathcal{C}$ if and only if $X \simeq D \text{Tr } (P_i/\tau_{P_j}(P_i))$ for some $i$ in $K - J$. This finishes the proof of (c).

(d) Follows trivially from (c).

For the sake of completeness we state without proof the following description of the indecomposable Ext-projectives and Ext-injectives in $\text{Fac } P_j$. This can be obtained by duality from Proposition 6.3.

**Proposition 6.4.** Let $J$ be a subset of $\{1, \ldots, n\}$, let $K$ be the subset of $\{1, \ldots, n\}$ consisting of all $i$ in $\{1, \ldots, n\}$ such that $(P_j, I_i) \neq 0$, let $\mathcal{C} = \text{Fac } P_j$, and let $A = \text{Sub } I_j$.

(a) $K \supset J$.

(b) The set of all $\tau_{\mathcal{C}}(I_k)$ with $k$ in $K$ is a complete set of nonisomorphic indecomposable Ext-injectives in $\text{Fac } P_j = \mathcal{C}$.

(c) The set $\{P_j\}_{j \in J} \cup \{\text{Tr } D(t_A I_i)\}_{i \in K - J}$ is a complete set of nonisomorphic Ext-projective modules in $\text{Fac } P_j$.

(d) The number of isomorphism classes of indecomposable Ext-projective is the same as the number of isomorphism classes of indecomposable Ext-injective modules.

As a consequence of Propositions 6.3 and 6.4, we have the following result which was also obtained independently by R. Bautista.

**Proposition 6.5.** Let $S_1, \ldots, S_n$ be a complete set of nonisomorphic simple $A$-modules and $J$ and $K$ subsets of $\{1, \ldots, n\}$. Then the subcategory $\mathcal{C}$
of mod $A$ consisting of all $M$ such that every simple submodule of $M$ is in $J$ and every simple submodule of $M/rM$ is in $K$ has almost split sequences.

Proof. Clearly $C = \text{Sub} I_J \cap \text{Fac} P_K$. Since $\text{Tr} DI_J = 0$, we have that $\text{Hom}_A(\text{Tr} DI_J, I_J) = 0$ and since $D Tr P_K = 0$, we have that $\text{Hom}(P_K, D Tr P_K) = 0$. Hence by Theorem 5.10, $C$ is closed under extensions and is functorially finite in mod $A$. Hence $C$ has almost split sequences.

We now give some examples involving preprojective and preinjective modules over hereditary artin algebras $A$.

**Proposition 6.6.** Suppose $A$ is a hereditary artin algebra. Let $M = \bigsqcup_{i=1}^t M_i$ and $N = \bigsqcup_{j=1}^s N_j$ where the $M_i$ and $N_j$ are indecomposable modules such that there are integers $m \geq 0$ and $n \geq 0$ with the property that $D Tr^m M_i$ is projective and $D Tr^n N_j$ is projective. Then the subcategories $\text{Sub} M$, $\text{Fac} M$, $\text{Sub} N \cap \text{Fac} M$ and $\text{Sub} M \cap \text{Fac} N$ of mod $A$ have almost split sequences.

Proof. The hypothesis on $M$ implies that $(M, D Tr M) = 0 = (\text{Tr} DM, M)$ and $(N, D Tr M) = 0 = (\text{Tr} DN, N)$ (see [3, Sect. 1]). The desired result now follows from Theorem 5.10.

For reasons similar to those used to establish Proposition 6.6 we have the following.

**Proposition 6.7.** Suppose $A$ is a hereditary artin algebra. Let $A = \bigsqcup_{i=1}^t A_i$ and $B = \bigsqcup_{j=1}^s B_j$ where $A_i$ and $B_j$ are indecomposable modules such that there are integers $m \geq 0$ and $n \geq 0$ such that $\text{Tr} D^m M_i$ and $\text{Tr} D^n N_j$ are injective. Then the subcategories $\text{Sub} M$, $\text{Fac} M$, $\text{Sub} N \cap \text{Fac} M$ and $\text{Sub} M \cap \text{Fac} N$ of mod $A$ have almost split sequences.

C. Riedtmann has informed us of another source of examples. Namely, she knows infinite families of selfinjective algebras of finite type $A$ such that $(\text{Tr} DM, M) = 0 = (M, D Tr M)$ for all indecomposable $A$-modules $M$.

