PERGAMON

Applied Mathematics Letters 16 (2003) 847–851

Applied Mathematics Letters

www.elsevier.com/locate/aml

Comments on the Feedback Stabilization for Bilinear Control Systems

H. Bounit
Ibnou Zohr University
Department of Mathematics
Faculty of Sciences, BP. S/28
Agadir 80000, Morocco
bounith@caramail.com

(Received July 2002; revised and accepted October 2002)

Abstract—Necessary and sufficient condition are given for state feedback stabilization of distributed control bilinear systems. The results are based on the theory of linear and nonlinear contraction semigroups.

Keywords—Infinite-dimensional systems, Semigroups of contractions, Canonical decomposition, Bounded feedback stabilization.

1. INTRODUCTION

Consider the evolutionary bilinear control system

$$\dot{x}(t) = Ax(t) + u(t)(Bu(t) + b), \qquad x(0) = x_0, \tag{1}$$

where the state $x(.)$ takes the value in the separable real Hilbert space H and the control function $u(.)$ is $\mathbb{R}$-valued and $b \in H$. Here $A : D(A) \longrightarrow H$ is a linear closed densely defined operator on a Hilbert space H, B is a bounded linear operator from H into itself. $\langle .,. \rangle$ and $\|.\|$ are, respectively, the usual inner product and the norm of H. The norm of the linear operator on H will be denoted also by $\|.\|$ for convenience and if there is no ambiguity.

We make the following assumptions regarding the unbounded operator A.

(A_1) A generates a C_0-semigroup of contractions (e^{tA}) on H;
(A_2) $R_\lambda(A) = (\lambda I - A)^{-1}$ is compact for some $\lambda > 0$.

In this work, we continue the study the problem of global feedback stabilization of systems (1). We give a necessary and sufficient condition for system (1) to be strongly stabilizable by an *a priori* bounded control.

Definition 1. *System (1) is said to be globally asymptotically strongly stabilizable (in short GASS) if there exists a continuous feedback control $u(.) : H \longrightarrow \mathbb{R}$ such that the following hold.*

(1) *System (1) with $u(t) = u(x(t))$ has a unique weak solution $x(t)$ defined on $\mathbb{R}^+$.*

0893-9659/03/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(03)00103-4

(2) *The origin is a Liapunov stable equilibrium point.*
(3) *The origin attracts every point in H (i.e.,* $\|x(t)\| \stackrel{t\to+\infty}{\longrightarrow} 0$*, whenever* $x_0 \in H$*).*

Then, for the system (1) with the feedback $u(t) = u(x(t))$, there exists a mild solution of the following form:

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}u(x(s))(Bx(s) + b)\,ds.$$

In the sequel of the paper, we use the following sets:

- $M = \{x \in H;\ \langle Be^{\tau A}x + b, e^{\tau A}x\rangle = 0 \text{ for every } \tau \in \mathbb{R}^+\}$,
- $N = \{x \in D(A);\ \langle Ae^{\tau A}x, e^{\tau A}x\rangle = 0 \text{ for every } \tau \in \mathbb{R}^+\}$.

Our main results in this paper are essentially based on the following canonical decomposition of a contraction semigroup due to Foguel [1] and Foias-Nagy [2] and the asymptotic behavior of nonlinear contraction semigroups due to Dafermos and Slemrod [3].

THEOREM 1. *(See [1].) Let* $(e^{tA})_{t\geq 0}$ *be a* C_0 *contraction semigroup on H. Then H can be decomposed into an orthogonal sum* $H = H_u \oplus H_{cnu}$ *where the subspaces* H_u *and* H_{cnu} *are* e^{tA}*-invariant. Moreover:*

(1) *the restriction of* e^{tA} *to* H_u *is a unitary semigroup,*
(2) *the restriction of* e^{tA} *to* H_{cnu} *is a completely nonunitary semigroup and weakly stable,*
(3) *the decomposition is unique and* H_u *can be characterized by*

$$H_u = \left\{x \in H;\ \left\|e^{tA}x\right\| = \left\|e^{tA^*}x\right\| = \|x\|,\ t \geq 0\right\} = \bar{K}_u, \qquad K_u = H_u \cap D(A).$$

