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Jones index theory for Hilbert C^*-bimodules and its equivalence with conjugation theory

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Abstract

We introduce the notion of finite right (or left) numerical index on a C^*-bimodule ${}_AX_B$ with a bi-Hilbertian structure, based on a Pimsner–Popa-type inequality. The right index of X can be constructed in the centre of the enveloping von Neumann algebra of A . The bimodule X is called of finite right index if the right index lies in the multiplier algebra of A. In this case the Jones basic construction enjoys nice properties. The C^*-algebra of bimodule mappings with a right adjoint is a continuous field of finite dimensional C^*-algebras over a compact Hausdorff space, whose fiber dimensions are bounded above by the index. If A is unital, the right index belongs to A if and only if X is finitely generated as a right module. A finite index bimodule is a bi-Hilbertian C^*-bimodule which is at the same time of finite right and left index.

Bi-Hilbertian, finite index C^*-bimodules, when regarded as objects of the tensor 2-C^*-category of right Hilbertian C^*-bimodules, are precisely those objects with a conjugate in the same category, in the sense of Longo and Roberts.
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0. Introduction

The theory of conjugation in abstract tensor C^*-categories appeared in the algebraic formulation of Quantum Field Theory [H].

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In this connection, Doplicher and Roberts showed in [DR1], [DR2] that any symmetric tensor C^*-category with conjugation can be embedded into a category of finite dimensional Hilbert spaces, and therefore the category is isomorphic to the representation category of a compact group.

However, some tensor C^*-categories with a unitary braiding, arising from low dimensional QFT, can not be embedded into categories of Hilbert spaces [LR].

Longo and Roberts studied in [LR] conjugation in tensor C^*-categories, and they showed that this notion is closely related to the Jones index theory for subfactors [J].

Yamagami showed in [Y] that a tensor C^*-category with simple unit object and countably many generators can be realized as a category of von Neumann algebra bimodules of finite Jones index if and only if it is rigid.

An interesting open problem is to decide which tensor C^*-categories with conjugation can be embedded into categories of Hilbert C^*-bimodules. A related problem is to ask which sort of bimodules should appear. In this paper we solve the latter problem.

The paper naturally splits into two parts. In the first part we introduce Jones index theory for general Hilbert bimodules over pairs of C^*-algebras, while in the second part we show its equivalence with conjugation theory.

In [KW1] the first and third-named authors studied Hilbert C^*-bimodules with finite Jones index in the case where the C^*-algebras are unital and the bimodules are finitely generated as right as well as left modules.

If A and B are C^*-algebras, an object of our category is a right Hilbert B-module X with an action of A on the left given by a nondegenerate *-homomorphism of A into the C^*-algebra of B-module maps of X into itself with adjoint. We will refer to such a bimodule as a *right* Hilbert A–B C^*-bimodule. The space of intertwiners from ${}_AX_B$ to ${}_AY_B$ is the set of adjointable bimodule maps.

A C^*-bimodule ${}_AX_B$ is called bi-Hilbertian if it is at the same time a right and a left Hilbert C^*-bimodule in such a way that the two Banach space norms arising from the two inner products are equivalent. Examples are Rieffel's imprimitivity bimodules [R1].

A bi-Hilbertian C^*-bimodule will be called of *finite right* (*or left*) *numerical index* if a suitable Pimsner–Popa-type [Pim,PiPo] inequality relating the two Banach space norms holds (see Definitions 2.8 and 2.9). A bimodule of finite numerical index is a bimodule which is at the same time of finite right and left numerical index.

If X is of finite right (or left) numerical index, we construct the right (or left) index element of X as a positive central element of A'' (or B'').

While the tensor product of two imprimitivity bimodules is still an imprimitivity bimodule, a tensor product of bi-Hilbertian bimodules cannot be made, in general, into a bi-Hilbertian bimodule in the natural way. However, if ${}_AX_B$ has finite right numerical index and ${}_BY_C$ has finite left numerical index then the algebraic tensor product bimodule $X \odot_B Y$ can be completed into a bi-Hilbertian bimodule in the natural way (Proposition 2.13).

Typical examples of bimodules of finite right numerical index arise from conditional expectations between C^*-algebras satisfying a Pimsner–Popa inequality. The work of Frank and Kirchberg [FK] shows that under this only assumption the index element of the conditional expectation lies in the enveloping von Neumann algebra of the bigger algebra. This reflects the fact that the Jones basic construction is not always possible in the C^*-algebraic setting.

We introduce in our theory an extra requirement: the right index element of ${}_AX_B$ should lie in the multiplier algebra of A, and therefore in its centre. When this assumption is satisfied, we say that X is of finite right index.

We prove that this property is in fact equivalent to other properties which would seem stronger a priori, such as, e.g., the fact that the left action of A on X has range into the compacts $\mathscr{K}(X_B)$ (Theorem 2.22).

Now this assumption guarantees the existence of the Jones basic construction (see Theorem 2.29), which takes the form of a positive, A-bilinear, strictly continuous map $F : \mathscr{K}(X_B) \to A$ satisfying a Pimsner–Popa inequality. Since left A-action lies in $\mathscr{K}(X_B)$, F extends uniquely to a A-bilinear map $\hat{F} : \mathscr{L}(X_B) \to M(A)$ between the corresponding multiplier algebras. The right index element of X coincides with $\hat{F}(I)$ and it can be reached by the strict limit of the image under F of an approximate unit of $\mathscr{K}(X_B)$.

Bases are a useful tool in Jones index theory, as they lead to a simple formula for the index element. Izumi proved in [I] that a conditional expectation satisfying the Pimsner–Popa inequality from a simple, unital C^*-algebra admits a finite quasi-basis in the sense of [W]. We generalize Izumi's result to C^*-bimodules ${}_AX_B$ of finite right numerical index: if A is unital, the right index element of X belongs A if and only if X is finitely generated as a right B-module. More generally, we show that bimodules with finite right index over σ-unital C^*-algebras admit countable bases (Corollary 2.24). In the general case we shall deal with *generalized bases* in the sense of Definition 1.3, which always exist, as shown in Proposition 1.4.

We illustrate our approach to index theory with a typical example of an inclusion of commutative unital C^*-algebras satisfying a Pimsner–Popa inequality, for which a finite quasi–basis in the sense of [W] does not exist. This class of examples arises from branched coverings, or orbifolds. It was first pointed out in [W, Section 2.8] and later analyzed by Frank and Kirchberg in [FK]. We show that this inclusion is in fact determined by a canonical nonunital subinclusion of finite right index in our sense (cf. Example 2.32).

Let us go back to our aim of comparing Jones index theory for Hilbert bimodules with conjugation theory. One of the main result of this paper is that these two approaches are equivalent (cf. Theorems 4.4 and 4.13). We show that a bi-Hilbertian C^*-bimodule has finite Jones index in our sense if and only if it has a conjugate object in the 2-C^*-category of right Hilbert C^*-bimodules with nondegenerate left actions.

Imprimitivity bimodules are finite index bimodules, with left and right index elements equal to the identities. They can be characterized, among general right Hilbert C^*-bimodules, as those objects with trivial minimal dimension (Corollary 4.14).

We show two applications of our characterization theorem. The first one is that if ${}_AX_B$ is of finite index, the set of Hilbert module mappings (i.e. those with an adjoint) on X_B commuting with the left action coincides, as an algebra, with the set of Hilbert module mappings on ${}_AX$ commuting with the right action (Corollary 4.6). Moreover, each one of these C^*-algebras is a continuous bundle of finite dimensional C^*-algebras over a compact space, in the sense of [KW] (Theorem 3.3).

As a second application, we show that the tensor product of two bi-Hilbertian bimodules of finite (resp. numerical) index is still of finite (resp. numerical) index (Theorem 5.1).

An index theory for Hilbert bimodules turns out to be more general than for conditional expectations in the case where the algebras are not σ-unital. In fact, it is known that if $E : B \to A$ is a conditional expectation satisfying a Pimsner–Popa inequality, an approximate unit of A must be an approximate unit of B as well. Therefore if A is σ-unital, B must be σ-unital as well. In particular, a unital C^*-algebra and a non-σ-unital one cannot be linked by a conditional expectation satisfying a Pimsner–Popa inequality. However, a II_1 factor can be strongly Morita equivalent to a non-σ-unital C^*-algebra (see [BGR]).

In Section 6 we will discuss further examples of bimodules of finite index arising from locally finite directed graphs and topological correspondences.

This paper is an extended version of an appendix contained in the draft of [KPW1].

1. Countable bases and generalized bases

Let A be a C^*-algebra and $X = X_A$ a right Hilbert C^*-module over A. We denote by $\mathscr{L}(X_A)$ the C^*-algebra of A-module maps on X with an adjoint.

A finite subset $\{u_i\}_i$ of X is called a *finite basis* if $x = \sum_i u_i(u_i|x)_A$ for $x \in X$. Our aim in this section is to generalize this notion to comprehend *countable bases* or, more generally, *generalized bases*. These are infinite bases, and they will be a good substitute of finite bases in the case where the finite generation property does not hold.

We denote by $\theta^r_{x,y}$ the *rank one operator* on X defined by $\theta^r_{x,y}(z) = x(y|z)_A$. The linear span of rank one operators is denoted by $FR(X_A)$ and called the ideal of *finite rank operators*. Its norm closure, $\mathscr{K}(X_A)$, is the C^*-algebra of *compact operators*, which is a closed ideal in $\mathscr{L}(X_A)$.

For a left Hilbert A-module X, we define the rank one operators by $\theta^l_{x,y}(z) = {}_A(z|y)x$, and the spaces of finite rank operators $FR({}_AX)$, compact operators $\mathscr{K}({}_AX)$ and adjointable left A-module maps $\mathscr{L}({}_AX)$ are defined similarly.

The right Hilbert module X_A has a finite basis if and only if $\mathscr{L}(X_A) = \mathscr{K}(X_A)$. If in addition A is unital, $\mathscr{L}(X_A) = \mathscr{K}(X_A)$ if and only if X_A is finite projective as a right module.

Definition 1.1. Let X be a right Hilbert A-module. We say that a sequence $\{u_i\}_{i \in \mathbb{N}} \subset X$ is a (*right*) *countable basis* for X if for any $x \in X$, $x = \sum_1^\infty u_n(u_n|x)_A$ in norm.

One can easily show that for $i \in \mathbb{N}$, $||u_i|| \leqslant 1$, and for any finite subset F of $\mathbb{N}$, $||\sum_{i \in F} u_i (u_i|x)_A|| \leqslant ||x||$. Furthermore, the sequence $(\sum_1^n \theta^r_{u_i,u_i})_{n \in \mathbb{N}}$ is an approximate unit for $\mathscr{K}(X_A)$.

Remark. Our notion of basis corresponds to the notion of *standard normalized tight frame* given by Franks and Larson [FL1], [FL2].

The following fact is an immediate consequence of Kasparov's stabilization trick.

Proposition 1.2. *Let X be a countably generated right Hilbert C^*-module over a σ-unital C^*-algebra A. Then X has a countable basis.*

Remark. The referee kindly pointed out to us that a countable right basis $\{u_i\}$ always converges unconditionally (in the sense that for any $x \in X$, the net associating $\sum_{i \in F} u_i (u_i|x)_A$ to each finite subset $F \subset \mathbb{N}$ is norm converging to x), as an easy consequence of the following estimates: for every $x \in X_A$, $a, b \in \mathscr{K}(X_A)$, with $0 \leqslant a \leqslant b \leqslant I$,

$$||x - bx||^2 = ||(x|(I-b)^2 x)_A|| \leqslant ||(x|(I-b)x)_A|| \leqslant ||(x|(I-a)x)_A|| \leqslant ||x||||x - ax||.$$

In the case where the right Hilbert A-module X is not countably generated, countable bases will be replaced in the sequel by *generalized bases*, in the following sense.

Definition 1.3. Consider a set Λ and, for each finite subset $\mu \subset \Lambda$, let u_μ be a finite subset of X with the same cardinality as μ. Let us endow the set of finite subsets of Λ with the partial order defined by inclusion. The net $\mu \to u_\mu$ will be called a *generalized* (*right*) *basis* of X if $\mu \to T_\mu := \sum_{y \in u_\mu} \theta^r_{y,y}$ is an increasing approximate unit of $\mathscr{K}(X_A)$ with norm $\leqslant 1$.