As another example of modules $M$ satisfying $(\text{Tr} DM, M) = 0$ or $(M, D Tr M) = 0$, we have the following.

**Proposition 6.8.** Let $S$ be a simple $A$-module such that $\text{Ext}_A^1(S, S) = 0$. Then the subcategories $\text{Sub}(D Tr S)$, $\text{Fac}(D Tr S)$ and $\text{Sub}(D Tr S) \cap \text{Fac}(D Tr S)$ of mod $A$ have almost split sequences.

Proof. Let $0 \to rP \to P \to S \to 0$ be exact with $P \to S \to 0$ a projective cover. Then $\text{Ext}_A^1(S, S) = 0$ if and only if $S$ is not a submodule of $rP/r^2P$. Since $rP/r^2P \approx \text{Soc} D Tr S$ (see [5, Proposition 5.3]), the assumption $\text{Ext}_A^1(S, S) = 0$ implies $(S, D Tr S) = 0$. Because $\text{Tr} D(D Tr S) = S$, we have
that \((\text{Tr } D(D \text{ Tr } S), D \text{ Tr } S) = 0\). Hence \(\text{Sub}(D \text{ Tr } S)\) has almost split sequences. The dual argument shows that \(\text{Ext}^1_A(S, S) = 0\) implies \((\text{Tr } DS, S) = 0\). So \((\text{Tr } DS, D \text{ Tr } (\text{Tr } DS)) = 0\) and hence \(\text{Fac}(\text{Tr } DS)\) has almost split sequences as does \(\text{Sub}(D \text{ Tr } S) \cap \text{Fac}(\text{Tr } DS)\).

There are many examples of artin algebras \(A\) having simple modules \(S\) satisfying \(\text{Ext}^1(S, S) = 0\). We just cite one family of examples. Suppose \(A\) is an artin algebra of finite type with \(M_1, \ldots, M_n\) a complete set of nonisomorphic indecomposable \(A\)-modules. Then \(\Gamma = \text{End}_A(M_i)\) has the property that \(\text{Ext}^1(S, S) = 0\) for all simple \(\Gamma\)-modules \(S\).

We now point out another source of examples of subcategories of \(\text{mod } A\) having almost split sequences.

Let \(A\) be an arbitrary artin algebra. The subcategory \(\text{Sub } A\) consists of the submodules of projective \(A\)-modules. Then \(I_0(\text{Sub } A)\) consists of those indecomposable projective modules \(P\) with the property that if \(0 \to P \to T\) is exact with \(T\) in \(\text{Sub } A\), then \(0 \to P \to T\) is a splittable monomorphism. We say that a projective module is maximal if it is in \(I_0(\text{Sub } A)\). We know by Proposition 5.3 that \(\text{Sub } A\) is closed under extensions if and only if for each maximal projective \(P\) we have that \(\text{Ext}^1(C, P) = 0\) for all \(C\) in \(\text{Sub } A\). But \(\text{Ext}^1_A(C, P) = 0\) for all \(C\) in \(\text{Sub } A\) if and only if \(\text{Ext}^2(C, P) = 0\) for all \(X\) in \(\text{mod } A\), or equivalently, \(\text{inj dim } P \leq 1\). Therefore we have shown that \(\text{Sub } A\) is closed under extensions if and only if \(\text{inj dim } P \leq 1\) for all maximal projective modules \(P\). Hence we have proven

**Proposition 6.9.** Let \(A\) be an arbitrary artin algebra. Then \(\text{Sub } A\) has almost split sequences if and only if \(\text{inj dim } P \leq 1\) for all maximal projective modules \(P\).

As an easy consequence of this result we have the following due to Bautista and Martinez [8].

**Corollary 6.10.** Let \(A\) be a 1-Gorenstein artin algebra; i.e., \(\text{pd } I_0(A) = 0\) where \(I_0(A)\) is the injective envelope of \(A\). Then \(\text{Sub } A\) has almost split sequences.

**Proof.** Since \(I_0(A)\) is projective, \(I_0(\text{Sub } A) = \text{Ind } I_0(A)\). Therefore \(\text{inj dim } P = 0\) for all maximal projective modules \(P\). Our desired result now follows trivially from Proposition 6.9.