THEOREM 2. *(See [3].) Let* $\mathcal{A}$ *be a maximal monotone operator on H. Assume that* $\mathcal{A}(0) = 0$ *and* $(\lambda I + \mathcal{A})^{-1}$ *is compact for any* $\lambda > 0$*. Then, for every* $x_0 \in H$*, the weak solution denoted by* $x(t, x_0)$ *of the Cauchy problem*

$$\dot{x}(t) = -\mathcal{A}x(t) + d(t), \qquad d(.) \in L^1\left(\mathbb{R}^+, H\right),$$
$$x(0) = x_0,$$

approaches as $t \longrightarrow +\infty$ *a compact subset* $\Omega(x_0)$ *of a sphere* $\{y \in H/\|y - a\| = \sigma\}$*, with* $\sigma \leq \|x_0 - a\| + \|d(.)\|_{L^1(R^+,H)}$*, and a is any element of H s.t.* $\mathcal{A}(a) = 0$*. Furthermore,* $\Omega(x_0)$ *is invariant under the semigroup* $T(t)$ *generated by* $-\mathcal{A}$*. If in addition* $x_0 \in D(\mathcal{A})$*, then the set* $\Omega(x_0)$ *is contained in* $D(\mathcal{A})$*.*

In the finite-dimensional case ($H = \mathbb{R}^n$), when A is antisymmetric, this bilinear control problem has been studied by Slemrod [4], and he showed that the so-called ad-condition $M = \{0\}$ is sufficient for the stabilization of system (1).

For a real Hilbert space, the authors in [5] have already studied the strong stabilization problem of systems (1) using the theory of nonlinear contraction semigroup [3] and LaSalle's invariance principle. It has been shown that condition (C_1): $M \cap N = \{0\}$ is sufficient for the GASS by the feedback law $u_r(x) = -r\langle Bx + b, x\rangle$ if either the semigroup $(e^{tA})_{t\geq 0}$ or the resolvent $R_\lambda(A)$ is compact on H.

For the complex Hilbert space and if $(e^{tA})_{t>0}$ is compact, based on Theorem 1 listed above the authors in [6] prove that condition (C_2): $M \cap H_u = \{0\}$ which is weaker than condition (C_1) is sufficient for GASS of system (1) by means of the feedback $u_r(x)$. We remark that the compactness of $(e^{tA})_{t>0}$ on a complex Hilbert space H is needed to guarantee that the strong Ω-limit set $\Omega(x_0)$ of the closed-loop system is nonempty. Note that the compactness of $(e^{tA})_{t>0}$ gives a very simple and nice description of H_u and H_{cnu} [7]. Moreover, the authors in [6] use this description and prove that $\Omega(x_0) \subseteq H_u = N \subseteq D(A)$.

The compactness of $(e^{tA})_{t>0}$ used in [6] can be considered a strong one. However, it is satisfied by finite-dimensional systems and a few class of operators (ex. parabolic). Note that if this condition is dropped, then for a general Hilbert space, it is well known that existence of the set $\Omega(x_0)$ associated to the closed-loop system is not guaranteed. Moreover, the set H_u is not necessarily contained in $D(A)$ (even if $(e^{tA})_{t>0}$ is compact on a real Hilbert space) and only $H_u \cap D(A) \subseteq N$.

Of course, the canonical decomposition given in Theorem 1 is still true even for a real Hilbert space. In this work, we are interested in particular in the bounded feedback stabilization (1) (i.e., $|u(x)| \leq r$, $\forall x \in H$) of a certain system of kind (1) and only the assumptions (A_1),(A_2) are assumed to give a weaker condition than (C_2) to resolve the GASS of system (1) by a judicious state feedback law.

2. MAIN RESULTS

We are now ready to present the main results of this paper.

2.1. Homogeneous Case: $b = 0$

PROPOSITION 1. *Assume that B is a self-adjoint dissipative (or positive) operator. Then $M \cap K_u$ is reduced to the origin if and only if system (1) is GASS by the bounded state feedback law*

$$u_r(x) = -\frac{r\langle Bx, x\rangle}{1+\varepsilon\langle Bx, x\rangle}, \qquad \varepsilon = \operatorname{sign}(\langle Bx, x\rangle). \tag{2}$$