Proposition 1.4. *Any right* (*or left*) *Hilbert C^*-module X admits a generalized right* (*or left*) *basis.*

Proof. This is essentially the proof of existence of an approximate unit of a C^*-algebra with entries in a dense ideal (cf. Proposition 1.7.2 in [Di]). Let Λ be X as a set, and, for each finite subset $\mu = \{x_1, \ldots, x_n\}$ of Λ let $u_\mu = \{(1/n + \sum_1^n \theta_{x_j,x_j})^{-1/2} x_i$, $i = 1, \ldots, n\}$. Then the proof of the cited result shows that the net $T_\mu := \sum_{y \in u_\mu} \theta_{y,y} = (\sum_1^n \theta_{x_j,x_j})(1/n + \sum_1^n \theta_{x_j,x_j})^{-1}$ is increasing, with norm $\leqslant 1$ and satisfies $\sum_1^n (T_\mu - I)\theta_{x_i,x_i}(T_\mu - I) \leqslant \frac{1}{4n}$. So for $i = 1, \ldots, n$, $||(T_\mu - I)x_i||^2 \leqslant ||\sum_1^n (T_\mu - I)\theta_{x_i,x_i} (T_\mu - I)|| \leqslant \frac{1}{4n}$. $\square$

2. Bimodules of finite index

In this section we study the notion of C^*-bimodules of finite right index. We start with a weak notion of finite index, based only on a Pimsner–Popa-type inequality, and we construct the right index element of ${}_AX_B$ as a positive central element in the enveloping von Neumann algebra A''. Later on we shall concentrate on those bimodules for which the index element lies in the multiplier algebra of A, and we perform, in this case, the analogue of the Jones basic construction with nice properties. We also prove that in the case where A is unital, the index element belongs to A if and only if X is finitely generated as a right B-module. This result will be stated in a more general form, which includes the non unital case.

2.1. Bimodules of finite right numerical index

Definition 2.1. Let A and B be C^*-algebras and $X = {}_AX_B$ a bimodule over the complex algebras A and B. We say that X is a *right Hilbert A–B bimodule* if

(1) X, as a right B-module, is endowed with a B-valued inner product making it into a right Hilbert B-module,
(2) for all $a \in A$, the map $\phi(a) : x \in X \mapsto ax \in X$ is adjointable, with adjoint $\phi(a)^* = \phi(a^*)$.

Therefore $\phi : a \in A \to \phi(a) \in \mathscr{L}(X_B)$ is a *-homomorphism from A to the algebra $\mathscr{L}(X_B)$ of right adjointable maps on X_B. The map ϕ will be referred to as the left action of A on X.

We introduce the notion of a *left* Hilbert A–B bimodule in a similar manner. Thus, if X is a left Hilbert A–B bimodule, the map $\psi : B \to \mathscr{L}({}_AX)$, $\psi(b) : x \in X \mapsto xb \in X$, for all $b \in B$, and referred to as the right action of B on X, is a *-antihomomorphism from B to the algebra $\mathscr{L}({}_AX)$ of left adjointable maps on ${}_AX$.

Notice that left and right actions on a right (or left) Hilbert bimodule are not assumed to be faithful. The following proposition gives a sufficient condition. Recall that a closed ideal J in a C^*-algebra B is called *essential* if each nonzero closed ideal of B has a nonzero intersection with J (see 3.12.7 in [P]).

Proposition 2.2. *Let X be a right pre-Hilbert B-module* (*resp. left pre-Hilbert A-module*)*. If the closed linear span in B* (*resp. A*) *of inner products $(x|y)_B$* (*resp. ${}_A(x|y)$*) *$x, y \in X$ is an essential ideal of B* (*resp. A*), *the equation $Xb = 0$ for some $b \in B$* (*resp. $aX = 0$ for some $a \in A$*) *implies $b = 0$* (*resp. $a = 0$*).

Definition 2.3. A A–B bimodule ${}_AX_B$ will be called *bi-Hilbertian* if it is endowed with a right as well as a left Hilbert A–B C^*-bimodule structure in such a way that the two Banach space norms arising from the two inner products are equivalent. Thus there exist constants λ, $\lambda' > 0$ such that, for $x \in X$,

$$\lambda' ||(x|x)_B|| \leqslant ||{}_A(x|x)|| \leqslant \lambda ||(x|x)_B||.$$

The inequality at the left-hand side always extends to finite sums, in the sense of the following proposition.

Proposition 2.4. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule, and let $\lambda' > 0$ satisfy $\lambda'||(x|x)_B|| \leqslant ||_A(x|x)||$, $x \in X$. Then for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in X$ we have*

$$\lambda' \left|\left| \sum_1^n \theta^r_{x_i,x_i} \right|\right| \leqslant \left|\left| \sum_1^n {}_A(x_i|x_i) \right|\right|.$$

Proof. Let $T \in M_n(B)$ be the positive matrix whose (i,j)th entry is $(x_i|x_j)_B$. Notice that by Lemma 2.1 in [KPW1], $||\sum_1^n \theta^r_{x_i,x_i}|| = ||T||$, which, in turn, equals the supremum of $||\sum_{i,j=1}^n b_i^*(x_i|x_j)_B b_j||$ over all the n-tuples $(b_1, \ldots, b_n)$ with elements in B such that $||\sum_j b_j^* b_j|| = 1$. Now the norm at the right hand side coincides with the norm of $(y|y)_B$, where $y = \sum_j x_j b_j$, therefore

$$\lambda'||(y|y)_B|| \leqslant ||_A(y|y)|| = \left|\left| \sum_{i,j} {}_A(x_i b_i b_j^*|x_j) \right|\right| \leqslant \left|\left| \sum_i {}_A(x_i|x_i) \right|\right|,$$

and the proof is now complete. $\square$

On the contrary, there may exist no $\lambda > 0$ for which $||\sum_A(x_i|x_i)|| \leqslant \lambda||\sum \theta^r_{x_i,x_i}||$ for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$, as the following elementary example shows.

Example 2.5. Let $A = B = \mathbb{C}$ and let $H = \ell^2(\mathbb{N})$ be an infinite dimensional Hilbert space, regarded as a bi-Hilbertian $\mathbb{C}$–$\mathbb{C}$ bimodule in the natural way. Let $e_1, e_2, \ldots$ be a countable orthonormal subset of H. Then for all $n \in \mathbb{N}$, $||\sum_{i=1}^n \theta^r_{e_i,e_i}|| = 1$, while $||\sum_{i=1}^n {}_{\mathbb{C}}(e_i|e_i)|| = n$.

In fact, the existence of such a constant λ will lead us to the notion of *finite right numerical index* of X. We anticipate a lemma.

Lemma 2.6. *Let ${}_AX_B$ be a right Hilbert C^*-bimodule, and let $x, y \in X \to {}_A(x|y)$ be an A-valued, biadditive, left A-linear, right A-antilinear form on X such that ${}_A(x|y)^* = {}_A(y|x)$ and ${}_A(x|x) \geqslant 0$ for all $x, y \in X$. If this form is continuous, in the sense that there is $\lambda > 0$ such that $||_A(x|x)|| \leqslant \lambda||(x|x)_B||$ for $x \in X$, and if the right B-action is adjointable with respect to this form (i.e. ${}_A(xb|y) = {}_A(x|yb^*)$, $x, y \in X$, $b \in B$), there exists a unique additive map $F : FR(X_B) \to A$ such that $F(\theta^r_{x,y}) = {}_A(x|y)$. F satisfies the following properties:*

1. *$F(T^*T) \geqslant 0$, for $T \in FR(X_B)$,*
2. *$F(T^*) = F(T)^*$, for $T \in FR(X_B)$,*
3. *$F(\phi(a)T) = aF(T)$, $F(T\phi(a)) = F(T)a$ for $a \in A$, $T \in FR(X_B)$,*
4. *if X is bi-Hilbertian and if $\lambda' > 0$ satisfies $\lambda'||(x|x)_B|| \leqslant ||_A(x|x)||$ for all $x \in X$ then $||F(T)|| \geqslant \lambda'||T||$ for any $T \in FR(X)$ that can be written as a finite sum of operators of the form $\theta^r_{x,x}$.*

Proof. Uniqueness is obvious. Let $\mu \subset \Lambda \to u_\mu$ be a generalized right basis of the right Hilbert module X_B, which exists by Proposition 1.4. Consider the linear map $F_\mu : T \in \mathscr{L}(X_B) \to \sum_{y \in u_\mu} {}_A(Ty|y) \in A$. Note that

$$F_\mu(\theta^r_{x,z}) = \sum_{y \in u_\mu} {}_A(x(z|y)_B|y) = {}_A\left(x \middle| \sum_{y \in u_\mu} y(y|z)_B\right).$$

Since $\lim_\mu \sum_{y \in u_\mu} y(y|z)_B = z$ in the norm defined by the right inner product, and since $||{}_A(x|x)|| \leqslant \lambda ||(x|x)_B||$, we also have that $\lim_\mu \sum_{y \in u_\mu} y(y|z)_B = z$ in the seminorm defined by the left inner product. Thus $\lim_\mu F_\mu(\theta^r_{x,z}) = {}_A(x|z)$. Let us define F as the pointwise norm limit of the net $\mu \to F_\mu$ on $FR(X_B)$. Obviously this limit does not depend on the generalized right basis. (1) follows from the fact that any element of the form T^*T, with $T \in FR(X)$, can be written as a finite sum of elements of the form $\theta^r_{x,x}$. Properties (2) and (3) are easy to check. (4) follows from Proposition 2.4. $\square$

The map F will be referred to as the *additive extension of the form* ${}_A(\cdot|\cdot)$ to the finite rank operators on X_B.

Notice that a bimodule satisfying the properties of the previous lemma is almost bi-Hilbertian. The only missing properties are the fact that the seminorm coming from the left-linear A-valued form is in fact a norm, and completeness of X with respect to this norm.

Proposition 2.7. *Let X be a right Hilbert A–B C^*-bimodule and let $x, y \to {}_A(x|y)$ be an A-valued form on X satisfying the same properties as in the previous lemma* (*with left seminorm not necessarily a Banach space norm*). *Then following properties are equivalent.*

(1) *There exists $\lambda > 0$ such that for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in X$,*

$$\left\| \sum_1^n {}_A(x_i|x_i) \right\| \leqslant \lambda \left\| \sum_1^n \theta^r_{x_i,x_i} \right\|,$$

(2) *there exists $\lambda > 0$ such that for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$,*

$$\left\| \sum_1^n {}_A(x_i|y_i) \right\| \leqslant \lambda \left\| \sum_1^n \theta^r_{x_i,y_i} \right\|,$$

(3) *$F(T) \geqslant 0$ for any $T \in FR(X_B) \cap \mathscr{K}(X_B)^+$ and $\sup_\mu ||F(\sum_{y \in u_\mu} \theta^r_{y,y})||$ is finite for some generalized right basis $\mu \to u_\mu$ of X_B.*

If one of these conditions is satisfied, the smallest constants for which (1) *and* (2) *hold, coincide and equal, in turn,* $\sup_\mu ||F(\sum_{y \in u_\mu} \theta^r_{y,y})||$. *In particular, the latter does not depend on the generalized right basis.*

Proof. (1) $\Rightarrow$ (2) Let $\mu \subset \Lambda \to u_\mu$ be a generalized basis of the right Hilbert module X_B. Consider the linear map $F_\mu : T \in \mathscr{L}(X_B) \to \sum_{y \in u_\mu} {}_A(Ty|y) \in A$, already considered in the proof of Lemma 2.6. We claim that $||F_\mu|| \leqslant \lambda$ for any μ. We show the claim. For any $T \in \mathscr{L}(X_B)$,

$$||F_\mu(T)|| = \left\| \sum_{y \in u_\mu} {}_A(Ty|y) \right\| \leqslant \left\| \sum_{y \in u_\mu} {}_A(Ty|Ty) \right\|^{1/2} \left\| \sum_{y \in u_\mu} {}_A(y|y) \right\|^{1/2}$$

by the Cauchy–Schwarz inequality of the left inner product (see, e.g., [B] Proposition 13.1.3). Now by our assumption the last term is bounded above by

$$\begin{aligned} &\lambda \left\| \sum_{y \in u_\mu} \theta^r_{Ty,Ty} \right\|^{1/2} \left\| \sum_{y \in u_\mu} \theta^r_{y,y} \right\|^{1/2} \\ &= \lambda \left\| T \sum_{y \in u_\mu} \theta^r_{y,y} \, T^* \right\|^{1/2} \left\| \sum_{y \in u_\mu} \theta^r_{y,y} \right\|^{1/2} \leqslant \lambda ||T||. \end{aligned}$$

We have already seen that $\lim_\mu F_\mu(\theta^r_{x,z}) = {}_A(x|z)$. Since, for any $x_1, \dots, x_n$, $z_1, \dots, z_n \in X$, $||F_\mu(\sum_1^n \theta^r_{x_i,z_i})|| \leqslant \lambda ||\sum_1^n \theta^r_{x_i,z_i}||$, the proof is completed taking the norm limit at the left-hand side.

