JOURNAL OF DIFFERENTIAL EQUATIONS 79, 31–61 (1989)

A Krein Space Approach to Symmetric Ordinary Differential Operators with an Indefinite Weight Function

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Received January 5, 1988; revised January 28, 1988

0. INTRODUCTION

In this paper we consider spectral properties of the differential problem

$$l(f)=(-1)^n\,(p_0 f^{(n)})^{(n)}+(-1)^{n-1}(p_1 f^{(n-1)})^{(n-1)}+\cdots+p_n f=\lambda r f \tag{0.1}$$

on a finite or infinite interval (a, b) with real, locally summable coefficients $1/p_0, p_1, ..., p_n$, r under the assumptions that $p_0>0$ and that the weight function r changes its sign on (a, b). If r is positive, problem (0.1) can be studied in the context of Hermitian and self-adjoint operators in the Hilbert space $L^2(r)$ with the inner product

$$[f, g]=\int_a^b f\bar{g}r\,dx \tag{0.2}$$

which leads, e.g., to the definition of a (matrix) spectral function and expansion theorems. In our situation, when r is positive and negative on sets Δ_+ and Δ_-, resp., of positive Lebesque measure, the inner product (0.2) is indefinite, and the space $L^2(|r|)$ equipped with this inner product becomes a Krein space (denoted also by $L^2(r)$). However, independent of

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0022-0396/89 $3.00

505/79/1-3

sign of r, the operators which are usually related to (0.1) do still have the same symmetry properties with respect to the inner product (0.2). Thus, if r is indefinite, with the equation in (0.1) Hermitian and self-adjoint, operators in the Krein space $L^2(r)$ can be associated. We mention that, although the inner product (0.2) is indefinite, the topological structure of $L^2(r)$ is determined by the Hilbert norm

$$\|f\|^2=\left(\int_a^b |f|^2\,|r|\,dx\right)^{1/2}.$$

For an arbitrary self-adjoint operator in a Krein space the spectrum can be rather general (see [4, 19] and also Sect. 1 of this paper). Fortunately, the self-adjoint operators A in the Krein space $L^2(r)$, which arise in connection with (0.1) under some rather weak assumptions about the functons $p_1, ..., p_n$ and r, have a particularly nice property. Namely, they are definitizable; that is, the resolvent set of A is nonempty and for some polynominal p the relation $[p(A)f, f]\geqslant 0$ holds for all functons f in the domain of $p(A)$. This implies that these operators have some spectral function with, possibly, a finite number of singularities, called "critical points." Outside of these critical points the spectral theory of Eq. (0.1) has much in common with the spectral theory of problem (0.1) in the case $r>0$.

A particular question which arises for problem (0.1) if the weight function r is indefinite, is that of half-range completeness of a certain system of root functions. It turns out that this question can be studied in a natural way in the context of the Krein space $L^2(r)$. Namely, for the half-range completeness and corresponding expansion theorems it is important to know whether ∞ is a regular or a singular critical point of the associated definitizable operator in $L^2(r)$. This question and, in particular, a related preprint of R. Beals' paper [3], where a second-order problem is considered, were the starting points of our studies. Later it became clear that also other spectral properties of problem (0.1) with an indefinite weight function (see, e.g., [22, 23, 21, 24] where a survey of the regular second-order problem (0.1) is given) can be obtained in a simple way from results about definitizable operators in Krein spaces. In the particular case of a second-order Sturm–Liouville operator with a nonnegative potential this Krein space method was already used in [10]. There also, the so-called Weyl's coefficient and a spectral function of this indefinite problem were introduced. A partial extension of these results to higher order problems was given in [9].

The present paper is organized as follows. In Section 1 we repeat some definitions from the theory of Hermitian operators in Krein spaces. The basic result is Proposition 1.1, which gives a sufficient condition for the definitizability of the self-adjoint extensions of a Hermitian operator in a

Krein space. These extensions have definitizing polynomials of a special form (see (1.2)), which has consequences for their spectrum. Some of these consequences are formulated in no. 1.3. In Section 2 we introduce the differential expression $l(f)$ (to be understood in the sense of M. G. Krein's quasi-derivatives) and the operators associated with problem (0.1). Theorem 2.1 is more or less a reformulation of Proposition 1.1 for the minimal operator of (0.1). In the following no. 2.2 we formulate assumptions about $p_o, ..., p_n, r$ which assure that the conditions of Theorem 2.1 are satisfied. Then, obviously, the spectral properties, which were established in no. 1.3 for the self-adjoint extensions A of A_0, hold true for the self-adjoint extensions of the minimal operator $A_0 = A_{\min}$ associated with (0.1). We do not formulate these properties explicitly as this would be just a repetition of the formulations of Section 1. We mention, however, that they generalize some statements which were proved in the second-order case in [22, 23]. Some more special spectral properties are contained in Propositions 2.9 and 2.10. In Section 3 we show that the critical point of the associated definitizable operators is not singular if r satisfies some regularity condition at its turning points. The construction of the operator X in Lemma 3.2 is inspired by corresponding results of Beals [3] in the second-order case. Finally, in the last section we show that if the spectrum of A is discrete and ∞ is not a singular critical point, full- and half-range expansions hold. For the second-order case under stronger assumptions about the differential operator results of this kind were proved in [3] and for more special cases in [17].

Some of the results of this paper were stated without proofs in [8]. As we have mentioned already, the definitizability of the self-adjoint extension A of A_{min} implies the existence of a projection or matrix spectral function with, possibly, a finite number of singularities. These questions, which are closely related to expansions of Green's kernel and of elements of $L^2(r)$ in root functions of A, will be considered elsewhere. Also, in this paper we suppose that the weight function r is different from zero a.e. on (a, b). This condition can be weakened which, however, makes the definition of the operators more complicated (cf. [6, 10, 12]).

1. A Class of Definitizable Operators in Krein Spaces

1.1. In this section we collect some definitions and statements from the theory of linear operators in a Krein space which will be used in this paper. The reader can find more details in [4, 2, 20]. A linear space $\mathscr{K}$, equipped with an inner product $[\cdot, \cdot]$ is called a *Krein space* if there exists a decomposition

$$\mathscr{K} = \mathscr{K}_+ [\dot{+}] \mathscr{K}_- \tag{1.1}$$

such that $(\mathscr{K}_{\pm}, \pm[\cdot,\cdot])$ are Hilbert spaces and $[\mathscr{K}_+, \mathscr{K}_-] = \{0\}$. The decomposition (1.1) defines projections $P_{\pm}$: If $f = f_+ + f_-$, $f_{\pm} \in \mathscr{K}_{\pm}$, is the representation of $f \in \mathscr{K}$ according to (1.1) we put $P_{\pm} f := f_{\pm}$. Then, with the operator $J := P_+ - P_-$, a Hilbert inner product $(\cdot,\cdot)$ on $\mathscr{K}$ can be introduced as follows:

$$(f, g) := [Jf, g] \qquad (f, g \in \mathscr{K}).$$

The operator J is called a *fundamental symmetry* of the Krein space $\mathscr{K}$. All the topological notions in $\mathscr{K}$ are to be understood with respect to the topology of this Hilbert inner product, if not otherwise stated explicitly.

The linear opeator A in the Krein space $(\mathscr{K}, [\cdot,\cdot])$ is called *Hermitian* if its domain $\mathscr{D}(A)$ is dense in $\mathscr{K}$ and $[Af, f]$ is real for all $f \in \mathscr{D}(A)$. This is equivalent to $A \subset A^+$, where A^+ denotes the Krein space adjoint of A defined on the set of all $g \in \mathscr{K}$ such that $f \mapsto [Af, g]$ is a continuous linear functional on $\mathscr{D}(A)$ by the relation

$$[Af, g] = [f, A^+ g] \qquad (f \in \mathscr{D}(A)).$$

It is easy to see that A is a Hermitian (in the Krein space $(\mathscr{K}, [\cdot,\cdot])$) if and only if the operator $B := JA$ or $B_1 := AJ$ is Hermitian in the Hilbert space $(\mathscr{K}, (\cdot,\cdot))$. The densely defined operator A in $(\mathscr{K}, [\cdot,\cdot])$ is called *self-adjoint* if $A = A^+$ or, equivalently, if the operator $B := JA$ is self-adjoint in the Hilbert space $(\mathscr{K}, (\cdot,\cdot))$. The self-adjoint operator A in $(\mathscr{K}, [\cdot,\cdot])$ is said to be *definitizable* if $\rho(A) \neq \varnothing$ and there exists a polynomial p such that

$$[p(A)f, f] \geqslant 0 \qquad \text{for all } f \in \mathscr{D}(A^k),$$

where k is the degree of p. Recall that a definitizable operator admits a spectral function with, possibly, some critical points (see [20, 2, 4] and also no. 2 below).

We say that the closed Hermitian operator A in the Krein space $(\mathscr{K}, [\cdot,\cdot])$ has defect m $(\leqslant +\infty)$, if there exists a self-adjoint extension $\tilde{A} \supset A$ in $(\mathscr{K}, [\cdot,\cdot])$ such that $m = \dim \mathscr{D}(\tilde{A})/\mathscr{D}(A)$. This is equivalent to the fact that the operator $\tilde{B} := J\tilde{A}$ is a self-adjoint extension of the Hermitian operator $B := JA$ in the Hilbert space $(\mathscr{K}, (\cdot,\cdot))$; that is, the operator B has equal defect numbers $m_+ = m_- = m$, or its defect index is (m, m).

Recall that an inner product on a linear space $\mathscr{L}$ is said to have a finite number κ of negative squares if it is negative definite on a κ-dimensional subspace of $\mathscr{L}$ and there exists no $(\kappa + 1)$-dimensional subspace with this property.

1.2. The definitizable operators we shall study in this paper arise as in the following proposition.

PROPOSITION 1.1. *Let A_0 be a closed Hermitian operator in the Krein space $(\mathscr{K}, [\cdot, \cdot])$ with the properties*:

(a$_0$) *A_0 has finite defect m_0.*

(a$_1$) *The Hermitian form $[A_0 f, g]$ $(f, g \in \mathscr{D}(A_0))$ has a finite number $(0 \leqslant \kappa_0 < +\infty)$ of negative squares.*

(a$_2$) *A_0 has a self-adjoint extension A_1 in the Krein space $(\mathscr{K}, [\cdot, \cdot])$ such that $\rho(A_1) \neq \varnothing$.*

Then each self-adjoint extension A of A_0 in $(\mathscr{K}, [\cdot, \cdot])$ is definitizable.

Proof. According to (a$_2$) there exists an open set $\Delta \subset \rho(A_1)$, $\Delta \neq \varnothing$, which we can assume to be symmetric with respect to the real axis. For each $z \in \Delta$, the range $\mathscr{R}(A_1 - zI)$ is the whole space $\mathscr{K}$, hence closed, and from (a$_0$) it follows that also the ranges $\mathscr{R}(A_0 - zI)$, $z \in \Delta$, are closed. Now let A be an arbitrary self-adjoint extension of A_0 in $(\mathscr{K}, [\cdot, \cdot])$. Then its ranges $\mathscr{R}(A - zI)$, $z \in \Delta$, are closed too, hence $\Delta \subset \rho(A) \cup \sigma_p(A) \cup \sigma_r(A)$ (for the definitions of these subsets of the spectrum of A see, e.g., [11]). Moreover, by (a$_0$) and (a$_1$) the Hermitian form $[Af, g]$ $(f, g \in \mathscr{D}(A))$ has a finite number κ_A $(\kappa_0 \leqslant \kappa_A \leqslant \kappa_0 + m_0)$ of negative squares. Therefore the statement follows from [20, I.3(c)] if we only show that $\rho(A) \neq \varnothing$.