PROOF OF PROPOSITION 1. The "if" part. It is obvious to see that the feedback (2) is such that $\sup_{x\in H} |u_r(x)| \leq r$. It is well known that [8] system (1),(2) has a unique weak solution $x(t)$ defined on $\mathbb{R}^+$. Moreover, the nonlinear operator $\mathcal{A} = -A + (r\langle B.,.\rangle/(1+\varepsilon\langle B.,.\rangle))B$ defined on H is a maximal monotone on $D(\mathcal{A}) = D(A)$, so $\overline{D(\mathcal{A})} = H$ and $(\lambda I + \mathcal{A})^{-1}$ is compact for every $\lambda > 0$ [5]. According to the result of Dafermos and Slemrod [3], $(-\mathcal{A})$ generates a nonlinear semigroup $T(t)$ of contraction defined for $t \geq 0$ and the unique mild solution of system (1),(2) given by $x(t, x_0) = T(t)(x_0)$, which satisfies the integral equation

$$T(t)(x_0) = e^{tA}x_0 + \int_0^t e^{(t-s)A}u_r(T(s)(x_0))BT(s)(x_0)\,ds \tag{3}$$

approaches as $t \longrightarrow \infty$ a compact, subset $\Omega(x_0) \subset \{y \in H/\|y\| = \sigma\}$, $\sigma \leq \|x_0\|$, which is $T(t)$-invariant. To end, we must prove that $\Omega(x_0)$ is reduced to the origin. Let $x_0 \in D(A)$. Then $\Omega(x_0) \subseteq D(A)$, the solution $T(t)(x_0)$ is a strong one, and $T(t)x_0$ lies in $D(A)$, $\forall t \geq 0$ [5].

Let $\bar{x} \in \Omega(x_0)$. Since $\Omega(x_0)$ is $T(t)$-invariant $V(t) = (1/2)\|T(t)\bar{x}\|^2 = (1/2)\|\bar{x}\|^2$, $\forall t \in \mathbb{R}^+$. Differentiating $V(t)$ along the trajectories of (1),(2) we obtain

$$\dot{V}(t) = \langle T(t)(\bar{x}), AT(t)(\bar{x})\rangle - \frac{r\langle BT(t)(\bar{x}), T(t)(\bar{x})\rangle^2}{1+\varepsilon\langle BT(t)(\bar{x}), T(t)(\bar{x})\rangle} = 0.$$

Hence,

$$\langle BT(t)(\bar{x}), T(t)(\bar{x})\rangle = 0; \quad \langle T(t)(\bar{x}), AT(t)(\bar{x})\rangle = 0, \qquad \forall t \in \mathbb{R}^+. \tag{4}$$

Using integral equation (3), we deduce that $T(t)(\bar{x}) = e^{tA}\bar{x}$ (i.e., $\Omega(x_0)$ is e^{tA}-invariant). Finally, equation (4) implies that $\bar{x} \in M$ and $\langle e^{tA}\bar{x}, Ae^{tA}\bar{x}\rangle = 0$, $\forall t \in \mathbb{R}^+$. From the fact that $\bar{x} \in D(A)$, we obtain

$$\begin{aligned} &\frac{d\|e^{tA}\bar{x}\|^2}{dt} = \langle e^{tA}\bar{x}, Ae^{tA}\bar{x}\rangle = 0, \qquad \forall t \in \mathbb{R}^+, \\ &\left\|e^{tA}\bar{x}\right\|^2 = \|\bar{x}\|^2. \end{aligned} \tag{5}$$

According to Theorem 1, there exist $\bar{x}_1 \in H_u$ and $\bar{x}_2 \in H_{cnu}$ such that $\bar{x} = \bar{x}_1 + \bar{x}_2$. Since H_u and H_{cnu} are e^{tA}-invariant orthogonal subspaces, a direct computation gives

$$\begin{aligned} \|\bar{x}\|^2 &= \|\bar{x}_1\|^2 + \|\bar{x}_2\|^2, \\ \left\|e^{tA}\bar{x}\right\|^2 &= \left\|e^{tA}\bar{x}_1\right\|^2 + \left\|e^{tA}\bar{x}_2\right\|^2. \end{aligned} \tag{6}$$

Combining (5) with (6), we deduce

$$\left\|e^{tA}\bar{x}_2\right\|^2 = \|\bar{x}_2\|^2. \tag{7}$$

Let $n > 0$ be a sufficiently large integer. The fact that $R_n(A)$ is compact and $\bar{x}_2 \in H_{cnu}$, using (2) in Theorem 1, we have

$$R_n(A)e^{tA}\bar{x}_2 \overset{t\to+\infty}{\longrightarrow} 0. \tag{8}$$

On the other hand,

$$nR_n(A)e^{tA}\bar{x}_2 \overset{n\to+\infty}{\longrightarrow} e^{tA}\bar{x}_2. \tag{9}$$