(2) $\Rightarrow$ (3) We first show that $F(T) \in A^+$ for any $T \in FR(X) \cap \mathscr{K}(X_B)^+$. If $T \in FR(X) \cap \mathscr{K}(X_B)^+$ and if $\mu \to u_\mu \subset X$ is a generalized basis of X, then the net $T^{1/2} \sum_{y \in u_\mu} \theta^r_{y,y} T^{1/2} = \sum_{y \in u_\mu} \theta^r_{T^{1/2}y, T^{1/2}y}$ converges to T in norm. Since, by (2) F is norm continuous, $F(T) = \lim_\mu \sum_{y \in u_\mu} F(\theta^r_{T^{1/2}y, T^{1/2}y}) \in A^+$ as, by definition, F takes elements of the form $\theta^r_{z,z}$ to positive operators in A. Furthermore for all μ,

$$\left\| F\left(\sum_{y \in u_\mu} \theta^r_{y,y} \right) \right\| \leqslant \lambda \left\| \sum_{y \in u_\mu} \theta^r_{y,y} \right\| \leqslant \lambda.$$

(3) $\Rightarrow$ (1) Let $x_1, \dots, x_n$ be elements of X, and set $T = \sum_1^n \theta^r_{x_i,x_i}$. Let $\mu \to u_\mu$ be a generalized right basis of X. Since $\sum_{y' \in u_\mu} \theta^r_{y',y'}$ is an approximate unit of $\mathscr{K}(X_B)$ and since the left inner product is continuous with respect to the right one, for all μ, the net $\mu' \to \sum_{y \in u_\mu, y' \in u_{\mu'}} {}_A(\theta^r_{y',y'} Ty|y)$ converges to $\sum_{y \in u_\mu} {}_A(Ty|y)$ in norm. On the other hand this net coincides with $F((\sum_{y' \in u_{\mu'}} \theta^r_{y',y'}) T (\sum_{y \in u_\mu} \theta^r_{y,y}))$. The form $S, T \in FR(X_B) \to F(ST^*)$ is left A-linear, right A-antilinear, symmetric and positive and therefore it satisfies the Cauchy–Schwarz inequality

$||F(ST^*)||^2 \leqslant ||F(SS^*)||||F(TT^*)||$. It follows that

$$\left\| F\left(\left(\sum_{y' \in u_{\mu'}} \theta^r_{y',y'} \right) T \left(\sum_{y \in u_\mu} \theta^r_{y,y} \right) \right) \right\|^2$$
$$\leqslant \left\| F\left(\left(\sum_{y' \in u_{\mu'}} \theta^r_{y',y'} \right) TT^* \left(\sum_{y' \in u_{\mu'}} \theta^r_{y',y'} \right) \right) \right\| \left\| F\left(\left(\sum_{y \in u_\mu} \theta^r_{y,y} \right)^2 \right) \right\|.$$

Now

$$\left(\sum_{y' \in u_{\mu'}} \theta^r_{y',y'} \right) TT^* \left(\sum_{y' \in u_{\mu'}} \theta^r_{y',y'} \right) \leqslant ||T||^2 \left(\sum_{y' \in u'_\mu} \theta^r_{y',y'} \right)$$

and $(\sum_{y \in u_\mu} \theta^r_{y,y})^2 \leqslant (\sum_{y \in u_\mu} \theta^r_{y,y})$, so, applying F, we deduce that the above term is bounded above by $||T||^2 \lambda_0^2$ where $\lambda_0 = \sup_\mu ||F(\sum_{y \in u_\mu} \theta^r_{y,y})||$. Passing first to the limit over μ' and then over μ we deduce that (1) holds with $\lambda = \lambda_0$.

It is now clear from the proof that if one of these three equivalent conditions holds, the best constants satisfying (1) and (2) coincide, a coincide in turn with

$$\sup_\mu \left\| F\left(\sum\nolimits_{y \in u_\mu} \theta^r_{y,y} \right) \right\|. \qquad \square$$

Definition 2.8. A C^*-bimodule satisfying one of the equivalent properties described in the previous proposition will be called of *finite right numerical index*. The corresponding smallest positive constant will be called the *right numerical index* of X, and denoted $r - I[X]$.

Let X be an A–B bimodule. The *contragradient bimodule* of X is the B–A bimodule $\bar{X} = \{\bar{x}; x \in X\}$ with complex conjugate vector space structure and bimodule structure given by

$$b \cdot \bar{x} = \overline{xb^*}, \quad \bar{x} \cdot a = \overline{a^* x}, \quad b \in B, a \in A.$$

If X is a right (left) Hilbert A–B C^*-bimodule, $\bar{X}$ becomes a left (right) Hilbert B–A C^*-bimodule with inner product given by:

$${}_B(\bar{x}|\bar{y}) = (x|y)_B, \quad ((\bar{x}|\bar{y})_A = {}_A(x|y).)$$

Therefore if ${}_A X_B$ is bi-Hilbertian, ${}_B \bar{X}_A$ is bi-Hilbertian as well.

Definition 2.9. We will say that ${}_A X_B$ is of *finite left numerical index* if the contragradient bimodule ${}_B \bar{X}_A$ is of finite right numerical index. Its left numerical index is defined by $\ell - I[X] := r - I[\bar{X}]$.

A bi-Hilbertian bimodule of finite left and right numerical indices will be simply called of finite numerical index. Its numerical index is defined by $I[X] := (r-I[X])(\ell - I[X])$.

Corollary 2.10. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule, and, for $n \in \mathbb{N}$, let us consider $Y_n := \oplus_1^n X$ as a $M_n(A)$-B bimodule in the natural way. Endow Y_n with the following forms:* ${}_{M_n(A)}(\underline{x}|\underline{y}) = ({}_A(x_i|y_j))$, $(\underline{x}|\underline{y})_B = \sum_1^n (x_i|y_i)_B$, *where* $\underline{x} = (x_1, \dots, x_n)$, $\underline{y} = (y_1, \dots, y_n)$.

1. *If for some $\lambda > 0$, $\|{}_A(x|x)\| \leqslant \lambda \|(x|x)_B\|$ then $\|{}_{M_n(A)}(\underline{x}|\underline{x})\| \leqslant \lambda \|(\underline{x}|\underline{x})_B\|$.*
2. *If X is of finite right numerical index, Y_n becomes a C^*-bimodule of finite right numerical index and $r - I[Y_n] = r - I[X]$ for all $n \in \mathbb{N}$.*
3. *If X is of finite numerical index, Y_n is bi-Hilbertian and of finite left numerical index (and hence of finite numerical index, by (1)): $\|(\underline{x}|\underline{x})_B\| \leqslant \ell - I[X] \|{}_{M_n(A)}(\underline{x}|\underline{x})\|$ and $\ell - I[X] = \ell - I[Y_n]$, for all $n \in \mathbb{N}$.*

Remark. Notice that the constants comparing the two norms on Y_n do not depend on n.

We omit the proof as it relies on routine computations.

The following result is the first step towards the Jones basic construction.

Corollary 2.11. *If X is a bi-Hilbertian A–B C^*-bimodule of finite right numerical index, the additive extension F of the left inner product to $FR(X_B)$ extends uniquely to a norm continuous map $F : \mathscr{K}(X_B) \to A$. One has: $\|F\| = r - I[X]$. This extension, still denoted by F, is positive, A-bilinear (in the sense that $F(\phi(a)T) = aF(T)$ and $F(T\phi(a)) = F(T)a$ for $a \in A$, $T \in \mathscr{K}(X_B)$) and has range contained in the closed ideal of left inner products. Moreover one has $\phi(F(T)) \geqslant \lambda' T$ for all $T \in \mathscr{K}(X_B)^+$, where λ' is the best constant for which $\lambda' \|(x|x)_B\| \leqslant \|{}_A(x|x)\|$.*

Proof. The only assertion that is not obvious yet is the inequality $\phi(F(T)) \geqslant \lambda' T$ for $T \in \mathscr{K}(X_B)^+$. Now part (4) in Proposition 2.6 implies that $\|F(T)\| \geqslant \lambda' \|T\|$ for $T \in \mathscr{K}(X_B)^+$. Left A-action is faithful on the norm closed ideal $\mathscr{J}$ generated by left inner products: $\phi(j) = 0$ for some $j \in \mathscr{J}^+$ implies $0 = {}_A(x|jy) = {}_A(x|y)j$ for $x, y \in X$ and therefore $j = 0$. Since $F(T) \in \mathscr{J}$ for all $T \in \mathscr{K}(X_B)$, ϕ is isometric on $\mathscr{J}$, therefore for all $T \in \mathscr{K}(X_B)^+$, $\|\phi(F(T))\| = \|F(T)\| \geqslant \lambda' \|T\|$. Arguing as in [FK], with the map $\phi \circ F$ in place of a conditional expectation, we deduce the desired inequality. □

Conditional expectations satisfying a Pimsner–Popa inequality provide typical examples of bimodules of finite right numerical index.

Proposition 2.12. *Let $A \subset B$ be an inclusion of C^*-algebras and let $E : B \to A$ be a conditional expectation with fixed point set A. Assume that $\|E(b)\| \geqslant \lambda \|b\|$, for all positive elements $b \in B$ and for some $\lambda > 0$.*

(1) *Consider ${}_BX_A = B$ as a B–A bimodule in the natural way, and with inner products $(x|y)_A = E(x^*y)$, ${}_B(x|y) = xy^*$. Since $\|(x|x)_A\| \leqslant \|{}_B(x|x)\| \leqslant \lambda^{-1}\|(x|x)_A\|$, X is bi-Hilbertian. By [FK] and [Po], 1.1.2 there is a constant $\lambda' > 0$ such that $E - \lambda'$ is completely positive. Let us choose the best such λ'. Then X has finite right numerical index and $r - I[X] = \lambda'^{-1}$.*
(2) *Consider now ${}_AY_B = B$ as a A–B bimodule with inner products $(x|y)_B = x^*y$ and ${}_A(x|y) = E(xy^*)$. Then the B–A antilinear map $X \to Y$ induced by the *-involution of B identifies Y with the contragradient $\bar{X}$ of X. Therefore X is of finite left numerical index and $\ell - I[X] = 1$.*

Proof. (1) For all $n \in \mathbb{N}$, and all $x_1, \dots, x_n \in X$,

$$\lambda'^{-1} \left\| \sum_1^n \theta^r_{x_i, x_i} \right\| = \lambda'^{-1} \|(E(x_i^* x_j))_{i,j}\|_{M_n(A)}$$
$$\geqslant \|(x_i^* x_j)_{i,j}\|_{M_n(B)} = \left\| \sum_1^n {}_B(x_i|x_i) \right\|,$$

i.e. X is of finite right numerical index in our sense and $r - I[X] \leqslant \lambda'^{-1}$. On the other hand, let $\mu \to u_\mu$ be a generalized basis of X_A, and set, for every μ and every $x \in X$, $x_\mu := \sum_{y \in u_\mu} yE(y^*x)$. Then

$$x_\mu^* x_\mu \leqslant \left\| \sum_{y \in u_\mu} yy^* \right\| \sum_{y \in u_\mu} E(x^*y)E(yx^*)$$
$$= E(x^* x_\mu) \left\| \sum_{y \in u_\mu} yy^* \right\| \leqslant \sup_\mu \left\| \sum_{y \in u_\mu} yy^* \right\| E(x^* x_\mu).$$

Taking the limit over μ, we are led to the inequality $r - I[X] \geqslant \lambda^{-1}$. Consider now the inclusion $M_n \otimes A \subset M_n \otimes B$ and the conditional expectation $E_n := \mathrm{id} \otimes E$, which satisfies $E_n(b) \geqslant \lambda' b$, $B \in M_n(B)^+$. Corollary 2.10 shows that $\oplus_1^n B$ is a $M_n(B)$–A bimodule with the same right index as B, hence, combining with the above argument, we deduce that $r - I[X] \geqslant \lambda'^{-1}$.

The proof of part (2) is easy, therefore we omit it. $\square$

In particular, a conditional expectation $E : B \to A$ between unital C^*-algebras admitting a finite quasi-basis $\{u_i\}$ in the sense of [W] satisfies $E(x) \geqslant \|\mathrm{Ind}[E]\|^{-1} x$, for $x \in B^+$, where $\mathrm{Ind}[E] = \sum_i u_i u_i^*$, as shown in Proposition 2.6.2 of [W].

Remark. If ${}_BX_A$ and ${}_AX_B$ arise from a conditional expectation E satisfying a Pimsner–Popa inequality, as the previous proposition, the corresponding map F_Y constructed in Corollary 2.11 reduces to E itself. More interestingly, $F_X : \mathscr{K}(X_A) \to B$ is related to the construction of the *dual* conditional expectation. However, if the index of E, as an element of $Z(B'')$ (cf. Definition 2.17), does not belong to the multiplier algebra of B, F_X is not a multiple of a conditional expectation.

2.2. *Tensoring bi-Hilbertian C^*-bimodules*

In this subsection we analyze the behaviour of bi-Hilbertian bimodules under taking their tensor products. We show that the algebraic tensor product $X \odot_B Y$ of bi-Hilbertian C^*-bimodules can be made into a bi-Hilbertian bimodule in a natural way if X is of finite right numerical index and Y is of finite left numerical index, and that this is also a necessary condition in general.

The problem of studying conditions under which $X \otimes_B Y$ is of finite index will be considered in Section 5 (cf. Theorem 5.1).