Assume $\rho(A) = \varnothing$. Then, as $z \in \sigma_r(A)$ implies $\bar{z} \in \sigma_p(A)$, in at least one of the half planes an infinite number of points of Δ belongs to $\sigma_p(A)$. Consider $n := m_0 + \kappa_0 + 1$ such (mutually different) eigenvalues with eigenvectors $f_1, ..., f_n$. These are neutral and mutually orthogonal in the Krein space $(\mathscr{K}, [\cdot, \cdot])$. Moreover, there exist elements $g_1, ..., g_n$ in $\mathscr{D}(A)$ such that $[Af_i, g_j] = \delta_{ij}$, $i, j = 1, 2, ..., n$. Indeed, the vectors $Af_1, ..., Af_n$ and hence also $JAf_1, ..., JAf_n$ are linearly independent. We choose a system $g'_1, ..., g'_n$ in $\mathscr{K}$ such that

$$([Af_i, g'_j] =) \qquad (JAf_i, g'_j) = \delta_{ij}, \qquad i, j = 1, 2, ..., n.$$

As $\mathscr{D}(A)$ is dense, the elements $g'_1, ..., g'_n$ can be changed slightly to elements $g''_1, ..., g''_n \in \mathscr{D}(A)$ such that still $\det([Af_i; g''_j])_1^n \neq 0$. Now $g_1, ..., g_n$ are easily obtained as linear combinations of the $g''_1, ..., g''_n$. On the $2n$-dimensional space spanned by $f_1, ..., f_n, g_1, ..., g_n$, the inner product $[A\ \cdot, \cdot]$ is given by the Gram matrix

$$\begin{pmatrix} O & I \\ I & \cdot \end{pmatrix}$$

with $n \times n$-blocks. Hence it is nondegenerated. As it contains an n-dimensional neutral subspace it also contains an n-dimensional negative subspace, which contradicts the fact that the form $[Af, g]$ has $< n$ negative squares. Thus $\rho(A) \neq \varnothing$, and the proposition is proved.

Remark 1.2. The condition (a_1) is obviously equivalent to the condition that for the closed Hermitian operator $B_0 = JA_0$ in the Hilbert space $(\mathscr{K}, (\cdot, \cdot))$, the Hermitian form $(B_0 f, g)$ $(f, g \in \mathscr{D}(B_0))$ has finite number κ_0 of negative squares. This is the case if and only if the operator B_0 has a self-adjoint extension B_1 in the Hilbert space $(\mathscr{K}, (\cdot, \cdot))$ such that $(-\infty, 0) \cap \sigma(B_1)$ consists of a finite number of eigenvalues of finite multiplicities. Similarly, κ_A which was defined as the number of negative squares of $[Af, g]$ $(f, g \in \mathscr{D}(A))$, coincides with the number of negative squares of the form (Bf, g) $(f, g \in \mathscr{D}(B) = \mathscr{D}(A))$ with $B := JA$, which is the total multiplicity of the negative eigenvalues of B.

Remark 1.3. In the proof of Proposition 1.1 we have actually shown that under conditions (a_0) and (a_1) the following statements are equivalent to (a_2):

(i) For some self-adjoint extension A_1 of A_0 and some $\lambda_0 \in \mathbb{C}$ the range $\mathscr{R}(A_1 - \lambda_0 I)$ is closed.

(ii) For some $\lambda_0 \in \mathbb{C}$ the range $\mathscr{R}(A_1 - \lambda_0 I)$ is closed.

(iii) For all self-adjoint extensions A of A_0 in the Krein space $(\mathscr{K}, [\cdot, \cdot])$ we have $\rho(A) \neq \varnothing$.

COROLLARY 1.4. *The conditions* (a_0), (a_1), (a_2) of *Proposition* 1.1 *are satisfied for the operator A_0 if the operator $B_0 = JA_0$ has a finite defect index and a self-adjoint extension B_1 in $(\mathscr{K}, (\cdot, \cdot))$ such that for some $\varepsilon > 0$ the set $(-\infty, \varepsilon) \cap \sigma(B_1)$ consists of a finite number of eigenvalues of finite multiplicities. If, in addition, the spectrum of B_1 is discrete then the spectrum of each self-adjoint extension A of A_0 in $(\mathscr{K}, [\cdot, \cdot])$ is discrete.*

Proof. To prove the first statement we observe that $0 \in \rho(B_1)$ or 0 is an isolated eigenvalue of $\sigma(B_1)$; hence, $\mathscr{R}(B_1) = (\ker B_1)^{(\perp)}$ where $(\perp)$ denotes the orthogonal complement in $(\mathscr{K}, (\cdot, \cdot))$. Therefore $\mathscr{R}(B)$, and also $\mathscr{R}(A) = J\mathscr{R}(B)$ are closed. Now the statement follows from Remark 1.3 (put $\lambda_0 = 0$). If the spectrum of B_1 is discrete, then B_0 has a self-adjoint extension B_2 in $(\mathscr{K}, (\cdot, \cdot))$ with discrete spectrum and $0 \in \rho(B_2)$. The operator B_2^{-1} is compact. Consequently, $B_2^{-1} J$ is a compact operator and the spectrum of $A_2 = JB_2$ is discrete. The operator A_2 is a self-adjoint extension of A_0 in $(\mathscr{K}, [\cdot, \cdot])$. Since for an arbitrary self-adjoint extension A of A_0 in $(\mathscr{K}, [\cdot, \cdot])$ we have $\rho(A) \neq \varnothing$, it follows that the spectrum of A is discrete.

We mention that for the equivalence of (i), (ii), and (iii) in Remark 1.3 the condition (a_1) is essential; that is, (a_0) and (a_2) do not necessarily imply that $\rho(A) \neq \varnothing$ holds for all self-adjoint extensions A and A_0. In other words, there exist closed Hermitian operators A_0 such that (a_0) and

(a_2) hold and A_0 has a self-adjoint extension A whose point spectrum covers the whole complex plane and a self-adjoint extension A_1 with $\rho(A_1) \neq \varnothing$. An example of a differential operator A_0 with those properties was given by F. V. Atkinson and A. B. Mingarelli [1] and also by the first author of this note (unpublished). An abstract version of this example is as follows:

Let C be a closed Hermitian operator in a Hilbert space $(\mathscr{H}, (\cdot, \cdot))$ with defect index $(1, 1)$ and such that the whole complex plane $\mathbb{C}$ consists of points of regular type. Then $\sigma_p(C^*) = \mathbb{C}$. Consider the operator

$$A_0 = \begin{pmatrix} C & O \\ O & C \end{pmatrix}$$

in the Krein space $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$ with indefinite inner product $[\cdot, \cdot]$ given by the fundamental symmetry

$$J := \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

Then A_0 is a closed Hermitian operator in $\mathscr{H} \oplus \mathscr{H}$ and also in $(\mathscr{K}, [\cdot, \cdot])$. As $\dim \mathscr{D}(C^*)/\mathscr{D}(C) = 2$ we can choose two elements $e_1, e_2 \in \mathscr{D}(C^*)$ which are linearly independent with respect to $\mathscr{D}(C)$. Define

$$\mathscr{D} := \left\{ \alpha \begin{pmatrix} e_1 \\ e_1 \end{pmatrix} + \beta \begin{pmatrix} e_2 \\ e_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : \alpha, \beta \in \mathbb{C}, u_1, u_2 \in \mathscr{D}(C) \right\}$$

and $B := JA_0^* \mid \mathscr{D}$. Then it is not hard to check that B is Hermitian in $\mathscr{H} \oplus \mathscr{H}$ and considering the dimension of the defect subspaces, it follows that it is even self-adjoint. Hence $A := JB$ is self-adjoint in the Krein space $(\mathscr{K}, [\cdot, \cdot])$. If $\lambda \in \mathbb{C}$ and $f \in \mathscr{D}(C^*)$ such that $C^* f = \lambda f$, then $\mathbf{f} = (f, f)^T \in \mathscr{D}$ and $A\mathbf{f} = \lambda \mathbf{f}$; hence the whole complex plane belongs to $\sigma_p(A)$. It is not hard to see that there also exists a self-adjoint extension A_1 of A_0 in $(\mathscr{K}, [\cdot, \cdot])$ such that $\rho(A_1) \neq \varnothing$.

Recall that for the closed Hermitian operator B_0 which is bounded from below in the Hilbert space $(\mathscr{K}, (\cdot, \cdot))$ the set $\mathscr{D}[B_0]$ is defined as a set of all $f \in \mathscr{K}$ for which there exists a sequence $(\varphi_n) \subset \mathscr{D}(B_0)$ such that $\varphi_n \to f$ in $\mathscr{K}$ and $(B_0(\varphi_n - \varphi_m), \varphi_n - \varphi_m) \to 0$ $(n, m \to +\infty)$. Then $\mathscr{D}[B_0]$ is the domain of the closure of the Hermitian form $(B_0 f, g)$ $(f, g \in \mathscr{D}(B_0))$. Denote this closure by $B_0(\cdot, \cdot)$. It is easy to see that the Hermitian form $(B_0 \cdot, \cdot)$ has a finite number κ_0 of negative squares on $\mathscr{D}(B_0)$ if and only if the Hermitian form $B_0(\cdot, \cdot)$ has κ_0 negative squares on $\mathscr{D}[B_0]$. For the Friedrichs extension B_{F} of B_0 in $(\mathscr{K}, (\cdot, \cdot))$ we have $\mathscr{D}[B_{\mathrm{F}}] = \mathscr{D}[B_0]$ and $B_{\mathrm{F}}(f, g) = B_0(f, g)$ $(f, g \in \mathscr{D}[B_0])$. Consequently $\kappa_{B_{\mathrm{F}}} = \kappa_0$, where $\kappa_{B_{\mathrm{F}}}$ is the number of negative squares of the Hermitian form $(B_{\mathrm{F}} f, g)$ $(f, g \in \mathscr{D}(B_{\mathrm{F}}))$. More information about the number of negative squares of the Hermitian form (Bf, g) $(f, g \in \mathscr{D}(B))$ for an arbitrary self-adjoint extension B of B_0 in $(\mathscr{H}, (\cdot, \cdot))$ in the case when B_0 has a positive lower bound can be found in [18].

1.3. In this section we prove some spectral properties of a definitizable operator A for which the form $[Af, g]$ $(f, g \in \mathscr{D}(A))$ has a finite number of negative squares. First let A be an arbitrary definitizable operator. The spectral function of A is denoted by E; for its definition and properties we refer the reader to [20, Theorem 3.1]. Here we repeat for the convenience of the reader the definitions of positive and negative type spectrum and of the critical points of A.

If $\mathscr{L}$ is a linear space with an inner product $[\cdot, \cdot]$, $\kappa_+(\mathscr{L}; [\cdot, \cdot])$ $(\kappa_-(\mathscr{L}; [\cdot, \cdot]))$, denotes the least upper bound $(\leqslant +\infty)$ of the dimensions of the positive (negative, resp.) subspaces of $\mathscr{L}$. Instead of $\kappa_\pm(\mathscr{L}; [\cdot, \cdot])$ we often write $\kappa_\pm(\mathscr{L})$. Now, if $\lambda \in \mathbb{R} \cup \{\infty\}$ we define $\kappa_\pm(\lambda; A)$ as the minimum $(\leqslant +\infty)$ of the numbers $\kappa_\pm(E(\Delta)\mathscr{H})$ where Δ runs through all neighborhoods of λ such that $E(\Delta)$ is defined, and we put

$$\kappa(\lambda; A) := \min\{\kappa_+(\lambda; A), \kappa_-(\lambda; A)\}.$$

If $\kappa(\lambda; A) = 0$ and $\kappa_+(\lambda; A) > 0$ $(\kappa_-(\lambda; A) > 0)$ then λ is said to be a *spectral point of positive* (*negative*, resp.) *type*; if $\kappa(\lambda; A) > 0$ then λ is said to be a *critical point* of A and $\kappa(\lambda; A)$ is called the *rank of indefiniteness* of λ. The set of the finite critical points of A is denoted by $c(A)$; $\tilde{c}(A) := c(A) \cup \{\infty\}$ if ∞ is a critical point of A, too, and, finally, $\tilde{c}_\infty(A)$ is the set of all $\lambda \in \tilde{c}(A)$ with $\kappa(\lambda; A) = +\infty$.

If a spectral point $\lambda \in \mathbb{R}$ of positive (negative) type is an eigenvalue of A then all its corresponding root vectors f have the property $[f, f] > 0$ $(< 0$, resp.). If $\kappa(\lambda; A)$ is finite and positive, then λ is an eigenvlue of A. In this case, if, e.g., $\kappa_-(\lambda; A)$ is finite, the root subspace $\mathscr{S}_\lambda(A)$ contains a $\kappa_-(\lambda; A)$-dimensional nonpositive subspace, and each Jordan chain of A in $\mathscr{S}_\lambda(A)$ is of length $\leqslant 2\kappa_-(\lambda; A) + 1$. Thus, if one of the numbers $\kappa_\pm(\lambda; A)$ is positive and finite, it can be calculated from the signature of $\mathscr{S}_\lambda(A)$ with respect to the inner product $[\cdot, \cdot]$.