Since $(e^{tA})_{t\geq 0}$ is a contraction semigroup and $R_n(A) = \int_0^\infty e^{-ns}e^{tA}\,ds$ is such that $R_n(A)e^{tA} = e^{tA}R_n(A)$, then $nR_n(A)e^{tA}\bar{x}_2$ converges to $e^{tA}\bar{x}_2$ as $n \to +\infty$ uniformly in t. It follows that

$$\lim_{n\to+\infty}\left(\lim_{t\to+\infty} nR_n(A)e^{tA}\bar{x}_2\right) = \lim_{t\to+\infty}\left(\lim_{n\to+\infty} nR_n(A)e^{tA}\bar{x}_2\right). \tag{10}$$

According to (8)–(10), we obtain

$$\lim_{t\to+\infty} e^{tA}\bar{x}_2 = 0.$$

This combined with (7) implies that $\bar{x}_2 = 0$. Finally, $\bar{x} = \bar{x}_1 \in K_u \cap M$, which is reduced to the origin. Hence, $\Omega(x_0) = \{0\}$ for all $x_0 \in D(A)$. Since $\overline{D(A)} = H$ and $T(t)$ is a contraction, the triangle inequality and an argument of density show that $\Omega(x_0) = \{0\}$ for all $x_0 \in H$.

The **"only if"** part. Let $\bar{x} \in M \cap K_u$ and $\bar{x}(t)$ be the solution of system (1),(2) emanating from $\bar{x}$ at $t = 0$. It is given by

$$\bar{x}(t) = e^{tA}\bar{x} - \int_0^t e^{(t-s)A}\frac{r\langle B\bar{x}(s),\bar{x}(s)\rangle}{1+\varepsilon\langle B\bar{x}(s),\bar{x}(s)\rangle}B\bar{x}(s)\,ds. \tag{11}$$

But $\bar{x}$ is such that $\langle e^{tA}\bar{x}, Be^{tA}\bar{x}\rangle = 0$ for all $t \geq 0$. It follows that $e^{tA}\bar{x}$ is also a solution of integral equation (11). Uniqueness of solution implies that $\bar{x}(t) = e^{tA}\bar{x}$. Due to the fact that $\bar{x} \in K_u$ and system (1),(2) is strongly stable, we have $\|\bar{x}\| = \|e^{tA}\bar{x}\| = \|\bar{x}(t)\| \overset{t\to+\infty}{\longrightarrow} 0$. This ends the proof of the proposition. ∎

2.2. Linear Case: $B = 0$

PROPOSITION 2. *$M \cap K_u$ is reduced to the origin if and only if system (1) is GASS by the bounded state feedback law*

$$u_r(x) = -\frac{r\langle b, x\rangle}{1+|\langle b,x\rangle|}. \tag{12}$$

PROOF OF PROPOSITION 2. The **"if"** part. The authors [9] showed that the nonlinear operator $\mathcal{A} = -A + r(\langle b,.\rangle/(1+|\langle b,.\rangle|))b$ is maximal monotone and $(\lambda I + \mathcal{A})^{-1}$ is compact on H. As the reader can see, the rest of the proof can be obtained by applying the same reasoning as in Proposition 1; it is omitted.

The **"only if"** part. Let $\bar{x} \in M \cap K_u$ and $\bar{x}(t)$ be the solution of system (1)–(12) emanating from $\bar{x}$ at $t = 0$. It is given by

$$\bar{x}(t) = e^{tA}\bar{x} - r\int_0^t e^{(t-s)A}\frac{\langle b,\bar{x}(s)\rangle}{1+|\langle b,\bar{x}(s)\rangle|}b\,ds. \tag{13}$$

But $\bar{x}$ is such that $\langle b, e^{tA}\bar{x}\rangle = 0$ for all $t \geq 0$. It follows that $e^{tA}\bar{x}$ is also a solution of integral equation (13). Uniqueness of solution implies that $\bar{x}(t) = e^{tA}\bar{x}$. Since $\bar{x} \in K_u$, then $\|e^{tA}\bar{x}\|$ is constant. The strong stability of system (1)–(12) gives $\|\bar{x}\| = \|e^{tA}\bar{x}\| = \|\bar{x}(t)\| \overset{t\to+\infty}{\longrightarrow} 0$. This ends the proof of Proposition 2. ∎

REMARK. We note that [9] for a general linear control system when b is replaced by a bounded linear operator mapping another real Hilbert space U of control into H, the affirmation analogous to Proposition 2 takes place. The problem of feedback stabilization with an *a priori* bounded control of linear control system has been initially studied by Slemrod [10] by using the energy stability method, while an earlier paper [11] treated a related of suboptimal control.