Let ${}_AX_B$ and ${}_BY_C$ be bi-Hilbertian C^*-bimodules. Then the algebraic tensor product $X \odot_B Y$ is an A–C bimodule in a natural way, also endowed with a right and a left pre-bi-Hilbertian structure:

$$(x_1 \otimes y_1 | x_2 \otimes y_2)_C = (y_1 | (x_1|x_2)_B y_2)_C,$$

$$_A(x_1 \otimes y_1 | x_2 \otimes y_2) = {}_A(x_1 {}_B(y_1|y_2) | x_2).$$

Therefore $X \odot_B Y$ can be made into a right Hilbert C-module $X \otimes^r_B Y$ completing with respect to the first inner product and also into a left Hilbert A-module $X \otimes^\ell_B Y$ completing with respect to the second inner product (always after dividing out by vectors of seminorm zero).

Under which conditions are these two seminorms equivalent on the algebraic tensor product $X \odot_B Y$?

Let Y denote the strong Morita equivalence ${}_B\ell^2(B)_{\mathscr{K}\otimes B}$ with inner products

$$_B(\underline{b}|\underline{b}') = \sum_j b_j b_j'^*, \quad (\underline{b}|\underline{b}')_{\mathscr{K}\otimes B} = \sum \delta_{i,j} \otimes b_i^* b_j',$$

where $\{\delta_{i,j}, i,j \in \mathbb{N}\}$ a complete set of matrix units for $\mathscr{K}$. The tensor product $X \otimes^r_B \ell^2(B)$ identifies with $\ell^2(X)$ with inner products ${}_A(\underline{x}|\underline{x}') = \sum_j {}_A(x_j|x_j')$, $(\underline{x}|\underline{x}')_{\mathscr{K}\otimes B} = \sum \delta_{i,j} \otimes (x_i|x_j')_B$. Therefore the left and right seminorms on $X \odot_B \ell^2(B)$ are equivalent if and only if X is of finite right numerical index. Similarly, the left and right seminorms on $\overline{\ell^2(B)} \odot_B Y$, with $\overline{\ell^2(B)}$ the inverse strong Morita equivalence, are equivalent if and only if Y is of finite left numerical index. We show that these necessary conditions on X and Y are also sufficient.

Proposition 2.13. *Let ${}_AX_B$ and ${}_BY_C$ be bi-Hilbertian C^*-bimodules. Assume that X is of finite right numerical index and that Y is of finite left numerical index. Let $F_X : \mathscr{K}(X_B) \to A$, $F_{\bar{Y}} : \mathscr{K}(\bar{Y}_B) \to C$ be the corresponding maps constructed in Corollary 2.11. Then*

(1) *the two seminorms arising from the left and right inner products on $X \odot_B Y$ as above are equivalent. Therefore $X \otimes^r_B Y = X \otimes^\ell_B Y (=: X \otimes_B Y)$ and it is a bi-Hilbertian A–C bimodule.*
(2) *Consider $\mathscr{K}(\bar{Y}_B, X_B)$ as a A–C bimodule with left and right inner products ${}_A(T|S) = F_X(TS^*)$* and *$(T|S)_C = F_{\bar{Y}}(T^*S)$. Then $\mathscr{K}(\bar{Y}_B, X_B)$ is complete in any of the induced norms, and becomes in this way a bi-Hilbertian C^*-bimodule.*
(3) *The map $x \otimes y \in X \otimes Y \to \theta^r_{x,\bar{y}} \in \mathscr{K}(\bar{Y}_B, X_B)$ extends to a bijective A–C bimodule map $U : {}_AX_B \otimes_B Y_A \to \mathscr{K}(\bar{Y}_B, X_B)$ preserving the left and right inner products.*

Proof. Routine computations show that the map $U : X \odot_B Y \to FR(\bar{Y}, X)$, $U(x \otimes y) = \theta^r_{x,\bar{y}}$ is a A–C bimodule map which preserves the corresponding left and right inner products.

Since, when X and Y are bi-Hilbertian, F_X and F_Y are faithful maps (see Corollary 2.11), the two seminorms have the same vectors of length zero (therefore U is an injective map). Furthermore the two norms $||F_{\bar{Y}}(T^*T)||^{1/2}$ and $||F_X(TT^*)||^{1/2}$ on $\mathscr{K}(\bar{Y}_B, X_B)$ are both equivalent to the operator norm, still by Corollary 2.11, and therefore they are equivalent. We have thus shown that $X \otimes^r_B Y$ and $X \otimes^\ell_B Y$ are isomorphic as Banach spaces. It is now straightforward to check that right and left actions are adjointable, and therefore $X \otimes_B Y$ is bi-Hilbertian. Since U is a bijective map which preserves both inner products, it extends to a bijective A–C bimodule map $U : X \otimes_B Y \to \mathscr{K}(\bar{Y}_B, X_B)$ still preserving the inner products, and the proof is now complete. □

2.3. Nondegeneracy of the left action

The following nondegeneracy property will be relevant for our purposes.

Definition 2.14. The left action ϕ of a C^*-algebra A on a right Hilbert C^*-module X_B will be called *nondegenerate* if AX is total in X.

We recall the following characterization of nondegeneracy, due essentially to Vallin [V], see also Proposition 2.5 in [L].

Proposition 2.15. *For a *-homomorphism $\phi : A \to \mathscr{L}(X_B)$ the following conditions are equivalent.*

(1) *ϕ is nondegenerate,*
(2) *ϕ is the restriction to A of a unital *-homomorphism $\hat{\phi} : M(A) \to \mathscr{L}(X_B)$, strictly continuous on the unit ball,*

(3) *for some approximate unit* $(u_\alpha)_\alpha$ *of* A, $(\phi(u_\alpha))_\alpha$ *converges strictly to the identity map on* X.

Note that if ϕ is nondegenerate, (3) must hold for all approximate units of A. We show that the left action of a bi-Hilbertian C^*-bimodule is automatically nondegenerate.

Proposition 2.16. *Let* ${}_AX_B$ *be a bi-Hilbertian* A–B C^**-bimodule. Then the left* (*right*) *action of* A (B) *on the underlying right Hilbert* C^**-module* X *is nondegenerate.*

Proof. If $\{u_\alpha\}$ is an approximate unit of the closed ideal of A generated by the left inner products, $u_\alpha x$ converges to x for all $x \in X$, in the norm arising from the left inner product. Therefore AX is total in X with respect to the norm defined by the left inner product. Since the two norms on X defined by the right and left inner product are equivalent, we also have that AX is total with respect to the norm arising from the right inner product. □

2.4. The index element and the Jones basic construction

If X is bi-Hilbertian and of finite right numerical index, one can extend the maps $\phi : A \to \mathscr{L}(X_B)$, $F : \mathscr{K}(X_B) \to A$ uniquely to normal positive maps $\phi'' : A'' \to \mathscr{K}(X_B)''$, $F'' : \mathscr{K}(X_B)'' \to A''$ between the corresponding enveloping von Neumann algebras. Since ϕ is nondegenerate, and the inclusion $M(A) \subset A''$ is unital, ϕ'' is a unital homomorphism. The same does not hold for F'': $F''(I)$ is, in general, neither the identity, nor invertible.

Definition 2.17. If ${}_AX_B$ is of finite right numerical index, the *right index element of* ${}_AX_B$, denoted, $r - \mathrm{Ind}[X]$ is the element $F''(I)$ of A''.

If in particular ${}_BX_A$ is the bimodule arising from a conditional expectation $E : B \to A$ as in Proposition 2.12, the corresponding right index element will be denoted by $\mathrm{Ind}[E]$. (We will give in Corollary 4.9 an alternative definition of $\mathrm{Ind}[E]$.)

If ${}_AX_B$ is of finite left numerical index, the left index element of X is $\ell - \mathrm{Ind}[X] := r - \mathrm{Ind}[\bar{X}]$.

Notice that the numerical indices and the index elements are related by

$$||r - \mathrm{Ind}[X]|| = r - I[X], \quad ||\ell - \mathrm{Ind}[X]|| = \ell - I[X].$$

If one of $r - \mathrm{Ind}[X]$ and $\ell - \mathrm{Ind}[X]$ is a scalar, or if $A = B$, the index element of X can be defined by $\mathrm{Ind}[X] := (r - \mathrm{Ind}[X])(\ell - \mathrm{Ind}[X])$.

Our next aim is to define an index element $\mathrm{Ind}[X]$ in the general case. We notice that for $c \in Z(B)$, the map $\psi(c) : x \in X \to xc \in X$ has the map $x \in X \to xc^* \in X$ as an adjoint with respect to the right inner product of X. Furthermore $\psi(c)$ commutes with all the elements of $\mathscr{L}(X_B)$, therefore $\psi(Z(B)) \subset Z(\mathscr{L}(X_B))$ (it is not difficult to see that $\psi(Z(B)) = Z(\mathscr{L}(X_B))$ in the case where X_B is full and B is unital). We need to consider an extension of this right action of $Z(B)$ on X to the centre of B''. Therefore we anticipate the following lemma.

Lemma 2.18. *Let X be a right Hilbert B-module, and let $\psi : Z(B) \to Z(\mathscr{L}(X_B))$ denote the right action of $Z(B)$ on X. Then there is a canonical extension of ψ to a unital surjective *-homomorphism $\psi_0 : Z(B'') \to Z(\mathscr{K}(X_B)'')$ with $\ker\psi_0 = (1-q)Z(B'')$, where q is the central projection of B'' corresponding to the weak closure in B'' of the ideal generated by right inner products.*

Proof. Let B_0 denote the norm closed ideal in B generated by the right inner products, and let π be a Hilbert space representation of B_0 on H_π. Consider the Stinespring induced representation $\tilde{\pi}$ of $\mathscr{K}(X_B)$ on the Hilbert space $K_\pi := X_B \otimes_{B_0} H_\pi$, defined by $T \to T \otimes 1_{H_\pi}$. Since B_0 and $\mathscr{K}(X_B)$ are strongly Morita equivalent, it is well known that the map $\pi \to \tilde{\pi}$ is a bijective correspondence between representations of B_0 and representations of $\mathscr{K}(X_B)$. Therefore the representation $\rho = \oplus_\pi \tilde{\pi}'' : \mathscr{K}(X_B)'' \to \mathscr{L}(\oplus_\pi K_\pi)$, where the sum is taken over all Hilbert space representations of B_0, is faithful and normal. It follows that $\rho(\mathscr{K}(X_B)'') = \rho(K(X_B))''$. On the other hand $Z(B'')$ acts on each K_π, and therefore on their direct sum by $\psi_0' : c \in Z(B'') \to 1_X \otimes (\oplus_\pi \pi'')(qc) \in \mathscr{L}(\oplus_\pi K_\pi)$. Notice that $qZ(B'')$ is contained in (the centre of) the weak closure of B_0 in B'', therefore $\pi''(qc)$ makes sense. Clearly, $\psi_0'(c) \in \rho(\mathscr{K}(X_B))'$. If A is a positive, bounded operator on $\oplus_\pi K_\pi$ commuting elementwise with $\rho(\mathscr{K}(X_B))$ then $A(x(y|z)_B \otimes \xi) = \theta^r_{x,y} \otimes 1_{\oplus_\pi H_\pi} A(z \otimes \xi)$ for all $x, y, z \in X$, $\xi \in \oplus_\pi H_\pi$, $\pi \in \mathrm{Rep}(B_0)$. Arguments similar to those used in the proof of Proposition 1.4, which, in turn, go back to Proposition 1.7.2 in [Di], show that there exists a bounded approximate unit $(u_\alpha)_\alpha$ of B_0 of the form $u_\alpha = \sum_{y \in F_\alpha} (y|y)_B$, with F_α a finite subset of X. This implies that A is of the form $1_X \otimes a$, with $a \in \mathscr{L}(\oplus_\pi H_\pi)$ any weak limit point of the net $\alpha \to \sum_{y \in F_\alpha} \ell_y^* A \ell_y$, with $\ell_z : \xi \in \oplus_\pi H_\pi \to z \otimes \xi \in \oplus_\pi K_\pi$. This also easily shows that a lies in the commutant of the image of B_0 under its universal representation, so A and $\psi_0'(c)$ commute. We thus obtain that $\psi_0'(Z(B'')) \subset Z(\rho(\mathscr{K}(X_B)''))$. Notice that, $\psi_0'(c) = 0$ if and only if each $\pi''(qc)$ annihilates the subspace qH_π, or, equivalently $\pi''(cq) = 0$ for all π, i.e. $cq = 0$. On the other hand if $A \in Z(\rho(\mathscr{K}(X_B)''))$ and we write $A = 1_X \otimes a$, a one moment thought shows that $a \in Z(B'')$. The *-homomorphism $\psi_0 := \rho^{-1}\psi_0'$ is then the desired extension of ψ. $\square$

Proposition 2.19. *Let X be of finite right numerical index. Then for any generalized right basis $\mu \to u_\mu$ of X, the net $\mu \to \sum_{y \in u_\mu} {}_A(y|y)$ is increasing and it converges strongly in A'' to* $r - \mathrm{Ind}[X]$. *This limit is therefore independent on the choice of the basis, and belongs to the centre of A''. If in addition X is bi-Hilbertian one has $\lambda r - \mathrm{Ind}[X] \geqslant p$ where p is the support projection of $r - \mathrm{Ind}[X]$ in A'', and λ is the best constant for* *which* $\lambda||_A(x|x)|| \geqslant ||(x|x)_B||, x \in X$. *Furthermore, if* $\mathscr{I}$ *denotes the weak closure in* A'' *of the span of left inner products* $_A(x|y), x, y \in X$, *one has*