If $\lambda \in \mathbb{R} \cup \{\infty\}$ is a critical point of A, it is called a *regular critical point* if $\sup \|E(\Delta)\| < +\infty$ where the supremum runs over all sufficiently small neighborhoods Δ of λ. If the critical point λ is not regular, it is called a *singular critical point*; the set of singular critical points of A is denoted by $c_s(A)$.

If p is a definitizing polynomial of minimal degree of the definitizable operator A, then the nonreal spectrum of A consists of the zeros of p. It is symmetric with respect to the real axis and the linear span of all root subspaces corresponding to $\lambda \in \mathbb{C}^+ \cap \sigma(A)$ is neutral (see [20, 2] where also other spectral properties of definitizable operators can be found).

Now suppose additionally that $[Af, g]$ $(f, g \in \mathscr{D}(A))$ has a finite number κ_A of negative squares. Then it has a definitizing polynomial p of the form

$$p(z) = zq_A(z)\bar{q}_A(z) \tag{1.2}$$

with a polynomial q_A, which can be chosen monic (that is, the coefficient of the highest power of z is one) and of minimal degree $\leqslant \kappa_A$. Then q_A is uniquely determined. In this case, a real number $\lambda \neq 0$ is a zero of q_A if and only if it is a critical point (and also an eigenvalue of A such that $\lambda[f, f] \leqslant 0$ for some corresponding eigenvector f), or if it is an eigenvalue of A with the property $\lambda[f, f] < 0$ for each corresponding eigenvector f. Also, $q_A(0) = 0$ implies that 0 is an eigenvalue of A with a corresponding Jordan chain of length $\geqslant 2$ and, consequently, a critical point of A. Moreover, we have $\tilde{c}_\infty(A) \subset \{0, \infty\}$. These facts follow easily from [20].

Now consider the root subspace $\mathscr{S}_0(A)$ equipped with the inner product $[A\cdot, \cdot]$; if $\lambda = 0$ is not an eigenvalue, then we put $\mathscr{S}_0(A) = \{0\}$. The dimension of the isotropic subspace of $\mathscr{S}_0(A)$ with respect to $[A\cdot, \cdot]$ is denoted by $\kappa_A^0(\leqslant +\infty)$.

PROPOSITION 1.5. *Let A be a self-adjoint operator in the Krein space $(\mathscr{K}, [\cdot, \cdot])$ with $\rho(A) \neq \varnothing$ and the following properties*:

(a) *The form $[Af, g]$ $(f, g \in \mathscr{D}(A))$ has a finite number κ_A of negative squares.*

(b) $\ker A$ *is of finite dimension.*

Then $(0 \leqslant)$ $\dim \mathscr{S}_0(A) < +\infty$ and in $\mathscr{D}(A)$ there exists a subspace $\mathscr{L}_0$ with $\dim \mathscr{L}_0 = \kappa_A + \kappa_A^0$ which is invariant under A and such that $\kappa_+(\mathscr{L}_0; [A\cdot, \cdot]) = 0$ (that is, $\mathscr{L}_0$ is $[A\cdot, \cdot]$-nonpositive). It can be chosen such that $\operatorname{Im} \sigma(A \mid \mathscr{L}_0) \geqslant 0$; then $\mathscr{L}_0$ contains all root subspaces of A corresponding to eigenvalues in the upper half plane $\mathbb{C}^+$. Moreover, $\sigma(A \mid \mathscr{L}_0) \cap (\mathbb{R} \setminus \{0\})$ consists of all real eigenvalues λ such that there exists an eigenvector f with $\lambda[f, f] \leqslant 0$, and $\sigma(A \mid \mathscr{L}_0)$ coincides—with the possible exception of 0—with the set of zeros of q_A.

Proof. As in the proof of Proposition 1.1 [20, I.3.(c)] implies that A is definitizable and according to [20, I.3.(b)] it has a definitizing polynomial p of the form (1.2). Therefore, by [20, Proposition II.2.1], all the Jordan chains corresponding to the possible eigenvalue zero are of finite length, and (b) implies $\dim \mathscr{S}_0(A) < +\infty$. We choose a set $\Delta := \{z \in \mathbb{C} : 0 < \varepsilon_1 < |z| < \varepsilon_2\}$ for some $\varepsilon_1, \varepsilon_2 > 0$, containing all the zeros $\lambda \neq 0$ of q_A, and denote by $E(\Delta)$ the corresponding spectral projection of A. Then $A_\Delta := A \mid E(\Delta)\mathscr{K}$ is a bounded and boundedly invertible self-adjoint operator in the Krein space $(E(\Delta)\mathscr{K}, [\cdot, \cdot])$. Also, $E(\Delta)\mathscr{K}$ equipped with the inner product $[Af, g]$ $(f, g \in E(\Delta)\mathscr{K})$, is a $\pi_{\kappa'_A}$-space, $0 \leqslant \kappa'_A \leqslant \kappa_A$, and the difference $\kappa''_A := \kappa_A - \kappa'_A$ is the number of negative squares of the inner product $[A\,\cdot, \cdot]$ on $E([-\varepsilon_1, \varepsilon_1])\mathscr{K}$. Then $\mathscr{S}_0(A) \subset E([-\varepsilon_1, \varepsilon_1])\mathscr{K}$, the inner product $[A\,\cdot, \cdot]$ has κ''_A negative squares on $\mathscr{S}_0(A)$, and it degenerates on a κ^0_A-dimensional subspace of $\mathscr{S}_0(A)$. We consider a decomposition of $\mathscr{S}_0(A)$: $\mathscr{S}_0(A) = \mathscr{L}_{00} \dotplus \mathscr{L}_1$ where $\mathscr{L}_{00}$ is the $[A\,\cdot, \cdot\,]$-isotropic part of $\mathscr{S}_0(A)$ and $\mathscr{L}_1$ is, hence, a finite-dimensional with nondegenerate inner product $[A\,\cdot, \cdot]$ having κ''_A negative squares there. Consider a corresponding matrix representation of $A \mid \mathscr{S}_0(A)$:

$$\begin{pmatrix} A_0 & A_{01} \\ 0 & A_1 \end{pmatrix}.$$

Then $\mathscr{L}_1$ contains a κ''_A-dimensional subspace $\mathscr{L}_{0,-}$ which is invariant under A_1 and such that the inner product $[A_1\,\cdot, \cdot]$ is nonpositive on $\mathscr{L}_{0,-}$. Therefore the subspace $\mathscr{L}_{00} \dotplus \mathscr{L}_{0,-}$ is of dimension $\kappa''_A + \kappa^0_A$, $[A\,\cdot, \cdot]$-nonpositive, and invariant under A. As A_Δ has also a κ'_A-dimensional $[A\,\cdot, \cdot]$-nonpositive invariant subspace $\mathscr{L}_-$, we can put $\mathscr{L}_0 = \mathscr{L}_{00} \dotplus \mathscr{L}_{0,-} \dotplus \mathscr{L}_-$. The special choice of $\mathscr{L}_0$ as indicated in the proposition is possible because $\mathscr{L}_-$ can be chosen in this way (see [14, Theorem 12.1]). The simple proofs of the last statements are left to the reader.

COROLLARY 1.6. *The operator A has at least κ_A eigenvalues (counted according to their algebraic multiplicities) in the closed upper half-plane $\mathbb{C}^+ \cup \mathbb{R}$ with the following property: If $\lambda \neq 0$ then there exists a corresponding eigenfunction f of A such that $\lambda[f, f] \leqslant 0$.*

If $\lambda < 0$ belongs to $\sigma(A \mid \mathscr{L}_0)$, then $\kappa_+(\lambda; A)$ coincides with the number of negative squares of the form $[AE(\Delta)f, g]$ $(f, g \in E(\Delta)\mathscr{K})$, where Δ is a small negative interval around λ such that it contains besides λ no other zeros of q_A. This follows easily from the fact that the operator $-A_\Delta := -A \mid E(\Delta)\mathscr{K}$ in $E(\Delta)\mathscr{K}$ has a self-adjoint square root $(-A_\Delta)^{1/2}$ in the Krein space $E(\Delta)\mathscr{K}$ and from the relation

$$[AE(\Delta)f, g] = -[E(\Delta)(-A_\Delta)^{1/2}f, (-A_\Delta)^{1/2}g] \qquad (f, g \in E(\Delta)\mathscr{K}).$$

Similarly, if $\lambda > 0$, $\lambda \in \sigma(A \mid \mathscr{L}_0)$, then $\kappa_-(\lambda; A)$ equals the number of negative squares of the form $[AE(\Delta)f, g]$ $(f, g \in (\Delta)\mathscr{K})$, where Δ is now a positive small interval around λ. These observations immediately give the following

COROLLARY 1.7. *We have*

$$\sum_{\lambda < 0} \kappa_+(\lambda; A) + \sum_{\lambda > 0} \kappa_-(\lambda; A) + \sum_{\substack{\lambda \in \sigma(A) \\ \operatorname{Im}\lambda > 0}} \dim \mathscr{S}_\lambda(A) \leqslant \kappa_A \qquad (1.3)$$

with equality if and only if $q(0) \neq 0$. *In particular, the equality in* (1.3) *holds if* $0 \notin \sigma_p(A)$.

PROPOSITION 1.8. *Let* A *be as in Proposition* 1.4 *and suppose additionally that* $\kappa_+(\mathscr{K}) = \kappa_-(\mathscr{K}) = +\infty$. *Then* A *has positive and negative spectrum, both of infinite multiplicities; that is, on each half axis* $(-\infty, 0)$ *and* $(0, +\infty)$ *there are infinitely many eigenvalues or points of continuous spectrum of* A. *Moreover, if* $q(\lambda) \neq 0$ *and* $\lambda \in \sigma(A) \cap (0, +\infty)$ ($\lambda \in \sigma(A) \cap (-\infty, 0)$) *then* λ *is a spectral point of positive (negative, resp.) type of* A.

Proof. The last statement follows from the fact that the definitizing polynomial p of A is nonnegative on $(0, +\infty)$ and nonpositive on $(-\infty, 0)$. If, e.g., the negative spectrum of A would consist of a finite number of eigenvalues with finite multiplicities, for the linear span $\mathscr{L}_-$ of the corresponding root subspaces, we would have $\kappa_-(\mathscr{L}_-; [\cdot, \cdot]) < +\infty$. As zero is not an eigenvalue or is an eigenvalue of finite algebraic multiplicity of A, for a small interval Δ around zero, on the subspace $E(\Delta)\mathscr{K}$ the inner product $[\cdot, \cdot]$ will have a finite number of negative squares. If we choose a large positive interval Δ_+ which contains all positive zeros of q, then the number of negative squares of $[\cdot, \cdot]$ on $E(\Delta_+)\mathscr{K}$ is given by the second term on the left-hand side in (1.3). Finally, as the total multiplicity of the nonreal eigenvalues is also finite, the assumption about the finite multiplicity of the negative spectrum implies that $\mathscr{K}$ itself has the property $\kappa_-(\mathscr{K}) < +\infty$, a contradiction. The proposition is proved.

2. THE DIFFERENTIAL OPERATORS

2.1. We consider the formal differential expression of order $2n$ on the interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, given by

$$l(f) := (-1)^n (p_0 f^{(n)})^{(n)} + (-1)^{n-1} (p_1 f^{(n-1)})^{(n-1)} + \cdots + p_n f,$$
$$f^{(j)} := d^j f/dx^j, \qquad j = 1, 2, \ldots .$$

The coefficients p_k, $k = 0, 1, 2, \ldots, n$ are supposed to be real valued functions on (a, b) such that $1/p_0, p_1, \ldots, p_n$ are locally integrable and $p_0 > 0$ a.e. on (a, b). These assumptions do not allow a direct definition of $l(f)$, even if derivatives of f up to the order $2n$ exist. In order to give a meaning to $l(f)$ according to M. G. Krein [18] (see also [25, par. 15]) quasi-derivatives $f^{[k]}$ of orders $k = 0, 1, \ldots, 2n$ are defined by the formulae

$$\begin{gathered} f^{[0]} := f, \qquad f^{[k]} := df^{[k-1]}/dx, \qquad k = 1, 2, \ldots, n-1, \\ f^{[n]} := p_0\, df^{[n-1]}/dx, \\ f^{[n+k]} := p_k f^{[n-k]} - df^{[n+k-1]}/dx, \qquad k = 1, 2, \ldots, n, \end{gathered} \tag{2.1}$$

and we put

$$l(f) := f^{[2n]}. \tag{2.2}$$

In this way, the differential expression l on (a, b) is defined for all functions f such that $f^{[0]}, f^{[1]}, ..., f^{[2n-1]}$ exist and are absolutely continuous over compact subintervals of (a, b). For such f the formulae (2.1), (2.2) define $l(f)$ a.e. on (a, b).