We now end this section by pointing out how, given a subcategory \(\text{Sub } M\) of \(\text{mod } A\) having almost split sequences, we can obtain other such subcategories. Suppose \(C = \text{Sub } M\) has almost split sequences or, what is the same thing, is closed under extensions. Then for each indecomposable injective module \(I\) in \(\text{mod } A\) we have the exact sequence \(0 \to A C \to I C \to I\) described in Section 4, where \(\bigcup_{I \in I_0} \text{Ind } A C\) consists of the indecomposable Ext-injective modules in \(C\) which are not part of a minimal cocover for \(C\).
Let $M_i$ be the sum of a complete set of nonisomorphic modules in $\bigcup_{i \in I_0} \text{Ind} A_i^1$. Then $\text{Ext}_A^1(C, M_i) = 0$ for all $C$ in $C$ since certainly $\text{Ext}_A^1(C, M) = 0$ for all $C$ in Sub $M_1 \subset$ Sub $M$. Hence Sub $M_1$ has almost split sequences. Proceeding by induction we obtain a sequence of subcategories with almost split sequences Sub $M_1 \supset$ Sub $M_1 \supset \cdots \supset$ Sub $M_i \supset \cdots$ naturally associated with the subcategory Sub $M$.

Finally it should be observed that $M_i = 0$ or equivalently $A_i^1 = 0$ for all $I$ in $I_0$ if and only if $\tau_C(I)$ is in $C$ for all $I$ in $I_0$ if and only if the indecomposable Ext-injective modules in $C$ are precisely the modules in $I_0(C)$.

A similar discussion can be carried out for the subcategories Fac $M$ with almost split sequences.

7. Further Examples

The examples of subcategories of mod $A$ having almost split sequences we gave in Section 6 were all of the form Sub $M$ or Fac $M$. In this section we give a different type of example, namely:

**Proposition 7.1.** Let $A$ be an arbitrary artin algebra and $C$ the subcategory of mod $A$ consisting of all modules $M$ such that $\text{Ext}_A^1(M, A) = 0$. Then $C$ is functorially finite in mod $A$ and has almost split sequences.

**Proof.** Since $C$ is clearly closed under extensions, if we show that $C$ is functorially finite in mod $A$ we will also have that $C$ has almost split sequences. Now for each $A^{op}$-module $N$ we have a natural exact sequence

$$0 \rightarrow \text{Ext}_A^1(\text{Tr} N, A) \rightarrow N \rightarrow N^{**}$$

where $N \rightarrow N^{**}$ is the usual morphism $N \rightarrow \text{Hom}_A(\text{Hom}_A(N, A), A)$ (see [1, Proposition 6.3]). Hence $N \rightarrow N^{**}$ is a monomorphism if and only if $\text{Ext}_A^1(\text{Tr} N, A) = 0$. Therefore $M$ is in $C$ if and only if $\text{Tr} M$ is in Sub $A^{op}$. Or equivalently $C$ is $\text{add}(\text{Tr}(\text{Sub} A^{op}) \cup \text{P}_0(A))$.

We now want to use this description combined with the fact that Sub $A^{op}$ is functorially finite in mod $A^{op}$, to show that $C$ is functorially finite in mod $A$. This is a trivial consequence of the following general considerations which are of interest in their own right.

We recall that a subcategory $C$ of mod $A$ is contravariantly (covariantly) finite in mod $A$ if and only if $\text{Ind} C$ is contravariantly (covariantly) finite in $\text{Ind}(\text{mod} A)$ when $\text{Ind} C$ being contravariantly (covariantly) finite in $\text{Ind}(\text{mod} A)$ means that for each $X$ in $\text{Ind}(\text{mod} A)$ there is a morphism $C' \rightarrow X$ ($X \rightarrow C'$) with $C'$ in $\text{add} \text{Ind} C$ such that $(C, C') \rightarrow (C, X) \rightarrow 0$ ($(C', C) \rightarrow (X, C) \rightarrow 0$) is exact for all $C$ in $\text{Ind} C$ (see [7, Sect. 3]). Hence to prove Proposition 7.1 it suffices to show $\text{Ind} C$ is functorially finite in $\text{Ind}(\text{mod} A)$. 