(1) $\ker \phi'' = (I-p)A''$,
(2) $\mathscr{I} = pA''$,
(3) *the range of* $F'' : \mathscr{K}(X)'' \to A''$ *is* pA'',
(4) *if* z' *denotes the inverse of* $r - \mathrm{Ind}[X]$ *in* pA'', *and* $E'' \coloneqq z'F'' : \mathscr{K}(X_B)'' \to pA''$. *Then* $\phi'' \circ E'' : \mathscr{K}(X_B)'' \to \phi''(A'')$ *is a conditional expectation with range* $\phi''(A'')$ *satisfying*

$$\lambda\phi''(r - \mathrm{Ind}[X]E''(T)) \geqslant T, \quad T \in \mathscr{K}(X_B)''^{+}.$$

Proof. The strong limit z defined as in the statement is independent of the generalized basis since it coincides with $F''(I)$. Since F'' is still A-bilinear, $F(\phi(a)) = F''(I)a = aF''(I)$ for all $a \in A$, which shows that $F''(I)$ lies in the centre of A''. Let us assume X bi-Hilbertian, and let λ be as in the statement. The estimate

$$||T|| \leqslant \lambda||F''(T)|| \leqslant r - I[X]||T||, \quad T \in \mathscr{K}(X)''^{+},$$

still holds, therefore if $T = \phi''(a)$, with $a \in A''^{+}$,

$$||\phi''(a)|| \leqslant \lambda||za|| \leqslant r - I[X]||\phi''(a)||.$$

If we consider the restriction ϕ_0 of ϕ'' to the centre of A'', we deduce that $\ker \phi_0$ coincides with the weakly closed ideal generated by $I - p$. Therefore for a positive central element a of A'', $||\phi_0(a)|| = ||pa|| \leqslant \lambda||za||$. Let us identify the centre of A'' with some $L^\infty(\Omega, \nu)$. We claim that for every $\varepsilon > 0$, the function $z - (\lambda^{-1} - \varepsilon)p$ can not take negative values on a measurable subset of $p\Omega$ with positive measure. Indeed, if $Y \subset \Omega$ where such a set, we would have, for some $\varepsilon < \lambda^{-1}$,

$$\lambda^{-1} - \varepsilon = (\lambda^{-1} - \varepsilon)||\xi_Y|| \geqslant ||z\xi_Y|| \geqslant \lambda^{-1}||p\xi_Y|| = \lambda^{-1},$$

where ξ_Y is the characteristic function of Y. Therefore $z \geqslant \lambda^{-1}p$. Now if $a \in A''$, $\phi''(a) = 0$ if and only if $\phi''(aa^*) = 0$ and this holds if and only if $paa^* = 0$, i.e. $pa = 0$, so $\ker \phi'' = (I-p)A''$, and (1) is proved. Let $\mathscr{I}$ be the weakly closed ideal of A'' defined as in the statement. Since the range of F'' is contained in $\mathscr{I}$ and since $F''(\phi''(I)) = z$, p must belong to $\mathscr{I}$, and therefore $pA'' \subset \mathscr{I}$. Conversely, there exists an increasing, norm bounded net $\alpha \to \sum_{w \in \alpha} {}_A(w|w)$, indexed by the set of finite subsets of X, which is a bounded approximate unit of the norm closed ideal generated by the left inner products. Its weak limit, say q, is the unit of $\mathscr{I}$. By Proposition 2.4 the net $\alpha \to \sum_{w \in \alpha} \theta^r_{w,w}$ is norm bounded. We have $\sum_{w \in \alpha} \phi(a)\theta^r_{w,w}\phi(a)^* \leqslant \lambda_0\phi(aa^*)$ for some $\lambda_0 > 0$ and for all $a \in A$. Applying F we obtain

$$\sum_{w \in \alpha} a_A(w|w)a^* \leqslant \lambda_0 aza^*.$$

Thus $\lambda_0^{-1}aqa^* \leqslant aza^*$. It follows that $\lambda_0^{-1}q \leqslant z$, hence $q \leqslant p$. Therefore $\mathscr{I} \subset pA''$, and the proof of (2) is complete. (3) We are left to show that pA'' is contained in the range of F''. Let z' be the inverse of $r - \text{Ind}[X]$ in pA''. For $a \in A''$, $F''(\phi''(z'a)) = r - \text{Ind}[X]z'a = pa$. (4) It is now clear that $\phi''E''$ is a conditional expectation with range $\phi''(pA'') = \phi''(A)$. (5) Since $\lambda||F''(T)|| \geqslant ||T||$ for T positive is $\mathscr{K}(X_B)''$ and since ϕ'' is isometric on $\mathscr{I}$, and therefore on the range of F'', we see that $\lambda||\phi''(r - \text{Ind}[X]E''(T))|| = \lambda||\phi''F''(T)|| = \lambda||F''(T)|| \geqslant ||T||$. Therefore $\lambda\phi''(r - \text{Ind}[X]E''(T)) \geqslant T$ (cf. [FK]). $\square$

Corollary 2.20. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite right numerical index. Then the following properties are equivalent.*

(1) *$r - Ind[X]$ is invertible,*
(2) *ϕ'' is faithful,*
(3) *the linear span of the left inner products is weakly dense in A''.*

Remark. Notice that if ${}_BX_A$ is the bimodule arising from a conditional expectation satisfying a Pimsner–Popa inequality, as in Proposition 2.12, $r - \text{Ind}[X] = \text{Ind}[E]$ must be invertible since the left inner product is full.

Definition 2.21. Notice that $\phi''(r - \text{Ind}[X])$ is invertible in $Z(\phi(A)' \cap \mathscr{K}(X_B)'')$ and $\psi_0(\ell - \text{Ind}[X])$ is invertible in $Z(\mathscr{K}(X_B)'')$. Therefore we define the index element of X as an element of $\mathscr{K}(X_B)''$, in fact central in $\phi(A)' \cap \mathscr{K}(X_B)''$, by $\text{Ind}[X] := \psi_0(\ell - \text{Ind}[X])\phi''(r - \text{Ind}[X])$.

2.5. *On the condition $r - \text{Ind}[X] \in M(A)$ and existence of finite bases*

Under which conditions does the index element lie in $M(A)$? By Kadison's function representation theorem (see, e.g., [P]), the real Banach space A_{sa} identifies isometrically with the real Banach space of continuous, vanishing at 0, affine functions on Q, the quasi-state space of A. We have the following characterization of the property $r - \text{Ind}[X] \in M(A)$.

Theorem 2.22. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite right numerical index. Then the following properties are equivalent:*

(1) *$r - Ind[X] \in M(A)$ (and hence it is a central element of $M(A)$),*
(2) *there is a generalized right basis $\mu \to u_\mu \subset X$ of X such that the net $\mu \to \sum_{y \in u_\mu} {}_A(y|y)$ is convergent in the strict topology of $M(A)$,*
(3) *for any generalized right basis $\mu \to u_\mu \subset X$ of X the net $\mu \to \sum_{y \in u_\mu} {}_A(y|y)$ is convergent in the strict topology of $M(A)$,*
(4) *the range $\phi(A)$ of the left action is included in $\mathscr{K}(X_B)$.*

If one of these conditions is satisfied, the strict limits defined in (3) *do not depend on the choice of the generalized right basis, and coincide with* $r - Ind[X]$.

Proof. (3) $\Rightarrow$ (2) is obvious. (4) $\Rightarrow$ (3) Let $\mu \to u_\mu$ be a generalized basis of X. Then $\sum_{y \in u_\mu} {}_A(y|y)$ converges strictly if and only if for all $a \in A$,

$$\sum_{y \in u_\mu} a_A(y|y) = \sum_{y \in u_\mu} {}_A(\phi(a)y|y) = F\left(\phi(a) \sum_{y \in u_\mu} \theta_{y,y}\right)$$

converges in norm. Therefore if (4) holds, convergence of the above net follows from norm continuity of F and the fact that $\sum_{y \in u_\mu} \theta^r_{y,y}$ is an approximate unit of $\mathscr{K}(X_B)$. (2) $\Rightarrow$ (4) By Corollary 2.11, for any $T \in \mathscr{K}(X_B)^+$, $\lambda||T|| \leqslant ||F(T)||$, where λ is the best positive constant for which $\lambda||(x|x)_B|| \leqslant ||_A(x|x)||$. If now (2) holds for some generalized basis $\mu \to u_\mu \subset X$, $a^* \sum_{y \in u_\mu} {}_A(y|y)a = F(\phi(a)^* (\sum_{y \in u_\mu} \theta^r_{y,y})\phi(a))$ is an increasing, norm converging net, and therefore the net $\phi(a)^*(\sum_{y \in u_\mu} \theta^r_{y,y})\phi(a)$ is increasing and norm converging in $\mathscr{K}(X_B)$. It follows that $\phi(a^*a) \in \mathscr{K}(X_B)$ for all $a \in A$. (2) $\Rightarrow$ (1) is obvious, since $r - \mathrm{Ind}[X]$ is the strict limit of a strictly convergent net. (1) $\Rightarrow$ (3) By Kadison's function representation (see, e.g., [P]) the selfadjoint part of A'' identifies isometrically, as a real Banach space, with the real Banach space $B_0(Q)$ of affine, bounded functions on the quasi-state space Q of A vanishing at 0. Under this identification, the selfadjoint elements of A correspond to the continuous functions. If $r - \mathrm{Ind}(A) \in M(A)$, for all $a \in A$, $a^*((r - \mathrm{Ind}[X]) - \sum_{y \in u_\mu} {}_A(y|y))a \in A$. On the other hand the net $r - \mathrm{Ind}[X] - \sum_{y \in u_\mu} {}_A(y|y)$ decreases weakly to 0 in A'', therefore for any $\phi \in Q$,

$$\phi\left(a^*\left((r - \mathrm{Ind}[X]) - \sum_{y \in u_\mu} {}_A(y|y)\right)a\right)$$

decreases to 0. By Dini's theorem, this net converges uniformly to 0 on Q, and therefore $||(r - \mathrm{Ind}[X] - \sum_{y \in u_\mu} {}_A(y|y))^{1/2}a||^2 \to 0$, which implies $||(r - \mathrm{Ind}[X] - \sum_{y \in u_\mu} {}_A(y|y))a||^2 \to 0$ as the net $\sum_{y \in u_\mu} {}_A(y|y)$ is norm bounded.

Assume now that one of these equivalent properties holds, and let

$$z := \lim_\mu \sum_{y \in u_\mu} {}_A(y|y)$$

for some generalized right basis $\mu \to u_\mu$ of X. Since, for all $a \in A$, $za = F(\phi(a))$, z is independent of the choice of the basis. $\square$

Remark. If a bimodule is given by a conditional expectation $E : B \to A$ as in Proposition 2.12(1), Theorem 2.22 reduces to a result obtained by Izumi [I].

Definition 2.23. A bi-Hilbertian A–B C^*-bimodule X will be called *of finite right index* if

(1) X is of finite right numerical index,
(2) $r - \mathrm{Ind}[X] \in M(A)$ (and hence $r - \mathrm{Ind}[X] \in Z(M(A))$).

Remark. Notice that property (2) above can be replaced by any of the equivalent conditions in Theorem 2.22.

Similarly, X is *of finite left index* if the contragradient bimodule ${}_B\bar{X}_A$ is of finite right index.

${}_AX_B$ will be called *of finite index* if it is of finite right as well as left indices.

We study the special case where the C^*-algebras are σ-unital or unital.

Corollary 2.24. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite right index.*

(1) *If A is σ-unital, X is countably generated as a right Hilbert module,*
(2) *if A and B are σ-unital, X_B admits a countable right basis, $\{u_i\}_{i\in\mathbb{N}}$, therefore*

$$r - \mathrm{Ind}[X] = \sum_{i\in\mathbb{N}} {}_A(u_i|u_i)$$

in the strict topology of A and $F(T) = \sum_{i\in\mathbb{N}} {}_A(Tu_i|u_i)$, $T \in \mathscr{K}(X_B)$ in norm.