In this paper we study spectral properties of the equation

$$l(f) - \lambda r f = 0 \qquad \text{on } (a, b), \tag{2.3}$$

where r is a real (weight) function on (a, b) which is locally integrable on (a, b) and *idefinite*; that is, the sets

$$\Delta_+ := \{x \in (a, b): r(x) > 0\} \qquad \text{and} \qquad \Delta_- := \{x \in (a, b): r(x) < 0\} \tag{2.4}$$

are both of positive Lebesque measure. For the sake of simplicity we assume that $r \neq 0$ a.e. on (a, b). The elements of the set $\bar{\Delta}_+ \cap \bar{\Delta}_-$ are called *turning points* of r.

Besides (2.3) we shall consider the equation

$$l(f) - \lambda\, |r|\, f = 0 \qquad \text{on } (a, b). \tag{2.5}$$

Problem (2.3) (or (2.5)) is called *regular* if $-\infty < a < b < +\infty$ and $1/p_0, p_1, ..., p_n,\ r \in L^1(a, b)$; otherwise it is called *singular*. The boundary point a (b) is called *singular* if $a = -\infty$ $(b = +\infty)$ or at least one of the functions $1/p_0, p_1, ..., p_n,\ r$ is not summable at a (b, resp.).

By $L^2(a, b; r)$ or, for short, $L^2(r)$ we denote the Krein space of all (equivalence classes of) measurable functions f defined on (a, b) for which $\int_a^b |f(x)|^2\, |r(x)|\, dx < +\infty$. The indefinite and definite inner products on $L^2(r)$ are

$$[f, g] := \int_a^b f\bar{g} r\, dx \qquad \text{and} \qquad (f, g) := \int_a^b f\bar{g}\, |r|\, dx, \qquad \text{resp.} \tag{2.6}$$

Evidently, the operator J

$$(Jf)(x) := (\operatorname{sgn} r(x))\, f(x) \qquad (x \in (a, b,)) \tag{2.7}$$

is the fundamental symmetry connecting the inner products in (2.6). By $L^2(|r|)$ we denote the Hilbert space $(L^2(a, b; r), (\cdot, \cdot))$.

Let $\mathscr{D}^0_{\min}$ be the set of all $f \in L^2(r)$ for which the differential expression l is defined, which vanish identically in neighborhoods of a and b and are such that $l(f)=|r|\ g$ holds with some $g\in L^2(r)$. On $\mathscr{D}^0_{\min}$ we define the operators $B^0_{\min}$ and $A^0_{\min}$ as follows: $\mathscr{D}(B^0_{\min})=\mathscr{D}(A^0_{\min})=\mathscr{D}^0_{\min}$,

$$B^0_{\min} f := g \quad \text{ if } \ f\in\mathscr{D}^0_{\min},\ \ l(f)=|r|\ g,\ \ g\in L^2(r),\ \ A^0_{\min} := JB^0_{\min}.$$

Evidently $A^0_{\min} f=g$ if and only if for $f\in\mathscr{D}^0_{\min}$, $g\in L^2(r)$, we have $l(f)=rg$. Since $r\neq 0$ a.e. on (a,b) it is easy to see that these definitions are correct. The closures of $A^0_{\min}$ and $B^0_{\min}$ in $L^2(r)$ exist and are denoted by $A_{\min}$ and $B_{\min}$, respectively. Obviously $A_{\min}=JB_{\min}$. The operator $A_{\min}$ ($B_{\min}$) is called the *minimal operator* associated with problem (2.3) ((2.5), resp.). Since the operator $B_{\min}$ is Hermitian in the Hilbert space $L^2(|r|)$, $A_0=A_{\min}$ is Hermitian with respect to the inner product $[\cdot,\cdot]$; i.e., it is Hermitian in the Krein space $L^2(r)$, and it has self-adjoint extensions in $L^2(r)$. In fact, A is a self-adjoint extension of $A_{\min}$ in $L^2(r)$ if and only if the operator $B:=JA$ is a self-adjoint extension of the operator $B_{\min}$ in the Hilbert space $L^2(|r|)$. Therefore, if there is more than one self-adjoint extension of $A_{\min}$ in $L^2(r)$, all these extensions are completely described by boundary conditions at a and b, which are the same for A and $B=JA$ (see [25]).

The defect index of $B_{\min}$ is (m,m), $0\leqslant m\leqslant 2n$; therefore the operator $A_0=A_{\min}$ satisfies condition (a_0) of Proposition 1.1. Thus Proposition 1.1 immediately yields the following:

THEOREM 2.1. *Suppose that the closed Hermitian operator* $A_0:=A_{\min}$ *satisfies assumptions* (a_1) *and* (a_2) *of Proposition* 1.1. *Then every self-adjoint extension* A *of* $A_{\min}$ *in* $L^2(r)$ *is definitizable.*

In no. 2.2 we shall give sufficient conditions for (a_1) and (a_2) to hold for $A_{\min}$.

To conclude this section, we mention that for a self-adjoint extension A of $A_{\min}$, associated with (2.3), the resolvent $(A-\lambda I)^{-1}$ in the Krein space $L^2(r)$ is an integral operator of Carleman type; that is, there exists a kernel $G(x,y;\lambda)$ $(x,y\in(a,b),\ \lambda\in\rho(A))$ such that

$$((A-\lambda I)^{-1} f)(x)=\int_a^b G(x,y;\lambda)\, f(y) r(y)\, dy.$$

The kernel G has the properties

$$G(x,y;\lambda)=G(y,x;\lambda)=\overline{G(x,y;\bar{\lambda})} \qquad (x,y\in(a,b),\ \lambda\in\rho(A)),$$

$$\int_a^b |G(x,y;\lambda)|^2\ |r(y)|\ dy<+\infty \qquad (x\in(a,b)).$$

If the operator $A_{\min}$ has defect index $(2n, 2n)$, it holds that

$$\int_a^b \int_a^b |G(x, y; \lambda)|^2 \, |r(x)| \, |r(y)| \, dx \, dy < +\infty.$$

These results can be proved in the same way as the corresponding statements for the case $r \equiv 1$ in [25, Par. 19, Theorem 1].

2.2. PROPOSITION 2.2. *If problem* (2.3) *is regular, conditions* (a_1) *and* (a_2) *of Proposition* 1.1 *are satisfied for the operator* $A_0 = A_{\min}$.

Proof. As was first shown by M. G. Krein (see [18, 25]) the assumption $p_0 > 0$ a.e. on (a, b) implies that $B_{\min}$ is bounded from below and that the spectrum of an arbitrary self-adjoint extension of $B_{\min}$ in $L^2(|r|)$ is discrete. Now, the statement follows from Remarks 1.2 and 1.4.

With the operator $A_0 = A_{\min}$ we introduce on $\mathscr{D}^0_{\min}$ the inner product

$$\{f, g\} := [A_0 f, g] = (B_{\min} f, g) = \int_a^b l(f) \, \bar{g} \, dx = \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \bar{g}^{(j)} \, dx. \quad (2.8)$$

Some properties of the inner product $\{\cdot, \cdot\}$ are given in Proposition 2.6 below. First we consider condition (a_1).

PROPOSITION 2.3 (cf. [13, Theorem I.28]). *Condition* (a_1) *of Proposition* 1.1 *is satisfied for the operator* $A_0 = A_{\min}$ *if and only if for each singular boundary point a or b of problem* (2.3) *there is a point* $a' \in (a, b)$ *or* $b' \in (a, b)$ *such that the inner product in* (2.8) *is positive on the set of all functions* $f \in \mathscr{D}^0_{\min}$ *which vanish outside of* (a, a') *or* (b', b).

Proof. We use I. M. Glazman's decomposition method [13]. Suppose, e.g., that only b is a singular boundary point. We first consider the inner product $\{\cdot, \cdot\}$ on the set $\mathscr{D}'$ of all functions $f \in \mathscr{D}^0_{\min}$ with the property $f^{[k]}(b') = 0$, $k = 0, 1, ..., 2n-1$. Let $\Delta' = [a, b']$ and $\Delta'' = [b', b)$. Then the spaces $L^2(\Delta'; |r|)$ and $L^2(\Delta''; |r|)$ can be considered as subspaces of $L^2(|r|)$ and

$$L^2(|r|) = L^2(\Delta'; |r|) \dotplus L^2(\Delta''; |r|),$$
$$\mathscr{D}' = (L^2(\Delta'; |r|) \cap \mathscr{D}') \dotplus (L^2(\Delta''; |r|) \cap \mathscr{D}'),$$

where both sums are orthogonal with respect to the Hilbert space inner product $(\cdot, \cdot)$ and the second sum is also orthogonal with respect to $\{\cdot, \cdot\}$. By Proposition 2.2 the inner product $\{\cdot, \cdot\}$ has a finite number, say κ_1, of negative squares on $\mathscr{D}' \cap L^2(\Delta'; |r|)$. It is easy to see that the inner product $\{\cdot, \cdot\}$ does not degenerate on $\mathscr{D}' \cap L^2(\Delta'; |r|)$ and $\mathscr{D}' \cap L^2(\Delta''; |r|)$. Since we assume that $\{\cdot, \cdot\}$ is positive on $\mathscr{D}' \cap L^2(\Delta''; |r|)$, it follows that $\{\cdot, \cdot\}$

has κ_1 negative squares on $\mathscr{D}'$. As $\dim \mathscr{D}^0_{\min}/\mathscr{D}' = 2n$, $\{\cdot, \cdot\}$ also has a finite number of negative squares on $\mathscr{D}^0_{\min}$.

Assume now that for each $c \in (a, b)$ there exists $f_c \in \mathscr{D}^0_{\min}$ which vanishes on (a, c) and $\{f_c, f_c\} < 0$. It follows that there exist functions $f_k \in \mathscr{D}^0_{\min}$ with disjoint supports and such that $\{f_k, f_k\} < 0$, $k = 1, 2, \dots$. Hence, $\{\cdot, \cdot\}$ has an infinite number of negative squares, which contradicts condition (a_1) of Proposition 1.1. The proposition is proved.

Proposition 2.3 shows that (if $p_0 > 0$ a.e. on (a, b)) condition (a_1) implies some restrictions about the coefficients p_j, $j = 1, 2, \dots, n$, only at the singular boundary point a or b.

Remark 2.4. If (a_1) holds, the operator $B_{\min}$ is always bounded from below in $L^2(|r|)$. According to the remark in [18, p. 347] the assumption $p_0 > 0$ a.e. on (a, b), which we have imposed from the beginning, is a consequence of (a_1).

Now we include also condition (a_2).

PROPOSITION 2.5. *Suppose that the operator $A_0 = A_{\min}$ satisfies condition (a_1) of Proposition 1.1. Then A_0 satisfies assumption (a_2) of Proposition 1.1 if for each singular boundary point a or b there is a point $a' \in (a, b)$ or $b' \in (a, b)$ such that the weight function r is of constant sign a.e. on (a, a') or (b', b).*

Proof. We suppose that b is the only singular boundary point of problem (2.3) and use again Glazman's method. Denote by $A'_{\min}$ and $A''_{\min}$ the minimal operators in $L^2(\Delta'; r)$ and $L^2(\Delta''; r)$ associated with problem (2.3) on the intervals $\Delta' = [a, b']$ and $\Delta'' = [b', b)$, respectively. By Proposition 2.2 and Theorem 2.1 there exists a $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $\mathscr{R}(A'_{\min} - \lambda_0 I)$ is closed. Since the operator $A''_{\min}$ is Hermitian in the Hilbert (or anti-Hilbert) space $L^2(\Delta''; r)$, the $\mathscr{R}(A''_{\min} - \lambda_0 I)$ is also closed. Hence, for $\hat{A} = A'_{\min} \oplus A''_{\min}$, the range $\mathscr{R}(\hat{A} - \lambda_0 I)$ is closed, too. As the factor space $\mathscr{R}(A_{\min} - \lambda_0 I)/\mathscr{R}(\hat{A} - \lambda_0 I)$ is finite dimensional, $\mathscr{R}(A_{\min} - \lambda_0 I)$ is closed, and the statement follows from Remark 1.3.