PROPOSITION 7.2. Let $\Lambda$ be an arbitrary artin algebra and $B$ a subcategory of $\text{Ind}(\text{mod} \Lambda)$.

(a) If $B$ is contravariantly finite in $\text{Ind}(\text{mod} \Lambda)$, then $\text{Tr} B \cup P_0(\Lambda^{op})$ is covariantly finite in $\text{Ind}(\text{mod} \Lambda^{op})$.

(b) If $B$ is covariantly finite in $\text{Ind}(\text{mod} \Lambda)$, then $\text{Tr} B \cup P_0(\Lambda^{op})$ is contravariantly finite in $\text{Ind}(\text{mod} \Lambda^{op})$.

Proof. (a) Let $X$ be in $\text{Ind}(\text{mod} \Lambda^{op})$. If $X$ is in $P_0(\Lambda^{op})$ there is nothing to prove. Suppose $X$ is not projective and let $Y = \text{Tr} X$. Since $B$ is contravariantly finite in $\text{Ind}(\text{mod} \Lambda)$, there is a morphism $f : B \to Y$ with $B$ in add $B$ such that $(B', B) \to (B', Y) \to 0$ is exact for all $B'$ in $B$. Because $\text{Tr} : \text{mod} \Lambda \to \text{mod} \Lambda^{op}$ is a duality, we have that $\text{Tr}(f) : \text{Tr} Y \to \text{Tr} B$ has the property that the induced morphism $\text{Hom}(\text{Tr} B, V) \to \text{Hom}(\text{Tr} Y, V) \to 0$ is exact for all $V$ in $\text{Tr} B$. Now we know that there is a morphism $\text{Tr} Y \to Q$ with $Q$ projective such that $(Q, M) \to (\text{Tr} Y, M) \to \text{Hom}(\text{Tr} Y, M) \to 0$ is exact for all $M$ in $\text{mod} \Lambda^{op}$. Namely, if $P \to (\text{Tr} Y)^* \to 0$ is a projective cover, then the composition $\text{Tr} Y \to (\text{Tr} Y)^* \to P^*$ is our desired morphism $\text{Tr} Y \to Q$. Hence the induced morphism $\text{Tr} Y \to \text{Tr} B \cup Q$ has the property that $(\text{Tr} B \cup Q, V) \to (\text{Tr} Y, V) \to 0$ is exact for all $V$ in $\text{mod} \Lambda^{op}$. Since $X - \text{Tr} Y$ was an arbitrary nonprojective module in $\text{Ind}(\text{mod} \Lambda^{op})$, we have shown that $\text{Tr} C \cup P_0(\Lambda^{op})$ is covariantly finite in $\text{Ind}(\text{mod} \Lambda)$.

(b) Proven similarly.

COROLLARY 7.3. Let $B$ be a subcategory of $\text{Ind}(\text{mod} \Lambda)$. Then the following are equivalent:

(a) $B$ is contravariantly (covariantly) finite in $\text{Ind}(\text{mod} \Lambda)$.

(b) $\text{Tr} B \cup P_0(\Lambda^{op})$ is covariantly (contravariantly) finite in $\text{Ind}(\text{mod} \Lambda)$.

(c) $\text{Tr} B$ is covariantly (contravariantly) finite in $\text{Ind}(\text{mod} \Lambda^{op})$.

(d) $D \text{Tr} B$ is contravariantly (covariantly) finite in $\text{Ind}(\text{mod} \Lambda)$.

Proof. (a) equivalent to (b). Shown in Proposition 7.2.

(b) equivalent to (c). We know by [7, Proposition 3.13] that since $P_0(\Lambda^{op})$ has only a finite number of nonisomorphic modules, $\text{Tr} C \cup P_0(\Lambda^{op})$ is covariantly (contravariantly) finite in $\text{Ind}(\text{mod} \Lambda^{op})$ if and only if $\text{Tr} C$ is covariantly (contravariantly) finite in $\text{Ind}(\text{mod} \Lambda^{op})$.

(c) equivalent to (d). Follows by duality.

Clearly Proposition 7.1 follows from Corollary 7.3.
REFERENCES