Proof. (1) Let $(u_i)_{i\in\mathbb{N}}$ be a countable approximate unit of A. By nondegeneracy of the left action, and the fact that the left action has range in $\mathscr{K}(X_B)$, $\phi(u_i) \in \mathscr{K}(X_B)$ is a countable approximate unit for $\mathscr{K}(X_B)$, so $\mathscr{K}(X_B)$ is σ-unital and this shows that X is countably generated as a right Hilbert B-module (see, e.g., [B]). (2) If in addition B is σ-unital, X_B admits a countable right basis by Proposition 1.2, therefore the formulas for $r - \mathrm{Ind}[X]$ and for F follow. □

Corollary 2.25. *Let ${}_AX_B$ be a bi-Hilbertian bimodule of finite right numerical index, and let A be a unital C^*-algebra. The following are equivalent:*

(1) *X admits a finite right basis,*
(2) *$r - \mathrm{Ind}[X] \in A$.*

Proof. (1) ⇒ (2) follows from the definition of $r - \mathrm{Ind}[X]$. Conversely, assume that (2) holds. By nondegenereracy of the left action, $\phi(I)$ must be the identity map on X. Since, by Theorem 2.22, the range of ϕ is included in the compacts, $I \in \mathscr{K}(X_B)$, so X_B admits a finite basis. □

Corollary 2.26. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule with finite right numerical index. If A is simple then $r - \mathrm{Ind}[X]$ is a scalar, and therefore X is of finite right index.*

Proof. Since A is a simple C^*-algebra, the only positive elements of $Z(A'')$ arising as strong limits of increasing nets in A must be scalar (see, e.g., Lemma 3.1 in [I]), so $r-\mathrm{Ind}[X]$ is a scalar, and therefore it belongs to $M(A)$. $\square$

2.6. The Jones basic construction

Proposition 2.27. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite right index. Then*

(1) *the map $F:\mathscr{K}(X_B)\to A$ extends uniquely to a strictly continuous map $\hat{F}:\mathscr{L}(X_B)\to M(A)$. One has $\hat{F}(I)=r-\mathrm{Ind}[X]$, $||\hat{F}||=r-I[X]$. $\hat{F}$ is still positive, $M(A)$-bilinear and satisfies*

$$\lambda' T\leqslant \hat{\phi}\hat{F}(T),\quad T\in\mathscr{L}(X_B)^+,$$

where λ' is the best constant for which $\lambda'||(x|x)_B||\leqslant||_A(x|x)||$, $x\in X$,

(2) *the support projection p of $r-\mathrm{Ind}[X]$ in A'' lies in fact the centre of $M(A)$ and satisfies $r-\mathrm{Ind}[X]\geqslant\lambda' p$,*

(3) $\ker\phi=(I-p)A$, $\ker\hat{\phi}=(I-p)M(A)$,

(4) *the norm closed subspace of A generated by the left inner products coincides with pA,*

(5) *the range of $F:\mathscr{K}(X_B)\to A$ is pA.*

Proof. (1) Let us restrict the map $F'':\mathscr{K}(X_B)''\to A''$ to a positive map $\hat{F}:\mathscr{L}(X_B)\to A''$. If $T\in\mathscr{L}(X_B)$, and $a\in A$, both $\phi(a)T$ and $T\phi(a)$ lie in $\mathscr{K}(X_B)$, hence $a\hat{F}(T)=F(\phi(a)T)\in A$, $\hat{F}(T)a=F(T\phi(a))\in A$. Therefore $\hat{F}(T)\in M(A)$. $\hat{F}$ is clearly a strictly continuous extension of F, and it is uniquely determined by this property. In particular, $\hat{F}(I)=r-\mathrm{Ind}[X]$. The remaining properties follow from the corresponding properties of F''. (2) Since, by Proposition 2.19, 0 is an isolated point in the spectrum of $r-\mathrm{Ind}[X]$, the fact that p actually lies in $M(A)$ (and therefore in its centre) is obvious. The following is an alternative argument. Let us restrict $\hat{F}$ to the image under $\hat{\phi}$ of the centre of $M(A)$. One has $\hat{F}(\hat{\phi}(a))=r-\mathrm{Ind}[X]a$ for all $a\in Z(M(A))$. By the inequality in (1),

$$\ker\hat{\phi}\restriction_{Z(M(A))}=\{a\in Z(M(A)):a(r-\mathrm{Ind}[X])=0\}.$$

In particular, regarding $Z(M(A))$ as the algebra of continuous function over its spectrum, for all $a\in Z(M(A))$, the norm of $\hat{\phi}(a)$ coincides with the norm of the restriction of a on the support K of $r-\mathrm{Ind}[X]$. If ξ is an element of the spectrum of $Z(M(A))$ such that $(r-\mathrm{Ind}[X])(\xi)\neq 0$, and $\varepsilon>0$, we can find an open set U containing ξ such that $(r-\mathrm{Ind}[X])(\xi')<(r-\mathrm{Ind}[X])(\xi)+\varepsilon$ for all $\xi'\in U$. Let a be a continuous function on the spectrum of $Z(M(A))$ with support in U, such that $0\leqslant a\leqslant 1$ and taking value 1 on a compact set containing ξ. We then have

$$(r - \mathrm{Ind}[X])(\xi) + \varepsilon > ||r - \mathrm{Ind}[X]a|| \geqslant \lambda' ||\hat{\phi}(a)|| = \lambda' ||a\restriction_K|| = \lambda',$$

so

$$r - \mathrm{Ind}[X](\xi) \geqslant \lambda'.$$

Therefore the support of $r - \mathrm{Ind}[X]$ is an open and closed subset of the spectrum of $Z(M(A))$, which implies that its characteristic function p belongs to $Z(M(A))$, and $r - \mathrm{Ind}[X] \geqslant \lambda' p$. (3) $\ker \phi = \ker \phi'' \cap A = (I - p)A'' \cap A = (I - p)A$ by Proposition 2.19. Similarly, $\ker \hat{\phi} = (I - p)M(A)$. (4) Let $\mathscr{J}$ be the norm closed subspace generated by the left inner products, with weak closure $\mathscr{I}$ in A''. We have $\mathscr{J} \subset \mathscr{I} \cap A = pA'' \cap A = pA$. Let z' be the inverse of $p(r - \mathrm{Ind}[X])$ in $pZ(M(A))$ regarded as an element of $Z(M(A))$. Then for all $a \in A$, $pa = r - \mathrm{Ind}[X]z'a = F(\phi(z'a)) \in \mathscr{J}$ since the range of F is contained in $\mathscr{J}$. Thus $pA = \mathscr{J}$. The last property is now clear. $\square$

Corollary 2.28. *If X is a bi-Hilbertian A–B C^*-bimodule of finite right index, the following properties are equivalent.*

(1) *The left inner product is full,*
(2) *$r - Ind[X]$ is invertible,*
(3) *ϕ is faithful.*

We next construct the analogue of the Jones basic construction in the C^*-algebra setting.

Corollary 2.29. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite right index. Consider the positive A-bilinear map $E : T \in \mathscr{K}(X_B) \to z'F(T) \in pA$, with z' the inverse of $(r - \mathrm{Ind}[X])p$ in $pZ(M(A))$. Then $\phi E : \mathscr{K}(X_B) \to \phi(A)$ is a conditional expectation with range $\phi(A)$ which satisfies*

$$\lambda\phi(r - Ind[X]E(T)) \geqslant T, \quad T \in \mathscr{K}(X_B)^+$$

where λ is the best constant for which $\lambda||{}_A(x|x)|| \geqslant ||(x|x)_B||$.

Proof. ϕE is clearly a positive, $\phi(A)$-bilinear map with range $\phi(pA) = \phi(A)$. For all $a \in A$, $\phi E(\phi(a)) = \phi(z'za) = \phi(pa) = \phi(a)$ by (3) of Proposition 2.27, hence ϕE is a conditional expectation. The remaining inequality follows from Corollary 2.11 and $\phi \circ F = \phi(r - \mathrm{Ind}[X])\phi \circ E$. $\square$

2.7. Examples

Example 2.30. Let Ω be a locally compact Hausdorff space, $H = (\Omega, H(\omega)_{\omega \in \Omega}, \Gamma)$ a continuous field of Hilbert spaces, with Γ the space of continuous sections of H. Let

$$X = \{x \in \Gamma : (\omega \to ||x(\omega)||) \in C_0(\Omega)\}$$

be the associated right Hilbert bimodule over $C_0(\Omega)$. Let us consider a field $\omega \to T(\omega) \in \mathscr{L}(H(\omega))$ of positive, trace-class operators on each $H(\omega)$ defining an element T of $\mathscr{L}(X_{C_0(\Omega)})$ (e.g. $T \in FR(X)$). We can then define a left inner product on X, continuous with respect to the right one, by

$$_{C_0(\Omega)}(x|y)(\omega) := (y(\omega)|T(\omega)x(\omega)).$$

Writing the left inner product in the form

$$_{C_0(\Omega)}(x|y)(\omega) = \mathrm{Tr}\; T(\omega)\theta_{x(\omega),y(\omega)},$$

shows that X is of finite right numerical index if and only if $\sup_\omega \mathrm{Tr} T(\omega)$ is finite. In this case, $r - \mathrm{Ind}[X](\omega) = \mathrm{Tr} T(\omega)$. Therefore X is of finite index if and only if $\omega \to \mathrm{Tr}\; T(\omega)$ is a bounded, continuous function on Ω (e.g. $T \in FR(X) \cap \mathscr{K}(X)^+$). Notice that the set of linear combinations of elements $T \in \mathscr{K}(X)^+$ for which $T(\omega)$ is trace-class and $\omega \in \Omega \to \mathrm{Tr}\; T(\omega)$ is continuous, is a *-ideal of $\mathscr{K}(X)$ by 4.5.2 in [Di], norm dense in $\mathscr{K}(X)$. Assume now that $\sup_\omega \dim H(\omega)$ is finite. Then $T = I$ defines a bi-Hilbertian bimodule of finite right (and left) numerical index, with $r - \mathrm{Ind}[X](\omega) = \dim H(\omega)$. However, it is not of finite index, unless the dimension function is continuous. However, if $T \in \mathscr{K}(X)^+$, $\omega \to \mathrm{Tr} T(\omega)$ is always bounded and continuous by [F], and therefore T does define a finite right index structure on X.

The following example is a generalization of index theory to finitely generated Hilbert bimodules, studied in [KW1].

Example 2.31. Let A and B be unital C^*-algebras and let X be a Hilbert A–B bimodule such that both left and right actions are unit preserving. Then X is a bi-Hilbertian bimodule of finite index if and only if X is of finite type in the sense of [KW1]. In fact, assume that X is bi-Hilbertian and of finite index, then X is necessarily finitely generated projective as a right module (or as a left module), since $\mathscr{K}(X_B)$ (or $\mathscr{K}(_AX)$) contains the identity map. The two norms defined by the two inner products of X are equivalent, thus X is of finite type. Conversely, assume that X is of finite type in the sense of [KW1]. Then it is clear that X_B is bi-Hilbertian and that the left A-action on X has range into $\mathscr{K}(X_B) = \mathscr{L}(X_B)$. Furthermore X_B is of finite right numerical index by Lemma 1.26 in [KW1]. Thus X is of finite right index. Similarly, X is of finite left index and therefore of finite index.

We conclude this section with a discussion of a conditional expectation satisfying a Pimsner–Popa with no finite quasi-basis. This example was pointed out in [W]. Later it was considered also in [FK]. We show that this inclusion is determined by a natural σ-unital subinclusion of finite index in the sense of Definition 2.23.

Example 2.32. Consider the C^*-algebra $C([-1,1])$ of continuous functions over the interval $[-1,1]$ and the C^*-subalgebra $C([-1,1])_e = \{f \in C([-1,1]) : f(-x) = f(x)\}$ of even functions. The conditional expectation $E : C([-1,1]) \to C([-1,1])_e$ associating to $f \in C([-1,1])$ the function $\frac{1}{2}(f(x)+f(-x))$ does not have a finite quasi-basis in the sense of [W] since $C([-1,1,])$ is not a finite projective module over $C([-1,1])_e$. It follows that the bi–Hilbertian bimodule ${}_{C([-1,1])}C([-1,1,])_{C([-1,1])_e}$ with inner products

$$ {}_{C([-1,1])}(f|g) = f\bar{g}, \quad (f|g)_{C([-1,1])_e} = E(\bar{f}g) $$

is not of finite right index in the sense of Definition 2.23 because the identity operator over $C([-1,1])_{C([-1,1])_e}$ is not compact. However, the Pimsner–Popa inequality $E(f) \geqslant \frac{1}{2}f$ holds for any $f \in C([-1,1])^+$. One can treat this example by our methods passing to a subinclusion in the following way. Consider the σ-unital C^*-subalgebra $C_0([-1,1]) = \{f \in C([-1,1]) : f(0) = 0\}$. Then the restriction of E still defines a conditional expectation $E : C_0([-1,1]) \to C_0([-1,1])_e$, where $C_0([-1,1])_e = C([-1,1])_e \cap C_0([-1,1])$, and therefore a bi-Hilbertian C^*-bimodule

$$ X = {}_{C_0([-1,1])}C_0([-1,1])_{C_0([-1,1])_e} $$

which we show to be of finite right index. We first show that the left action $C_0([-1,1])$ has range included in the compacts. Set, for $f \in C_0([-1,1])$,

$$ f_e(x) = E(f)(x) = \frac{f(x)+f(-x)}{2} \in C_0([-1,1])_e, $$

$$ f_o(x) = \frac{f(x)-f(-x)}{2} \in C_0([-1,1])_o \in \{f \in C_0([-1,1]) : f(-x) = -f(x)\}. $$

Clearly $f = f_e + f_o$ and $E(\bar{f}_e g_o) = 0$, $f,g \in C_0([-1,1])$. Therefore the right Hilbert bimodule X_B splits into the direct sum of the subspaces of even and odd functions: $X = X_e \oplus X_o$, $X_e := \{f \in X : f(-x) = f(x)\}$, $X_o = \{f \in X : f(-x) = -f(x)\}$. Similarly, as a vector space, $C_0([-1,1]) = C_0([-1,1])_e \oplus C_0([-1,1])_o$. For $f,g \in X = C_0([-1,1])$, $\theta^r_{f,g}(h_e + h_o) = f\bar{g}_e h_e + f\bar{g}_o h_o$. Therefore if, for $n \in \mathbb{N}$, u_n is a positive continuous function in $C_0([-1,1])_e$ such that $u_n(x) = 1$ for $|x| \geqslant \frac{2}{n}$ and $u_n(x) = 0$ for $|x| \leqslant \frac{1}{n}$, the sequence $\theta^r_{f,u_n} + \theta^r_{xf,\frac{u_n}{x}}$ is norm converging to the multiplication operator by f. Therefore by Theorem 2.22, (2) of Definition 2.23 holds. We are left to show that X_B is of finite right numerical index, and this follows from the Pimsner–Popa inequality.