Let B be a self-adjoint extension of $B_{\min}$ in $L^2(|r|)$. Conditions which imply that $(-\infty, \varepsilon) \cap \sigma(B)$, $\varepsilon > 0$, consists of a finite number of eigenvalues are given in [13, nos. 12, 39, 40] and [25, Par. 24]. In light of Corollary 1.4 these conditions imply that the operator $A_0 = A_{\min}$ satisfies (a_1) and (a_2) of Proposition 1.1 and consequently each self-adjoint extension A of $A_{\min}$ in $L^2(r)$ is definitizable. If the operator $B_{\min}$ is bounded from below in $L^2(|r|)$ and the spectrum of B is discrete, then the spectrum of each self-adjoint extension A of $A_{\min}$ in $L^2(r)$ is discrete.

If the operator $B_{\min}$ is associated with a regular problem (2.3) then each self-adjoint extension B of $B_{\min}$ in $L^2(|r|)$ is determined by linearly independent boundary conditions

$$\sum_{k=1}^{2n} \alpha_{jk} f^{[k-1]}(a) + \sum_{k=1}^{2n} \beta_{jk} f^{[k-1]}(b) = 0, \qquad j = 1, ..., 2n. \tag{2.9}$$

If the rank of the matrix

$$\mathbb{A} = \begin{pmatrix} \alpha_{1,n} & \cdots & \alpha_{1,2n} & \beta_{1,n} & \cdots & \beta_{1,2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{2n,n} & \cdots & \alpha_{2n,2n} & \beta_{2n,n} & \cdots & \beta_{2n,2n} \end{pmatrix}$$

is d, by linear transformation of conditions (2.9) we can always obtain a matrix $\mathbb{A}$ having only zeros in the first $2n-d$ rows. Then the boundary conditions

$$\sum_{k=1}^{n} \alpha_{jk} f^{[k-1]}(a) + \sum_{k=1}^{n} \beta_{jk} f^{[k-1]}(b) = 0, \qquad j = 1, ..., 2n-d, \tag{2.10}$$

are called the *essential boundary conditions* of B. (See [18, 5].)

It was shown by M. G. Krein [18], that the set $\mathscr{D}[B]$ consists of those functions $f \in L^2(|r|)$ for which $f, f', ..., f^{(n-1)}$ are absolutely continuous on $[a, b]$, $\int_a^b |f^{(n)}|^2 p_0 \, dx < +\infty$, and which satisfy the essential boundary conditions (2.10). In [18] the case $r \equiv 1$ was considered, but the methods used there extend to the case $r > 0$ a.e. on $[a, b]$. In particular the set $\mathscr{D}[B_{\min}] = \mathscr{D}[B_F]$, B_F being the Friedrichs extension of $B_{\min}$ in $L^2(|r|)$, is determined by the essential boundary conditions

$$f^{(k)}(a) = f^{(k)}(b) = 0, \qquad k = 0, 1, ..., n-1.$$

PROPOSITION 2.6. *The number κ_0 of negative squares of the inner product $\{\cdot, \cdot\}$ in* (2.8) *does not depend on the weight function r. If problem* (2.3) *is regular then the number $\kappa_B = \kappa_A$, $B = JA$, of the inner product $(Bf, g) = [Af, g]$ $(f, g \in \mathscr{D}(B) = \mathscr{D}(A))$ is completely determined by the differential expression l and the essential boundary conditions of the self-adjoint extension B; i.e., $\kappa_B = \kappa_A$ does not depend on the weight function r.*

Proof. By the above characterization of the sets $\mathscr{D}[B_{\min}]$ and $\mathscr{D}[B]$ in the regular case of problem (2.3) these sets do not depend on the weight function r. In [18, part II, Par. 8] it is shown that we have

$$B(f, g) = \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \overline{g^{(j)}} \, dx + \Gamma_B(f, g) \qquad (f, g \in \mathscr{D}[B]),$$

where $\Gamma_B(\cdot,\cdot)$ is the Hermitian form defined on

$$\{\hat{f} = (f(a), ..., f^{(n-1)}(a), f(b), ..., f^{(n-1)}(b)) : f \in \mathscr{D}[B]\}.$$

Hence the Hermitian form $B(\cdot,\cdot)$ does not depend on r. It is easy to see that $\mathscr{D}[B_{\min}] = \mathscr{D}[B_{\mathrm{F}}]$ is the domain of the closure of the Hermitian form $\{\cdot,\cdot\}$ in (2.8). Hence the number κ_0 is equal to the number of negative squares of the form $B_{\mathrm{F}}(\cdot,\cdot)$ on $\mathscr{D}[\mathrm{B}_{\mathrm{F}}]$ which does not depend on r. Analogously κ_B does not depend on r.

Suppose that problem (2.3) is singular and let a linear subspace $\mathscr{L}$ of $\mathscr{D}^0_{\min}$ be such that $\dim \mathscr{L} = \kappa_0$ and $\{f, f\} < 0$ $(f \in \mathscr{L})$. It follows that there exist $a', b' \in (a, b)$ such that the functions from $\mathscr{L}$ vanish outside of (a', b'). Since problem (2.3) is regular on $[a', b']$, the already proved part of Proposition 2.6 implies that κ_0 does not depend on r. The proposition is proved.

More information about the number κ_B in the case when $B_{\min}$ has a positive lower bound can be found in [18, Part II, Par. 9].

2.3. If the Hermitian operator $A_0 = A_{\min}$ in Theorem 2.1 satisfies assumptions (a_1) and (a_2) of Proposition 1.1, then for each of its self-adjoint extensions all the conclusions of Proposition 1.5, Corollary 1.6, Corollary 1.7, and Proposition 1.8 hold (observe that the positivity of the Lebesque measure of Δ_+ and Δ_- implies for $\mathscr{K} = L^2(r)$ that $\kappa_+(\mathscr{K}) = \kappa_-(\mathscr{K}) = +\infty$).

Under more special conditions the results and methods of the previous sections yield more information about the spectrum of the self-adjoint extensions of $A_{\min}$ in $L^2(r)$. As examples, we prove two more propositions.

PROPOSITION 2.7. *Suppose that for each singular boundary point a or b of problem* (2.3) *there exists an $a' \in (a, b)$ or a $b' \in (a, b)$ with the following properties*:

(a) *The inner product $\{\cdot,\cdot\}$ in* (2.8) *is positive on the set of all $f \in \mathscr{D}^0_{\min}$ which vanish outside of (a, a') or (b', b)*;

(b) *The weight function r is positive a.e. on (a, a') or (b', b), respectively.*

Then for every self-adjoint extension of $A_{\min}$ in $L^2(r)$ the set $\sigma(A) \cap (-\infty, 0)$ is discrete with the only accumulation point $-\infty$.

Proof. It follows from Proposition 2.5 and Theorem 2.1 that every self-adjoint extension A of $A_{\min}$ in $L^2(r)$ is definitizable. For the sake of simplicity we suppose that b is the only singular boundary point of problem (2.3) and we use the notation from the proof of Proposition 2.3. Denote by

$B'_{\min}$ ($B''_{\min}$) the minimal operator associated with problem (2.3) on the interval Δ' (Δ'') and by B'_F (B''_F) the corresponding Friedrichs extension in $L^2(\Delta', |r|)$ ($L^2(\Delta'', |r|)$, resp.). Put $A' = JB'_F$ and $A'' = JB''_F$. The space $L^2(\Delta'', r)$ is a Hilbert space and $A = B_F$ is a self-adjoint operator in this Hilbert space. Since the inner product $\{\cdot, \cdot\}$ in (2.8) has a finite number of negative squares on $\mathscr{D}(A_{\min}) \cap L^2(\Delta''; r)$, the essential spectrum of A'' is contained in $[0, +\infty)$.

Problem (2.3) is regular on Δ' and Proposition 2.2 and Theorem 2.1 imply that the operator A' is definitizable in $L^2(\Delta', r)$ and that its spectrum is discrete. The operator $A' \oplus A''$ is definitizable in $L^2(r)$ and its spectrum in $(-\infty, 0)$ is discrete with the only accumulation point $-\infty$. Therefore $\kappa_-(0; A' \oplus A'') < +\infty$.

Now, let A be an arbitrary self-adjoint extension of $A_{\min}$ in $L^2(r)$. There exists a $\lambda \in \rho(A) \cap \rho(A' \oplus A'')$ and for such a λ we have

$$\dim((A - \lambda I)^{-1} - (A' \oplus A'' - \lambda I)^{-1}) < +\infty.$$

It follows by the arguments used in the proof of Theorem 1 in [15] that $\kappa_-(0; A) < +\infty$. The operators A and $A' \oplus A''$ have the same essential spectrum. Consequently, the spectrum of A in $(-\infty, 0)$ is discrete. Since $\kappa_-(0; A) < +\infty$, zero is not an accumulation point of $\sigma(A) \cap (-\infty, 0)$. The proposition is proved.

For a special case the structure of $\sigma(A)$ as in Proposition 2.7 was established in [21, Lemma 1].

Proposition 2.8 (Cf. [25, Par. 24, Theorem 3]). *Assume that for problem* (2.3) *the boundary point a is regular and that*

$$\lim_{x \uparrow b} p_n(x)/r(x) = \zeta > 0.$$

Further, suppose that there exists a $c \in (a, b)$ *such that*

$$p_1 \geqslant 0, \ldots, p_{n-1} \geqslant 0, \quad r \geqslant 0 \qquad \text{a.e. on } (c, b).$$

Then for each self-adjoint extension A *of* $A_{\min}$ *in* $L^2(r)$ *the spectrum* $\sigma(A)$ *is discrete in* $(-\infty, \zeta)$.

Proof. For $\varepsilon > 0$ there exists a $b' \in (c, b)$ such that

$$p_n(x)/r(x) > \zeta - \varepsilon \qquad \text{for almost all } x \in (b', b).$$

We use the notation of the proof of Proposition 2.7 and consider again the operators A' and A''. Let $f \in \mathscr{D}^0_{\min}$, $f(x) = 0$ on $[a, b']$. Then we have

$$\begin{aligned}(A''f,f)=[A''f,f]=[A_{\min}f,f]&=\int_{b'}^{b}(A_{\min}f)\bar{f}r\,dx\\&=\int_{b'}^{b}l(f)\,\bar{f}\,dx=\sum_{j=0}^{n}\int_{b'}^{b}p_j\,|f^{(n-j)}|^2\,dx\\&\geqslant\int_{b'}^{b}(p_n/r)\,|f|^2\,r\,dx=\int_{a}^{b}(p_n/r)\,|f|^2\,|r|\,dx\geqslant(\zeta-\varepsilon)(f,f).\end{aligned}\tag{2.11}$$

It follows that the operator $A_{\min}$ satisfies the assumptions of Proposition 2.3. Moreover, inequality (2.11) implies $\sigma(A'')\subset[\zeta-\varepsilon,+\infty)$. Therefore $\sigma(A'\oplus A'')$ is discrete in $(-\infty,\zeta-\varepsilon)$. Since A and $A'\oplus A''$ have the same continuous spectrum, and $\varepsilon>0$ is arbitrary, the statement follows.

COROLLARY 2.9. *If, in addition to the assumptions of Proposition 2.8, we have $\zeta=+\infty$, then for each self-adjoint extension A of $A_{\min}$ in $L^2(r)$ the spectrum $\sigma(A)$ is discrete.*

This corollary can be proved in the same way as Proposition 2.8. It follows from [25, Par. 24, Theorem 2] and Corollary 1.4, as well.

3. REGULARITY OF CRITICAL POINT INFINITY

3.1. In this section we show that ∞ is not a singular critical point of the definitizable operator A, associated with the differential problem (2.3), if the number of turning points of the weight function r is finite and p_0 and r satisfy some assumptions in neighborhoods of these turning points.

DEFINITION 3.1. A nonnegative (nonpositive) function w is said to be *n-simple from the right at* x_0 if there exists a $\delta>0$ such that w is defined at least on $[x_0,x_0+\delta]$ and

$$w(x)=(x-x_0)^\tau\rho(x)\qquad(w(x)=-(x-x_0)^\tau\rho(x),\ \text{resp.})$$

holds a.e. on $[x_0,x_0+\delta]$ with some $\tau>-1$, $\rho\in C^n[x_0,x_0+\delta]$, $\rho(x_0)>0$, and, if $n>1$, $\rho'(x_0+)=\cdots=\rho^{(n-1)}(x_0+)=0$. A function w is said to be *n-simple from the left at* x_0 if the function $x\mapsto w(-(x-x_0)+x_0)$ is n-simple from the right at x_0. A function w, defined in a neighborhood of x_0, is said to be *n-simple* at x_0 if it is n-simple from the right and n-simple from the left at x_0 (with, possibly, different numbers τ).