2.3. Nonhomogeneous Case

PROPOSITION 3. *Assume that $(e^{tA})_{t>0}$ is compact. Then $M \cap H_u$ is reduced to the origin if and only if system (1) is GASS by the bounded state feedback law*

$$u_r(x) = -\frac{r\langle Bx+b, x\rangle}{1+\langle Bx+b, x\rangle^2}. \tag{14}$$

PROOF OF PROPOSITION 3. The "if" part. Let $x_0 \in H$ and $x(t)$ be the solution of system (1)–(14) emanating from x_0 at $t = 0$. According to Lemma 5.5 [12], it is easy to verify that the solution is bounded on H. Moreover, for all $t_1 \geq t_2 \|x(t_1)\| \leq \|x(t_2)\|$. Because $(e^{tA})_{t>0}$ is compact on H, applying Theorem 4.1 [13] the authors in [5] have shown that the Ω-limit set $\Omega(x_0)$ associated to system (1)–(14) is nonempty, e^{tA}-invariant, and $\Omega(x_0) \subseteq M$. Now let us prove that $\Omega \subseteq H_u$. From the fact that $\varphi(t) = \|x(t)\|$ is a decreasing nonnegative function, it follows that

$$\exists\, c \geq 0, \qquad \lim_{t\to+\infty} \varphi(t) = c. \tag{15}$$

Let $\bar{x} \in \Omega(x_0)$. On the contrary to the complex case, we remark that even if $x_0 \in D(A)$ and $(e^{tA})_{t>0}$ is compact, do not assure that $\Omega(x_0) \subseteq D(A)$. By definition of $\Omega(x_0)$, we obtain

$$\exists\, t_n \overset{n\to+\infty}{\longrightarrow} +\infty, \qquad \lim_{n\to+\infty} x(t_n) = \bar{x}. \tag{16}$$

Combining (15) and (16) and the fact that $\Omega(x_0)$ is e^{tA}-invariant, we deduce $\|e^{tA}\bar{x}\| = \|\bar{x}\| = c$ for all $t \geq 0$. To finish the proof, it is sufficient to refer to Proposition 1. It is proved.

The "only if" part. It is obtained in a similar way as above. This ends the proof of Proposition 3. ∎

REFERENCES

1. S.R. Foguel, Powers of a contraction in a Hilbert space, *Pacific J. Math* **13**, 551–562, (1963).
2. B.Sz. Nagy and C. Foias, *Analyse Harmonique des Operateurs de L'espace de Hilbert*, Masson et Cie, Akadi MiaiKiads, Budapest, (1967).
3. C.M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contractions semigroups, *J. Funct. Analy.* **13**, 97–106, (1973).
4. M. Slemrod, Stabilization of bilinear control systems with application to conservative problems in elasticity, *SIAM. J. Control and Optim.* **16**, 131–141, (1978).
5. H. Bounit and H. Hammouri, Feedback stabilization for a class of semilinear distributed system, *Nonlinear Analysis: Theory, Methods and Applications* **37**, 953–969, (1999).
6. L. Berrahmoune, Y. El Boukfaoui and M. Erraoui, Remarks on the feedback stabilization of systems affine in control, *European J. Contr.* **7** (1), (2001).
7. K.J. Engel and R. Nagel, *One Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, (2000).
8. J.M. Ball and M. Slemrod, Feedback stabilization of distributed semilinear control systems, *App. Math. Optim.* **5**, 169–179, (1979).
9. H. Bounit and H. Hammouri, Bounded feedback stabilization and global separation principle of distributed parameter systems, *IEEE Trans. Automat. Contr.* **42** (3), (1997).
10. M. Slemrod, Feedback stabilization of a linear control system in Hilbert space with an *a priori* bounded control, *Math. Control Signals Systems* **2**, 265–285, (1989).
11. M. Slemrod, An application of maximal dissipative sets in control theory, *J. Math. Analy. Appl.* **46**, 369–387, (1974).
12. J.M. Ball, On the asymptotic behavior of generalized processes with application to nonlinear evolutions equations, *J. Differential Equations* **27**, 224–265, (1978).
13. J.M. Ball, A class of semilinear equations of evolution, *Israel. J. Math.* **20**, 23–36, (1975).
14. J.M. Ball, Strongly continuous semi-groups, weak solutions and the variation of constants formula, In *Proc. Amer. Soc.*, Volume 63, pp. 370–373, (1977).