Remark. There is a more immediate proof of the fact that X is of finite right index, which does not appeal to property (4) in Theorem 2.22, but only to the equivalent property (1) of the same theorem, which is easier to check in practice, but perhaps less instructive. Indeed, the inclusion $C_0([-1,1])_e \subset C_0([-1,1])$ is isomorphic to the inclusion $C_0((0,1]) \otimes (\mathbb{C} \subset \mathbb{C}^2)$ which is obviously of finite right index. We owe this isomorphism to the referee.

3. Continuous bundles of finite dimensional C^*-algebras arising from bimodules of finite right index

Let X be a right Hilbert A–B bimodule with nondegenerate left action ϕ, and let us consider the extension $\hat{\phi} : M(A) \to \mathscr{L}(X_B)$ of ϕ to the multiplier algebra (see Proposition 2.15). Restricting $\hat{\phi}$ to the centre $Z(M(A))$ of $M(A)$ yields a unital *-homomorphism $\hat{\phi} : Z(M(A)) \to \mathscr{L}(X_B)$, still denoted by $\hat{\phi}$.

Proposition 3.1. *If X is a right Hilbert A–B bimodule with nondegenerate left action* (*this being the case if, e.g., X is bi-Hilbertian, by Proposition* 2.16), *the range of $\hat{\phi} : Z(M(A)) \to \mathscr{L}(X_B)$ is actually included in the centre of ${}_A\mathscr{L}(X_B)$, the algebra of right adjointable maps on X_B commuting with the left action. Therefore ${}_A\mathscr{L}(X_B)$ becomes a $Z(M(A))$-algebra in the sense of* [Ka].

Adopting a standard procedure we can represent ${}_A\mathscr{L}(X_B)$ as a *semicontinuous field of C^*-algebras* $\omega \to \mathscr{L}_\omega$ over the spectrum Ω of $Z(M(A))$ in the sense of [Ka]. Let, for $\omega \in \Omega$, J_ω be the closed two-sided ideal of ${}_A\mathscr{L}(X_B)$ generated by the image under $\hat{\phi}$ of $C_\omega(\Omega)$, the continuous functions on Ω vanishing at ω. The fiber at ω is the quotient C^*-algebra $\mathscr{L}_\omega := {}_A\mathscr{L}(X_B)/J_\omega$. We will show that the field $\omega \to \mathscr{L}_\omega$ is in fact continuous in the case where X is bi-Hilbertian and of finite right index (see Theorem 3.3).

Let ${}_AX_B$ be bi-Hilbertian and of finite right index. In Proposition 2.27 we have constructed a $M(A)$-bilinear, positive, strictly continuous map $\hat{F} : \mathscr{L}(X_B) \to M(A)$ satisfying a Pimsner–Popa inequality and with range the ideal $pM(A)$, with p the support projection of $r - \mathrm{Ind}[X]$. Restricting $\hat{F}$ to the C^*-subalgebra ${}_A\mathscr{L}(X_B)$ yields a map, still denoted $\hat{F}$, with the same properties, and with range the ideal $pZ(M(A))$ of the commutative C^*-algebra $Z(M(A)) = C(\Omega)$. We write $\Omega = \Omega_0 \cup \Omega_1$, with Ω_0 corresponding to the projection p and Ω_1 to $I - p$. The map $\hat{F}$ makes ${}_A\mathscr{L}(X_B)$ into a right Hilbert $C(\Omega)$-module (in fact a Hilbert $C(\Omega_0)$-module) by $(S|T) = \hat{F}(S^*T)$. Since $\hat{F}$ is norm continuous and satisfies a Pimsner–Popa inequality, the operator norm and the Hilbert module norm are equivalent, therefore ${}_A\mathscr{L}(X_B)$ is complete in the Hilbert module norm. Since the inner product is evaluated on a commutative C^*-algebra, we can represent ${}_A\mathscr{L}(X_B)$ as a *continuous field of Hilbert spaces* over Ω in the sense of [Di]. For each $\omega \in \Omega$, the fiber Hilbert space at ω is given by $H_\omega =$

${}_A\mathscr{L}(X_B)/M_\omega$, where M_ω is the norm closed subspace of ${}_A\mathscr{L}(X_B)$, in the Hilbert module norm, generated by ${}_A\mathscr{L}(X_B)\hat{\phi}(C_\omega(\Omega))$. For each $\omega \in \Omega_1$, $M_\omega = {}_A\mathscr{L}(X_B)$ since $\hat{\phi}$ annihilates $(I-p)Z(M(A)) = C(\Omega_1)$, therefore $H_\omega = 0$, as expected. Since the C^*-algebra norm and the Hilbert module norm are equivalent, $J_\omega = M_\omega$ as vector spaces, and they are isomorphic as Banach spaces. In particular, $\mathscr{L}_\omega = 0$ for $\omega \in \Omega_1$. Let $\pi_\omega : {}_A\mathscr{L}(X_B) \to \mathscr{L}_\omega$ and $p_\omega : {}_A\mathscr{L}(X_B) \to H_\omega$ denote the corresponding quotient maps in the C^*-algebraic and Banach space sense.

Lemma 3.2. *If ${}_AX_B$ is a bi-Hilbertian bimodule of finite right index, for all $T \in {}_A\mathscr{L}(X_B)$ and for all $\omega \in \Omega_0$,*

$$\lambda'^{1/2}||\pi_\omega(T)|| \leqslant ||p_\omega(T)|| \leqslant (r - \mathrm{Ind}[X])(\omega)^{1/2}||\pi_\omega(T)||,$$

where λ' is the best positive scalar for which $||{}_A(x|x)|| \geqslant \lambda'||(x|x)_B||$, $x \in X$.

Proof. The positive $M(A)$-bilinear map $\hat{F} : \mathscr{L}(X_B) \to M(A)$ satisfies $\hat{\phi}\hat{F}(T) \geqslant \lambda' T$ for all $T \in \mathscr{L}(X_B)^+$, by Proposition 2.27. Therefore if $T \in {}_A\mathscr{L}(X_B)^+$, evaluating π_ω on this estimate yields $\pi_\omega(\hat{\phi}\hat{F}(T)) \geqslant \lambda'\pi_\omega(T)$ which shows that

$$\begin{aligned} ||p_\omega(T)||^2 &= (p_\omega(T), p_\omega(T)) = \hat{F}(T^*T)(\omega) \\ &= |\pi_\omega(\hat{\phi}\hat{F}(T^*T))| \geqslant \lambda'|\pi_\omega(T^*T)| = \lambda'||\pi_\omega(T)||^2. \end{aligned}$$

Consider the map $G_\omega : \mathscr{L}_\omega \to \mathbb{C} = C(\Omega)/C_\omega(\Omega)$ associating $\hat{F}(T)(\omega)$ to $\pi_\omega(T)$. This map is well defined: $\pi_\omega(T) = 0$ implies that $T \in J_\omega$, therefore $\hat{F}(T)$ belongs to $F(J_\omega)$ which is contained in the closed linear span of $C_\omega(\Omega)F({}_A\mathscr{L}(X_B))$ in the C^*-algebra norm. Clearly the latter space is contained in $C_\omega(\Omega)$. Now G_ω is a positive functional on the C^*-algebra $\mathscr{L}_\omega$ taking the unit of $\mathscr{L}_\omega$ to $(r - \mathrm{Ind}[X])(\omega)$, and therefore $||G_\omega|| = (r - \mathrm{Ind}[X])(\omega)$. Thus for all $T \in {}_A\mathscr{L}(X_B)$,

$$\begin{aligned} ||p_\omega(T)||^2 = |\hat{F}(T^*T)(\omega)| &= ||G_\omega(\pi_\omega(T^*T))|| \\ &\leqslant (r - \mathrm{Ind}[X])(\omega)||\pi_\omega(T^*T)|| \\ &= (r - \mathrm{Ind}[X])(\omega)||\pi_\omega(T)||^2. \qquad \square \end{aligned}$$

We are now ready to show the following result.

Theorem 3.3. *Let X be a bi-Hilbertian A–B C^*-bimodule of finite right index, and let Ω be the spectrum of $Z(M(A))$. Then for each $\omega \in \Omega$, the quotient C^*-algebra $\mathscr{L}_\omega$ is finite dimensional, and*

$$\dim(\mathscr{L}_\omega) \leqslant [\lambda'^{-1}(r - \mathrm{Ind}[X])(\omega)]^2,$$

where λ' is the best constant for which $||_A(x|x)||\geqslant\lambda'||(x|x)_B||$ and $[\mu]$ denotes the integral part of the real number μ. In particular, the fibers are trivial on Ω_1. Furthermore the collection of epimorphisms $\pi_\omega : {}_A\mathscr{L}(X_B)\to\mathscr{L}_\omega$, $\omega\in\Omega$, defines a continuous bundle of C^*-algebras in the sense of [KW].

Proof. Let us consider the positive map $\hat{F}:\mathscr{L}(X_B)\to M(A)$, which satisfies $\hat{\phi}(\hat{F}(T))\geqslant\lambda' T$ for $T\in\mathscr{L}(X_B)^+$ by Corollary 2.11. We restrict $\hat{\phi}\hat{F}$ to a map ${}_A\mathscr{L}(X_B)\to\hat{\phi}(Z(M(A))$ satisfying a corresponding inequality. Evaluating π_ω on this inequality yields $\pi_\omega(\hat{\phi}\hat{F}(T))\geqslant\lambda'\pi_\omega(T)$, $T\in{}_A\mathscr{L}(X_B)^+$. On the other hand for each ω in the support projection of $r-\mathrm{Ind}[X]$, $\pi_\omega(\hat{\phi}\hat{F}(T))=G_\omega(\pi_\omega(T))$, where G_ω is the positive functional of $\mathscr{L}_\omega$ defined as in the proof of the previous lemma: $G_\omega(\pi_\omega(T))=\hat{F}(T)(\omega)$. Therefore $g_\omega:=((r-\mathrm{Ind}[X])(\omega))^{-1}G_\omega$ is a state of $\mathscr{L}_\omega$ satisfying

$$(r-\mathrm{Ind}[X])(\omega)g_\omega(\pi_\omega(T))\geqslant\lambda'\pi_\omega(T),\quad T\in{}_A\mathscr{L}(X_B)^+.$$

It is well known that this condition implies that $\mathscr{L}_\omega$ is a finite dimensional C^*-algebra with at most $[\lambda'^{-1}(r-\mathrm{Ind}[X])(\omega)]$ minimal orthogonal projections, therefore $\dim(\mathscr{L}_\omega)\leqslant[\lambda'^{-1}(r-\mathrm{Ind}[X])(\omega)]^2$.

We are left to show that $\omega\in\Omega\to\pi_\omega$ is a continuous bundle in the sense of axioms (i)–(iii) of Definition 1.1 in [KW]. If T is positive and satisfies $\pi_\omega(T)=0$ for all $\omega\in\Omega$ then $T\in J_\omega$ for all ω in Ω and therefore $F(T)=0$ which implies $T=0$ by the Pimsner–Popa inequality. This shows axiom (i). Axiom (ii) is obvious. We are left to show that for all $T\in{}_A\mathscr{L}(X_B)$, the function $\omega\in\Omega\to||\pi_\omega(T)||$ is continuous. We will appeal to the continuity criteria discussed in Section 2 of [KW]. This function is upper semicontinuous by Lemma 2.3 in [KW] and it is lower semicontinuous by Lemma 2.2 in the same paper. Indeed, if $\Omega'\subset\Omega$ is a closed subset of Ω and $D\subset\Omega'$ is dense in Ω' then the condition $\pi_\omega(T)=0$ for all $\omega\in D$ and some $T\in{}_A\mathscr{L}(X_B)^+$ implies $T\in J_\omega$ for all $\omega\in D$, thus $F(T)(\omega)=0$ for all $\omega\in D$ and therefore for all $\omega\in\Omega'$ by continuity of the function $F(T)$. Now evaluating π_ω on both sides of the inequality $\hat{\phi}\hat{F}(T)\geqslant\lambda' T$ shows that $\pi_\omega(T)=0$ for all $\omega\in\Omega'$. □

Remark. Notice that the estimate given in Theorem 3.3 cannot be improved in general. In fact, if H is the finite index $\mathbb{C}$–$\mathbb{C}$ bimodule defined as in Example 2.30, with Ω a one point space, then ${}_{\mathbb{C}}\mathscr{L}(H_{\mathbb{C}})=M_n(\mathbb{C})$, which is the only fiber. In this case $\lambda'^{-1}=||T^{-1}||$, so the corresponding estimate reduces to $n\leqslant||T^{-1}||\mathrm{Tr}(T)$ which becomes an equality for $T=I$.