LEMMA 3.2. *Let $\Delta=[\alpha,\beta]$ and let $w,p\in L^1(\Delta)$, $w,p>0$ a.e. on Δ. Denote by $\mathscr{D}$ the set of all $f\in L^2(\Delta;w)$ for which $f,f',...,f^{(n-1)}$ are* *absolutely continuous on Δ and such that $\int_\Delta |f^{(n)}|^2 p\,dx < +\infty$. Suppose that the function w is n-simple from the right at α and that p and $1/p$ are essentially bounded in a neighborhood of α. Then there exists a bounded and boundedly invertible, positive operator X in $L^2(\Delta; w)$ such that*

$$X\mathscr{D} \subset \{f \in \mathscr{D} : f(\alpha) = f'(\alpha) = \cdots = f^{(n-1)}(\alpha) = 0\}$$

and such that the function $f - Xf$ vanishes in the fixed neighborhood of β for every $f \in L^2(\Delta; w)$.

Proof. Without loss of generality we can suppose that $\alpha = 0$; that is, $\Delta = [0, \beta]$. Choose $\varphi \in C^n(\Delta)$, $0 \leqslant \varphi \leqslant 1$, which is equal to 1 in a neighborhood of zero and vanishes outside of $[0, \delta/2]$, where $\delta < \beta$ is from Definition 3.1 and such that p and $1/p$ are essentially bounded in $[0, \delta]$. We define a linear operator Y in $L^2(\Delta; w)$ as

$$(Yu)(x) := \sum_{j=1}^{2n} \alpha_j s_j u(s_j x)\varphi(x), \qquad x \in \Delta, \tag{3.1}$$

where s_j, $1 \leqslant s_j < 2$, $j = 1, ..., 2n$, are mutually different and $\alpha_1, ..., \alpha_{2n}$ are reals to be chosen below. Put

$$h_j(x) := (w(x)/w(s_j x))\varphi(x) = (\rho(x)/s_j^\tau \rho(s_j x))\varphi(x),$$
$$x \in \Delta, \quad j = 1, 2, ..., 2n,$$

and let

$$c_1 = \max\{|h_j^{(k)}(x)| : x \in \Delta, k = 0, 1, ..., n, j = 1, 2, ..., 2n\},$$
$$c_2 = \max\{|\varphi^{(k)}(x)| : x \in \Delta, k = 0, 1, ..., n,\},$$
$$c_3 = \text{ess sup}\{p(x) : x \in (0, \delta)\}/\text{ess inf}\{p(x) : x \in (0, \delta)\}.$$

The operator Y is bounded in $L^2(\Delta; w)$. Indeed,

$$\begin{aligned}
(Yu, Yu)_\Delta &= \int_\Delta \left|\varphi(x) \sum_{j=1}^{2n} \alpha_j s_j u(s_j x)\right|^2 w(x)\,dx \\
&\leqslant 2n \int_0^{\delta/2} |\varphi(x)|^2 \left(\sum_{j=1}^{2n} |\alpha_j|^2 s_j^2 |u(s_j x)|^2\right) w(x)\,dx \\
&\leqslant 2n \sum_{j=1}^{2n} |\alpha_j|^2 s_j^2 \int_0^{\delta/2} |u(s_j x)|^2 h_j(x) w(s_j x)\,dx \\
&\leqslant 2nc_1 \sum_{j=1}^{2n} |\alpha_j|^2 s_j^2 \int_0^{\delta/2} |u(s_j x)|^2 w(s_j x)\,dx \\
&< 2nc_1 \left(\sum_{j=1}^{2n} |\alpha_j|^2 s_j\right) (u, u)_\Delta.
\end{aligned}$$

It is easy to see that the adjoint Y^* of the operator Y in $L^2(\Delta; w)$ is given by

$$(Y^*u)(x) = \sum_{j=1}^{2n} \alpha_j (h_j u)(x/s_j), \qquad x \in \Delta. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$(Yu)(x) = (Y^*u)(x) = 0, \qquad u \in L^2(\Delta, w), \quad x \in [\delta, \beta]. \tag{3.3}$$

If $u \in \mathscr{D}$, then the functions $(Yu)^{(k)}$, $(Y^*u)^{(k)}$, $k = 0, 1, ..., n-1$, are absolutely continuous on Δ and we have

$$(Yu)^{(k)}(x) = \sum_{i=0}^{k} \binom{k}{i} \varphi^{(k-i)}(x) \sum_{k=1}^{2n} \alpha_j s_j^{i+1} u^{(i)}(s_j x), \qquad x \in \Delta,$$

$$(Y^*u)^{(k)}(x) = \sum_{j=1}^{2n} \alpha_j s_j^{-k} \left(\sum_{i=0}^{k} \binom{k}{i} h_j^{(i)} u^{(k-i)} \right)(x/s_j), \qquad x \in \Delta,$$

$$k = 0, 1, ..., n-1, n.$$

Moreover,

$$\int_\Delta |(Yu)^{(n)}|^2 p\, dx < +\infty, \qquad \int_\Delta |(Y^*u)^{(n)}|^2 p\, dx < +\infty.$$

We prove, e.g., the second inequality:

$$\begin{aligned}
&\int_\Delta \left| \sum_{j=1}^{2n} \alpha_j s_j^{-n} \left(\sum_{i=0}^{n} \binom{n}{i} h_j^{(i)} u^{(n-i)} \right)(x/s_j) \right|^2 p(x)\, dx \\
&\quad \leqslant 2n(n+1) c_1^2 \int_0^{\delta/2} \sum_{j=1}^{2n} |\alpha_j|^2 s_j^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2 |u^{(n-i)}(x/s_j)|^2 \\
&\qquad \times (p(x)/p(x/s_j))\, p(x/s_j)\, dx \\
&\quad \leqslant 2n(n+1) s_1^2 c_3 \int_0^{\delta/2} \sum_{j=1}^{2n} \sum_{i=0}^{n} |\alpha_j|^2 s_j^{-2n+1} \binom{n}{i}^2 |u^{(n-i)}(x)|^2 p(x)\, dx \\
&\qquad + 2n(n+1) c_1^2 c_3 \left(\sum_{j=1}^{2n} |\alpha_j|^2 s_j^{-2n+1} \right) \int_\Delta |u^{(n)}(x)|^2 p(x)\, dx.
\end{aligned}$$

This expression is finite since $u \in \mathscr{D}$. Thus, $Y\mathscr{D} \subset \mathscr{D}$ and $Y^*\mathscr{D} \subset \mathscr{D}$.

Now we determine $\alpha_1, ..., \alpha_{2n} \in \mathbb{R}$ such that for $u \in \mathscr{D}$ we have $(Yu)^{(k)}(0+) = u^{(k)}(0+)$, $(Y^*u)^{(k)}(0+) = -u^{(k)}(0+)$, $k = 0, 1, ..., n-1$. The first n equalities are equivalent to

$$\sum_{j=1}^{2n} \alpha_j s_j^{k+1} = 1, \qquad k = 0, 1, ..., n-1. \tag{3.4}$$

Because of $\rho'(0+) = \cdots = \rho^{(n-1)}(0+) = 0$ we have $h_j^{(i)}(0+) = 0$, $j = 1, 2, ..., 2n$, $i = 1, ..., n-1$; i.e.,

$$(Y^*u)^{(k)}(0+) = \sum_{j=1}^{2n} \alpha_j s_j^{-k} u^{(k)}(0+) h_j(0+), \qquad k = 0, 1, ..., n-1,$$

where $h_j(0+) = s_j^{-\tau}$. Hence $(Y^*u)^{(k)}(0+) = -u^{(k)}(0+)$, $k = 0, 1, ..., n-1$ is equivalent to

$$\sum_{j=1}^{2n} \alpha_j s_j^{-k-\tau} = -1, \qquad k = 0, 1, ..., n-1. \tag{3.5}$$

Equations (3.4), (3.5) determine $\alpha_1, ..., \alpha_{2n}$ uniquely, since the determinant of this system is a generalized Vandermonde determinant which is different from zero.

The operator Y^*Y maps $\mathscr{D}$ into $\mathscr{D}$,

$$(Y^*Yu)^{(k)}(0+) = -u^{(k)}(0+), \qquad \text{for} \quad k = 0, 1, ..., n-1, \quad u \in \mathscr{D},$$

and it is a bounded and nonnegative operator in $L^2(\Delta; w)$. Moreover, (3.3) implies

$$(Y^*Yu)(x) = 0, \qquad x \in [\delta, \beta], \quad u \in L^2(\Delta; w).$$

Thus, the operator $X = Y^*Y + I$ has all the properties stated in the lemma.

Remark 3.3. If w is n-simple from the left at β, a corresponding result holds.

If $n = 1$, the conclusion of Lemma 3.2 holds if the function w (defined on $\Delta = [0, \beta]$) instead of 1-simplicity at zero from the right has the following properties: w is absolutely continuous in a neighborhood of zero, there exists $s_1 \neq 1$, $s_1 > 0$, such that $\lim_{x \downarrow 0} w(x)/w(s_1 x) \neq s_1$ and such that the function $(w(x)/w(s_1 x))'$ is bounded in a neighborhood of zero. In this case we can choose $s_2 = 1$ and the function φ such that $\operatorname{supp} \varphi \subset [0, \delta_1]$ where $\delta_1 \max\{s_1, 1/s_1\} < \delta$. The rest of the proof remains unchanged.

We also mention that the condition $\rho'(0+) = \cdots = \rho^{(n-1)}(0+) = 0$ was used in the proof of Lemma 3.2 only in order to assure that the system (3.4), (3.5) has a solution $\alpha_1, ..., \alpha_{2n}$.

3.2. In order to prove that ∞ is not a singular critical point of a definitizable extension A of the operator $A_{\min}$ we use the following criterium given in [7, Proposition 3.5].

PROPOSITION 3.4. *The point ∞ is not a singular critical point of the definitizable operator A in the Krein space $(\mathscr{K}, [\cdot, \cdot])$ if and only if there* *exists a positive, bounded, and boundedly invertible operator W in the Krein space $\mathcal{K}$ such that*

$$W\mathcal{D}[JA] \subset \mathcal{D}[JA].$$

If problem (2.3) is singular on (a, b), the boundary conditions at the singular boundary point(s), which determine a self-adjoint extension A of $A_{\min}$ in $L^2(r)$, depend on the values of the first $2n-1$ quasi-derivatives of $f \in \mathcal{D}(A_{\max})$ in a neighborhood of this (these) point(s). In this case, we call the boundary conditions which determine a self-adjoint extension A of $A_{\min}$ in $L^2(r)$ *separated* if the following is true: If $f \in \mathcal{D}(A)$, $g \in \mathcal{D}(A_{\max})$, $f = g$ in the neighborhood of one endpoint a or b, and $g = 0$ in the neighborhood of the other endpoint then $g \in \mathcal{D}(A)$.

A characterization of the set $\mathcal{D}[JA]$, for the regular case of problem (2.3), is given in no. 2.2. If problem (2.3) is singular, to our knowledge, there is no explicit description of $\mathcal{D}[JA]$. Some particular cases for which a characterization is available are mentioned in Remark 3.7. Denote by $\hat{\mathcal{D}}$ the set of all $f \in L^2(r)$ such that $f, f', ..., f^{(n-1)}$ are locally absolutely continuous on (a, b) and $f^{(n)} \in L^2_{\text{loc}}(a, b; p_0)$. The set $\mathcal{D}[JA]$ is said to be *separated* if the following is true: If $f \in \mathcal{D}[JA]$, $g \in \hat{\mathcal{D}}$, $f = g$ in the neighborhood of the one endpoint a or b, and $g = 0$ in the neighborhood of the other endpoint then $g \in \mathcal{D}[JA]$. We call the essential boundary conditions (2.10) *separated* if each equality in (2.10) contains derivatives at only one boundary point. Obviously, if problem (2.3) is regular, the set $\mathcal{D}[JA]$ is separated if and only if the essential boundary conditions of A are separated.

The following lemma is not hard to prove.

LEMMA 3.5. *Suppose that the Hermitian operator $A_{\min}$ is associated with a (singular) problem* (2.3) *on (a, b). Then we have*:

(i) $\mathcal{D}[JA] \subset \hat{\mathcal{D}}$.

(ii) *If $f \in \mathcal{D}[JA]$ and f vanishes on a closed interval $[a_1, b_1] \subset (a, b)$, then there exist functions $f_n \in \mathcal{D}(A)$, $n = 1, 2, ...$, which vanish on $[a_1, b_1]$ and are such that $f_n \to f$ in $L^2(|r|)$, $[(A(f_n - f_m)), f_n - f_m] \to 0$ $(n, m \to +\infty)$.*

(iii) *If the boundary conditions which determine A are separated, then the set $\mathcal{D}[JA]$ is separated.*

The main result of this section is:

THEOREM 3.6. *Assume that in problem* (2.3) *the weight function r has a finite number of turning points at which it is n-simple, and that p_0, $1/p_0$ are essentially bounded in neighborhoods of these turning points. Further, suppose that the Hermitian operator $A_{\min}$, associated with* (2.3), *satisfies conditions*

(a_1), (a_2) *of Proposition* 1.1. *Then* ∞ *is not a singular critical point of the self-adjoint extension* A *of* $A_{\min}$ *in* $L^2(r)$ *in each of the following cases*:

(i) *The set* $\mathscr{D}[JA]$ *is separated.*

(ii) *The number of turning points of* r *is even and* A *is an arbitrary self-adjoint extension of* $A_{\min}$ *in* $L^2(r)$.

(iii) *Problem* (2.3) *is regular*, r *is* n-*simple from the right at* a *and* n-*simple from the left at* b, *and* A *is an arbitrary self-adjoint extension of* $A_{\min}$ *in* $L^2(r)$.

Proof. Let

$$a < t_1 < t_2 < \cdots < t_{2\nu-1} < t_{2\nu} < t_{2\nu+1} < b$$

be a partition of the interval (a, b) such that t_{2j}, $j = 1, ..., \nu$, are the turning points of r. Then

$$\begin{aligned} L^2(|r|) = L^2(a, t_1; |r|) &\dotplus L^2(t_1, t_2; |r|) \\ &\dotplus \cdots \dotplus L^2(t_{2\nu}, t_{2\nu+1}; |r|) \dotplus L^2(t_{2\nu+1}, b; |r|), \end{aligned}$$

the sum being orthogonal with respect to $(\cdot, \cdot)$. Let A be a self-adjoint extension of $A_{\min}$ in $L^2(r)$ and $B = JA$. According to Theorem 2.1 the operator A is definitizable and the operator B is bounded from below in $L^2(|r|)$. In what follows we use the properties of $\mathscr{D}[B] = \mathscr{D}[JA]$ stated in no. 2.2. If $[\alpha, \beta] \subset (a, b)$ we denote by $\mathscr{D}[B] \mid [\alpha, \beta]$ the set of the restrictions of the functions from $\mathscr{D}[B]$ to $[\alpha, \beta]$. It follows from Lemma 3.2 and Remark 3.3 that for $i = 1, ..., 2\nu$ there exists a positive, bounded, and boundedly invertible operator X_i in $L^2(t_i, t_{i+1}; |r|)$ such that

$$\begin{aligned} &X_i(\mathscr{D}[B] \mid [t_1, t_{i+1}]) \\ &\quad \subset \{ f \in \mathscr{D}[B] \mid [t_1, t_{i+1}] : f(\hat{t}) = f'(\hat{t}) = \cdots = f^{(n-1)}(\hat{t}) = 0 \}, \end{aligned}$$

where $\hat{t} \in \{t_i, t_{i+1}\}$ is a turning point of r, and X_i does not change the functions in a neighborhood of the other boundary point of $[t_i, t_{i+1}]$. We introduce the operator

$$X = I_1 \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_{2\nu} \oplus I_2, \tag{3.6}$$

where I_1 and I_2 are the identities on $L^2(a, t_1; |r|)$ and $L^2(t_{2\nu+1}, b; |r|)$, respectively. Then X is positive, bounded, and boundedly invertible in $L^2(|r|)$ and $X\mathscr{D}[B] \subset \mathscr{D}_0[B]$, where $\mathscr{D}_0[B]$ denotes the set of all functions $f \in \mathscr{D}[B]$ such that

$$f(t_{2j}) = f'(t_{2j}) = \cdots = f^{(n-1)}(t_{2j}) = 0, \qquad j = 1, 2, ..., \nu.$$

Suppose now that problem (2.3) is regular. It is easy to see that for $f \in \mathscr{D}_0[B]$ the function Jf has absolutely continuous derivatives up to the order $n-1$, that $Jf^{(k)} = (Jf)^{(k)}$, $k = 0, 1, ..., n$, and, consequently, that $\int_a^b |(Jf)^{(n)}|^2 p_0 \, dx < +\infty$. Hence $Jf \in \mathscr{D}_0[B]$ if and only if Jf satisfies the essential boundary conditions of A.

If the number of turning points is even, the function r has the same sign on $[a, t_1]$ and $[t_{2\nu+1}, b]$. Hence

$$(Jf)(x) = f(x) \qquad \text{or} \qquad (Jf)(x) = -f(x) \qquad \text{for} \quad x \in [a, t_1] \cup [t_{2\nu+1}, b].$$

In both cases Jf satisfies the essential boundary conditions if $f \in \mathscr{D}_0[B]$. Therefore $J\mathscr{D}_0[B] \subset \mathscr{D}_0[B]$. Put $W := JX$. It follows that $W\mathscr{D}[B] \subset \mathscr{D}_0[B] \subset \mathscr{D}[B]$ and that W is a positive, bounded, and boundedly invertible operator in $L^2(r)$. According to Proposition 3.4 it follows that $\infty \notin c_s(A)$, and (ii) is proved if (2.3) is regular.

If $\mathscr{D}[B]$ is determined by separated essential boundary conditions, it is obvious that we have $Jf \in \mathscr{D}_0[B]$ for $f \in \mathscr{D}_0[B]$ and it follows again that $W\mathscr{D}[B] \subset \mathscr{D}_0[B] \subset \mathscr{D}[B]$. This proves (i) in the regular case of (2.3).

Suppose now that problem (2.3) is singular and put $\Delta = [a', b'] \subset (a, b)$. Problems (2.3) and (2.5) restricted to Δ are regular. Denote the associated minimal operators by $A_{\min,\Delta}$, $B_{\min,\Delta}$: $A_{\min,\Delta} = JB_{\min,\Delta}$. It follows from the definition of the set $\mathscr{D}[B_{\min,\Delta}]$ that $\mathscr{D}[B_{\min,\Delta}] \subset \mathscr{D}[B]$, if we consider the functions $f \in \mathscr{D}[B_{\min,\Delta}]$ to be zero on $(a, a') \cup (b', b)$. Recall that $f \in \mathscr{D}[B_{\min,\Delta}]$ satisfies the essential boundary conditions

$$f(a') = f'(a') = \cdots = f^{(n-1)}(a') = f(b') = f'(b') = \cdots = f^{(n-1)}(b') = 0.$$

Now let $a < a' < t_1' < t_1$, $t_{2\nu+1} < t_{2\nu+1}' < b' < b$ and choose the function $\Phi \in C^n(a, b)$ such that $0 \leqslant \Phi \leqslant 1$, $\Phi(x) = 1$ $(x \in [t_1', t_{2\nu+1}'])$, and $\Phi(x) = 0$ $(x \in (a, a'] \cup [b', b))$. If $f \in \mathscr{D}[B]$, according to Lemma 3.5(i) the functions $(\Phi f)^{(k)}$, $k = 0, 1, ..., n-1$, are locally absolutely continuous and $\int_\Delta |(\Phi f)^{(n)}|^2 p_0 \, dx < +\infty$. Hence $\Phi f \in \mathscr{D}[B_{\mathrm{F},\Delta}] \subset \mathscr{D}[B]$, where $B_{\mathrm{F},\Delta}$ is the Friedrichs extension of $B_{\min,\Delta}$. Since $f_0 = f - \Phi f$ vanishes on $[t_1', t_{2\nu+1}']$ we have $f_0 \in \mathscr{D}_0[B]$. It follows that $Xf_0 = f_0$ and $X(\Phi f) \in \mathscr{D}_0[B_{\mathrm{F},\Delta}] \subset \mathscr{D}_0[B]$; hence also $Xf = Xf_0 + X(\Phi f) = f_0 + X(\Phi f) \in \mathscr{D}_0[B]$. Evidently $W(\Phi f) = JX(\Phi f) \in \mathscr{D}_0[B_{\mathrm{F},\Delta}] \subset \mathscr{D}_0[B]$, and whether $Wf = Wf_0 + W(\Phi f) = Jf_0 + W(\Phi f)$ belongs to $\mathscr{D}_0[B]$ or not depends only on Jf_0. If the number of turning points of r is even we have $Jf_0 = f_0$ or $Jf_0 = -f_0$ and in both cases $Jf_0 \in \mathscr{D}_0[B]$ since $f_0 \in \mathscr{D}_0[B]$. Consequently $Wf \in \mathscr{D}_0[B] \subset \mathscr{D}[B]$. It follows that $W\mathscr{D}[B] \subset \mathscr{D}[B]$ and the proof of (ii) is complete.

Let $\mathscr{D}[JA] = \mathscr{D}[B]$ be separated. We have $f_0 = f_{0,a} + f_{0,b}$, where $f_{0,a}(x) = f_0(x)$ $(x \in (a, t_1))$, $f_{0,a}(x) = 0$ $(x \in [t_1, b))$, $f_{0,b}(x) = 0$ $(x \in (a, t_1])$, $f_{0,b}(x) = f_0(x)$ $(x \in (t_1, b))$, and the functions $f_{0,a}$ and $f_{0,b}$ belong to $\mathscr{D}[B]$ since $f_0 \in \mathscr{D}[B]$. Obviously, $Jf_{0,a} = f_{0,a}$ or $Jf_{0,a} = -f_{0,a}$ and $Jf_{0,b} = f_{0,b}$ or

$Jf_{0,b} = -f_{0,b}$. Consequently, $Jf_{0,a}$, $Jf_{0,b} \in \mathscr{D}[B]$. Thus, $Jf_0 = Jf_{0,a} + Jf_{0,b} \in \mathscr{D}[B]$. This implies that $Wf = Jf_0 + W(\Phi f)$ belongs to $\mathscr{D}[B]$, i.e., $W\mathscr{D}[B] \subset \mathscr{D}[B]$. This completes the proof of (i).

It remains to prove (iii). In this case, according to Lemma 3.2 and Remark 3.3, there exist positive, bounded, and boundedly invertible operators X_0 and $X_{2\nu+1}$ in $L^2(a, t_1; |r|)$ and $L^2(t_{2\nu+1}, b; |r|)$, respectively, such that

$$X_0(\mathscr{D}[B] \mid [a, t_1)] \subset \{f \in \mathscr{D}[B] \mid [a, t_1]: f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0\}$$

and

$$X_{2\nu+1}(\mathscr{D}[B] \mid t_{2\nu+1}, b]) \subset \{f \in \mathscr{D}[B] \mid [t_{2\nu+1}, b]: f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0\}.$$

Now, in definition (3.6) of X, we replace the operator I_1 by X_0 and I_2 by $X_{2\nu+1}$. Then X is positive, bounded, and boundedly invertible and

$$X\mathscr{D}[B] \subset \{f \in \mathscr{D}_0[B]: f(a) = \cdots = f^{(n-1)}(a) = f(b) = \cdots = f^{(n-1)}(b) = 0\}.$$

The last set we denote by $\mathscr{D}_0'[B]$. It is obvious that $J\mathscr{D}_0'[B] \subset \mathscr{D}_0[B]$. Hence, for $W = JX$ we have $W\mathscr{D}[B] \subset \mathscr{D}_0'[B] \subset \mathscr{D}[B]$, and W is a positive, bounded, and boundedly invertible operator in $L^2(r)$. This completes the proof of the theorem.

The remarks in the last two paragraphs of no. 3.1 allow one, in some situations, to prove the conclusion of Theorem 3.6 under weaker conditions about the behavior of the weight function at its turning points.

Remark 3.7. For indefinite Sturm–Liouville problems which are considered in [3, 17, 26], it is not difficult to see that for the corresponding operator A the set $\mathscr{D}[JA]$ is separated. Classes of $2n$th order singular problems for which the corresponding operator A has the separated set $\mathscr{D}[JA]$ can be found in [27].

4. Full- and Half-Range Completeness

4.1. In this section we suppose that for the operator $A_0 = A_{\min}$, associated with the differential problem (2.3), conditions (a_1) and (a_2) of Proposition 1.1 are satisfied and that the spectrum of one and hence of all self-adjoint extensions A of $A_{\min}$ in $L^2(r)$ is discrete. Then the operator A is definitizable in $L^2(r)$ and all finite critical points of A are regular.

Moreover, we suppose that ∞ is not a singular critical point of the self-adjoint extension A which we consider here. Recall that sufficient conditions for this last assumption to be satisfied were given in Theorem 3.6.

Denote by λ_j^+ (λ_j^-), $j=1,2,\ldots$, the nondecreasing (nonincreasing) sequence of the real eigenvalues of A such that $\kappa_-(\lambda_j^+;A)=0$ ($\kappa_+(\lambda_j^-;A)=0$, resp.); that is, λ_j^+ (λ_j^-) are the eigenvalues of positive (negative, resp.) type of A. Then

$$\{\lambda_j^+,\lambda_j^-,j=1,2,\ldots\}=(\sigma(A)\cap\mathbb{R})\setminus c(A).$$

It follows from Corollary 1.6 that the total number of negative eigenvalues among the λ_j^+, $j=1,2,\ldots$, and positive eigenvalues among the λ_j^- is at most κ_A. Since $\lambda_j^\pm\notin c(A)$, $j=1,2,\ldots$, the eigenvalues $\lambda_j^\pm$ of A are semi-simple; that is, the corresponding root subspaces and geometric eigenspaces coincide.

A set $\{v_j,j=1,2,\ldots\}$ in the Hilbert space $\mathscr{H}$ is called a *basis* if each element of $\mathscr{H}$ is the limit (in norm) of a unique series $\sum a_jv_j$, $a_j\in\mathbb{C}$. The basis $\{v_j,j=1,2,\ldots\}$ is a *Riesz basis* if there exists a bounded and boundedly invertible operator T in $\mathscr{H}$ such that $\{Tv_j,j=1,2,\ldots\}$ is an orthonormal basis in $\mathscr{H}$.

Denote by $\mathscr{K}_c$ the finite-dimensional linear span of the root subspaces of A corresponding to the critical and nonreal eigenvalues of A. Then $\mathscr{K}_c$ is a nondegenerate, hence orthocomplemented, subspace of $L^2(r)$: $L^2(r)=\mathscr{K}_c[\dot{+}]\mathscr{K}_1$, with a nondegenerate subspace $\mathscr{K}_1$, and we have $A\mathscr{K}_c\subset\mathscr{K}_c$, $A\mathscr{K}_1\subset\mathscr{K}_1$. As $\infty\notin c_s(A)$, it is not a singular critical point of the operator $A_1:=A\mid\mathscr{K}_1$ and A_1 has no finite critical points; moreover $\sigma(A_1)=\{\lambda_j^+,\lambda_j^-,j=1,2,\ldots\}$. It follows from Propositions II.5.2 and II.5.6 in [20] that the closed linear span of $\ker(A-\lambda_j^+I)$ ($\ker(A-\lambda_j^-I)$), $j=1,2,\ldots$, is a maximal uniformly positive (negative, resp.) subspace of $(\mathscr{K}_1,[\cdot,\cdot])$; denote it by $\mathscr{M}_+$ ($\mathscr{M}_-$, resp.). Then the decomposition $\mathscr{K}_1=\mathscr{M}_+[\dot{+}]\mathscr{M}_-$ holds and both subspaces $\mathscr{M}_\pm$ are invariant under A. The spaces $(\mathscr{M}_\pm,\pm[\cdot,\cdot])$ are Hilbert spaces and $A\mid\mathscr{M}_\pm$ are self-adjoint operators in these Hilbert spaces with

$$\sigma(A\mid\mathscr{M}_\pm)=\{\lambda_j^\pm,j=1,2,\ldots\}.$$

The norm topology of $(\mathscr{M}_\pm,\pm[\cdot,\cdot])$ coincides with the norm topology of $(\mathscr{M}_\pm,(\cdot,\cdot))$.

Let $\{e_j^\pm,j=1,2,\ldots\}$ be an orthonormal basis of $(\mathscr{M}_\pm,\pm[\cdot,\cdot])$ which consists of eigenfunctions of $A\mid\mathscr{M}_\pm$. Then for arbitrary $g\in\mathscr{K}_1$ we have

$$g=\sum_{j=1}^{+\infty}\frac{[g,e_j^+]}{[e_j^+,e_j^+]}e_j^+ +\sum_{j=1}^{+\infty}\frac{[g,e_j^-]}{[e_j^-,e_j^-]}e_j^-, \tag{4.1}$$

where both sums converge in the topology of $(\mathcal{K}_1, [\cdot,\cdot])$, and therefore also in $L^2(|r|)$. If $f_1, ..., f_k$ is a basis of $\mathcal{K}_c$, it follows from (4.1) and $L^2(r) = \mathcal{K}_c\ [\dot{+}]\ \mathcal{K}_1$ that the functions

$$f_1, ..., f_k, e_j^+, e_j^-, \qquad j = 1, 2, ..., \tag{4.2}$$

form a basis of $L^2(|r|)$. Obviously, $f_1, ..., f_k$ can always be chosen such that they are root vectos of A.

The basis in (4.2) is even a Riesz basis. Indeed, using the decomposition $\mathcal{K}_1 = \mathcal{M}_+\ [\dot{+}]\ \mathcal{M}_-$ a Hilbert space inner product $(\cdot,\cdot)_1$ can be defined on $\mathcal{K}_1$ and the norm induced by $(\cdot,\cdot)_1$ on $\mathcal{K}_1$ is equivalent on $\mathcal{K}_1$ to the norm of $L^2(|r|)$. The system $e_j^\pm$, $j = 1, 2, ...$, is orthonormal in $(\mathcal{K}_1, (\cdot,\cdot)_1)$. The inner product $(\cdot,\cdot)_1$ can be extended on $L^2(r)$ in such a way that $\mathcal{K}_c$ and $\mathcal{K}_1$ are orthogonal, $f_1, ..., f_k$ form an orthonormal basis in $(\mathcal{K}_c, (\cdot,\cdot)_1)$, and the norm induced by $(\cdot,\cdot)_1$ is equivalent to the norm of $L^2(|r|)$. The basis (4.2) is orthonormal in $(L^2(|r|); (\cdot,\cdot)_1)$; hence it is a Riesz basis of $L^2(|r|)$.

Summing up, we have

PROPOSITION 4.1. *The functions* (4.2) *form a Riesz basis of* $L^2(|r|)$. *Each* $f \in L^2(r)$ *has a unique expansion of the form*

$$f = \sum_{j=1}^{k} a_j f_j + \sum_{j=1}^{+\infty} \frac{[f, e_j^+]}{[e_j^+, e_j^+]} e_j^+ + \sum_{j=1}^{+\infty} \frac{[f, e_j^-]}{[e_j^-, e_j^-]} e_j^-, \tag{4.3}$$

with $a_1, ..., a_k \in \mathbb{C}$ *and both sums converge in the norm of* $L^2(|r|)$.

Because of Theorem 3.6 Proposition 4.1 contains the full-range expansions considered in [16, 17].

4.2. Let

$$L^2(r) = \mathcal{K}_+\ [\dot{+}]\ \mathcal{K}_- \tag{4.4}$$

be the fundamental decomposition corresponding to the fundamental symmetry J of (2.7). Then, with the sets $\Delta_\pm$ of (2.4)

$$\mathcal{K}_\pm = \{f \in L^2(r): f\chi_{\Delta_\mp} = 0 \text{ a.e. on } (a, b)\} = L^2(\Delta_\pm; r).$$

Denote by $P_\pm$ the orthogonal projections onto $\mathcal{K}_\pm$ in $L^2(r)$. Then $(P_\pm f)(x) = 0$ $(x \in \Delta_\mp)$ and $(P_+ f)(x) = f(x)$ $(x \in \Delta_\pm)$ for $f \in L^2(r)$.

PROPOSITION 4.2. *Let* $\mathcal{L}_+ (\mathcal{L}_-)$ *be a nonnegative (nonpositive) subspace of* $(\mathcal{K}_c, [\cdot,\cdot])$ *such that*

$$\dim \mathcal{L}_+ = \kappa_+(\mathcal{K}_c, [\cdot,\cdot]) =: \kappa_+ \qquad (\dim \mathcal{L}_- = \kappa_-(\mathcal{K}_c; [\cdot,\cdot]) =: \kappa_-)$$

and let $g_1^+, ..., g_{\kappa_+}^+$ $(g_1^-, ..., g_{\kappa_-}^-)$ *be a basis of* $\mathscr{L}_+$ $(\mathscr{L}_-$, *resp.*). *Then the functions*

$$P_+ g_1^+, ..., P_+ g_{\kappa_+}^+ \quad \textit{and} \quad P_+ e_j^+, \quad j=1, 2, ..., \tag{4.5}$$

$$(P_- g_1^-, ..., P_- g_{\kappa_-}^- \quad \textit{and} \quad P_- e_j^-, \quad j=1, 2, ...), \tag{4.6}$$

form a Riesz basis of $\mathscr{K}_+ = L^2(\Delta_+; r)$ $(\mathscr{K}_- = L^2(\Delta_-; -r)$, *resp.*).

We mention that according to a finite-dimensional version of a theorem of L. S. Pontrjagin the subspaces $\mathscr{L}_+$ and $\mathscr{L}_-$ can be chosen such that they are invariant under A. Hence also $g_1^+, ..., g_{\kappa_+}^+$ and $g_1^-, ..., g_{\kappa_-}^-$ can be chosen as root functions of A (corresponding to the critical points or to nonreal eigenvalues of A). Here, of course, the functions g_j^+ need not be linearly independent of g_j^-.

Proof of Proposition 4.2. The subspace $\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+$ is maximal non-negative closed subspace of $L^2(r)$ [4]. It follows that $P_+(\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+) = \mathscr{K}_+$ and the restriction $P_1 := P_+ \mid (\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+)$ is a bounded and boundedly invertible operator between the Hilbert spaces $(\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+, (\cdot, \cdot))$ and $(\mathscr{K}_+, (\cdot, \cdot))$ [4, Lemma IV.7.1]. In the same way as in the proof of Proposition 4.1 we find that $\{g_1^+, ..., g_{\kappa_+}^+, e_j^+, j=1, 2, ...\}$ is a Riesz basis in $\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+$. Since a bounded and boundedly invertible operator maps a Riesz basis onto a Riesz basis, the basis (4.5) is a Riesz basis of $\mathscr{K}_+ = L^2(\Delta_+; r)$. The proof of the second part of Proposition 4.2 is analogous.

It follows from (4.3) and the proof of Proposition 4.2 that the unique expansion of the function $f_+ \in \mathscr{K}_+ = L^2(\Delta_+; r)$ with respect to the Riesz basis (4.5) has the form

$$f_+ = \sum_{j=1}^{\kappa_+} a_j P_+ g_j^+ + \sum_{j=1}^{+\infty} \frac{[P_1^{-1} f_+, e_j^+]}{[e_j^+, e_j^+]} P_+ e_j^+, \tag{4.7}$$

with $a_1, ..., a_{\kappa_+} \in \mathbb{C}$, depending on f_+, and the sum converges in $L^2(\Delta_+; r)$. If K_+ denotes the angular operator of $\mathscr{L}_+ \,[\dot{+}]\, \mathscr{M}_+$ with respect to the decomposition (4.4) we have $P_1^{-1} f_+ = f_+ + K_+ f_+$ $(f_+ \in \mathscr{K}_+ = L^2(\Delta_+; r))$. The expansions of the functions $f_- \in \mathscr{K}_- = L^2(\Delta_-; -r)$ with respect to the Riesz basis (4.6) are analogous.

On account of Theorem 3.6 Proposition 4.2 generalizes the half-range expansions considered in [3, 16, 17].

Remark 4.3. Propositions 4.1 and 4.2 of this section can easily be formulated for an arbitrary definitizable operator A in a Krein space which has no singular critical points: $c_s(A) = \varnothing$. This holds true if A also has a continuous spectrum. Then, of course, the sums in the expansions (4.3) and (4.7) have to be replaced by integrals.

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