4. On the equivalence between finite index and conjugate equations

Our next aim is to show an equivalence between the notion of C^*-bimodule of finite index in the sense of Section 2 and Longo–Roberts conjugate object in the C^*-category of right Hilbert bimodules [LR].

4.1. The C^*-categories $\mathscr{H}_{\mathscr{A}}$, ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ and the W^*-categories $\mathscr{H}^w_{\mathscr{A}}$, ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$

Definition 4.1. Let $\mathscr{A}$ be a fixed set of C^*-algebras. We will denote by $\mathscr{H}_{\mathscr{A}}$ the category with objects and arrows defined as follows. Objects of $\mathscr{H}_{\mathscr{A}}$ are right Hilbert C^*-bimodules X over elements of $\mathscr{A}$ for which the left action is nondegenerate. The set of arrows (X, Y) in $\mathscr{H}_{\mathscr{A}}$ between two objects ${}_AX_B$ and ${}_AY_B$ is the set $(X, Y) := \mathscr{L}(X_B, Y_B)$ of (right) adjointable maps from X to Y. Given two objects ${}_AX_B$ and ${}_BY_C$ of $\mathscr{H}_{\mathscr{A}}$, their tensor product $X \otimes_B Y$ is still a nondegenerate right Hilbert C^*-bimodule, and therefore it is an object of $\mathscr{H}_{\mathscr{A}}$. For any $T \in (X, Y)$, the map taking a simple tensor $x \otimes y \in X \otimes_B Y$ to $T(x) \otimes y$, and denoted by $T \otimes I_Y$, extends to an adjointable map on $X \otimes_B Y$. For any C^*-algebra $A \in \mathscr{A}$, let ι_A be A, regarded as a right Hilbert bimodule over A itself, in the natural way. Since left action on ι_A is nondegenerate, ι_A is an object of $\mathscr{H}_{\mathscr{A}}$. For any right Hilbert A–B C^*-bimodule X, the tensor product Hilbert bimodule $X \otimes_B \iota_B$ identifies naturally with X. In general, $\iota_A \otimes_A X$ identifies with the right Hilbert sub-bimodule of X generated by AX, which, in the case where the left action is nondegenerate, still coincides with X (cf. Definition 2.14). Therefore $\{\iota_A, A \in \mathscr{A}\}$ is the set of left and right units for the $\otimes$-product between objects. One can summarize the structure of $\mathscr{H}_{\mathscr{A}}$, and say that $\mathscr{H}_{\mathscr{A}}$ is a *semitensor* 2-C^*-category (in the sense of [DPZ]).

If we want a *tensor* 2-C^*-category we need to restrict the arrow spaces, and consider only bimodule maps. Namely, let ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ be the subcategory of $\mathscr{H}_{\mathscr{A}}$ with the same objects and arrows $(X, Y) := {}_A\mathscr{L}(X_B, Y_B)$, the set of right adjointable maps from X to Y commuting with the left action. This is now a tensor 2-C^*-category.

In the sequel we will consider also the W^*-categories $\mathscr{H}^w_{\mathscr{A}}$ ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$ with the same objects, and set of arrows between two objects X and Y obtained completing the corresponding arrow spaces of $\mathscr{H}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ in a suitable weak topology. Choose, for each unit object $\iota_B \in \mathscr{H}_{\mathscr{A}}$, a state ω_B of B, and let us endow X with the inner product

$$(x, x')_{\omega_B} = \omega_B((x|x')_B), x, x' \in X.$$

Completing X, after dividing out by vectors of seminorm zero, with respect to this inner product, yields a Hilbert space $H_{\omega_B}(X)$. For each $T \in \mathscr{L}(X_B, Y_B)$, let $\mathscr{F}_\omega(T) \in \mathscr{B}(H_{\omega_B}(X), H_{\omega_B}(Y))$ be the operator which acts by left multiplication by T. We get in this way a *-functor $\mathscr{F}_\omega : \mathscr{H}_{\mathscr{A}} \to \mathscr{H}$ to the category $\mathscr{H}$ of Hilbert spaces. Consider now the *universal* *-functor $\mathscr{F} = \oplus_\omega \mathscr{F}_\omega : \mathscr{H}_{\mathscr{A}} \to \mathscr{H}$, where the direct sum is taken over all choice functions $\omega : B \in \mathscr{A} \to \omega_B$. $\mathscr{F}$ is faithful on arrows and strictly continuous on the unit ball of each arrow space.

Define (X, Y) to be the completion of $\mathscr{F}(\mathscr{K}(X_B, Y_B))$ in the weak topology of the bounded operators from $\oplus_\omega H_{\omega_B}(X)$ to $\oplus_\omega H_{\omega_B}(Y)$, and let $\mathscr{H}^w_{\mathscr{A}}$ be the W^*-category with arrows these W^*-closed subspaces. Since any operator in $\mathscr{L}(X_B, Y_B)$ is the strict limit of a norm bounded net from $\mathscr{K}(X_B, Y_B)$, $\mathscr{F}(\mathscr{L}(X_B, Y_B)) \subset (X, Y)$, therefore $\mathscr{H}_{\mathscr{A}}$ becomes a C^*-subcategory of $\mathscr{H}^w_{\mathscr{A}}$ under $\mathscr{F}$. The universal functor enjoys the following universality property.

Proposition 4.2. *A *-functor $\mathscr{G} : \mathscr{H}_{\mathscr{A}} \to \mathscr{H}$ to the category of Hilbert spaces, strictly continuous on the unit ball of each arrow space of $\mathscr{H}_{\mathscr{A}}$, extends uniquely to a *-functor $\mathscr{G}'' : \mathscr{H}^w_{\mathscr{A}} \to \mathscr{H}$, normal on the arrow spaces.*

Proof. Let us first assume that each Hilbert space $\mathscr{G}(\iota_B)$ is cyclic for $\mathscr{G}((\iota_B, \iota_B))$. Let ξ_B be a normalized cyclic vector. Then, identifying X with the subspace of intertwiners $\mathscr{K}(\iota_B, X_B) \subset (\iota_B, X)$, $\mathscr{G}(X)\xi_B$ is a subspace of the Hilbert space $\mathscr{G}(X)$ associated to the object X. We claim that $\mathscr{G}(X)\xi_B$ is the whole $\mathscr{G}(X)$. Let $\eta \in \mathscr{G}(X)$ be a vector orthogonal to $\mathscr{G}(X)\xi_B$. For all $x \in X$, $\mathscr{G}(x^*)\eta$ is orthogonal to $\mathscr{G}((\iota_B, \iota_B))\xi_B$ and hence it is zero. Since $\mathscr{G}$ is strictly continuous on the unit ball of (X, X), we conclude that $\eta = 0$. We therefore have an identification of the Hilbert space $\mathscr{G}(X)$ with $\mathscr{F}_\omega(X)$ where $\omega : B \to \omega_{\xi_B}$, and also an identification of $\mathscr{G}$ with $\mathscr{F}_\omega$. Now every *-functor $\mathscr{G} : \mathscr{H}_{\mathscr{A}} \to \mathscr{H}_{\mathbb{C}}$ is the direct sum cyclic *-functors, therefore $\mathscr{G}$ is a direct sum of some $\mathscr{F}_\omega$, and the rest now follows easily.

In particular, if $R_Y : \mathscr{H}_{\mathscr{A}} \to \mathscr{H}_{\mathscr{A}}$ is the *-functor which tensors on the right by an object $Y \in \mathscr{H}_A$, the normal extension of $\mathscr{F} \circ R_Y$, with $\mathscr{F}$ the universal *-functor, makes $\mathscr{H}^w_{\mathscr{A}}$ into a semitensor 2-W^*-category.

The subcategory ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$ of $\mathscr{H}^w_{\mathscr{A}}$ with the same objects and arrows

$$(X, Y) := \{T \in \mathscr{F}(\mathscr{K}(X_B, Y_B))'' : T\mathscr{F}(\phi(a)) = \mathscr{F}(\phi'(a))T, a \in A, x \in X\},$$

(where ϕ and ϕ' denote respectively the left actions of A on X and Y, and $\mathscr{F}$ is the universal *-functor) is now a tensor 2-W^*-category. $\square$

Remark. The functor of R_Y may not be injective on arrows in any of these categories. In other words, if ${}_AX_B$, ${}_AX'_B$ and ${}_BY_C$ are right Hilbert C^*-bimodules, the natural *-homomorphism

$$T \in \mathscr{L}(X_B, X'_B) \to T \otimes I_Y \in \mathscr{L}((X \otimes_B Y)_C, (X' \otimes_B Y)_C)$$

may not be injective. In fact, if $X = X' = \iota_B$ and $b \in B \subset \mathscr{L}(\iota_B) = M(B)$, under the identification of $\iota_B \otimes_B Y$ with Y, $b \otimes I_Y$ corresponds to the left action of B on Y evaluated in b, which may vanish.

4.2. Conjugation in ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$

In the sequel $\mathscr{T}$ will denote either ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ or ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$. Following [LR], we can introduce the notion of conjugation in the tensor C^* (or W^*) category $\mathscr{T}$.

Definition 4.3. Let $X = {}_AX_B$ be an object of $\mathscr{T}$. An object $Y = {}_BY_A$ of $\mathscr{T}$ is called a *conjugate* of X if there exist intertwiners $R \in (\iota_B, Y \otimes_A X) \in \mathscr{T}$ and

$\bar{R} \in (\iota_A, X \otimes_B Y) \in \mathscr{T}$ such that

$$\bar{R}^* \otimes I_X \circ I_X \otimes R = I_X,$$

$$R^* \otimes I_Y \circ I_Y \otimes \bar{R} = I_Y.$$

We adopt the convention that the $\otimes$-product is evaluated before $\circ$-product. We emphasize that, if $\mathscr{T} = {}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$, R and $\bar{R}$ are C^*-bimodule maps, i.e. they commute with left as well as right actions of the appropriate C^*-algebras. Therefore in this case R^*R and $\bar{R}^*\bar{R}$ are elements of ${}_B\mathscr{L}(\iota_B) = Z(M(B))$ and ${}_A\mathscr{L}(\iota_A) = Z(M(A))$ respectively. If, instead, $\mathscr{T} = {}_{\mathscr{A}}\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$, we can only conclude that R^*R and $\bar{R}^*\bar{R}$ are central elements of B'' and A'' respectively.

The above equations will be referred to as *the conjugate equations*. Clearly, if Y is a conjugate of X then X is a conjugate of Y.

The *dimension of X relative to the pair* $(R, \bar{R})$ is defined by $\dim_{R,\bar{R}} X = ||R||||\bar{R}||$. The *minimal dimension* of X, denoted by $\dim X$, is the infimum of all relative dimensions $\dim_{R,\bar{R}} X$.

4.3. From finite index to conjugation

Theorem 4.4. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule. Then left actions on the underlying right Hilbert C^*-bimodules X and $\bar{X}$ are nondegenerate, and therefore these are objects of ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$.*

(1) *If X is of finite numerical index, $\bar{X}$ is a conjugate of X in ${}_{\mathscr{A}}\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$. More specifically, if $\{u_\mu\}_\mu$ and $\{v_\nu\}_\nu$ are, respectively, a generalized right and left basis of X, the nets $\bar{R}_\mu := \sum_{y \in u_\mu} y \otimes \bar{y}$ and $R_\nu := \sum_{z \in v_\nu} \bar{z} \otimes z$ converge strongly under the universal *-functor to intertwiners $\bar{R} \in (\iota_A, X \otimes^r_A \bar{X})$ and $R \in (\iota_B, \bar{X} \otimes^r_B X)$ of ${}_{\mathscr{A}}\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$ which do not depend on the choice of the bases, and solve the conjugate equations. The following relations also hold for $x, x' \in X$,*

$$\bar{R}^*(\theta^r_{x,x'} \otimes I_{\bar{X}})\bar{R} = {}_A(x|x'), \quad R^*(\theta^r_{\bar{x},\overline{x'}} \otimes I_X)R = (x|x')_B,$$

$$R^*R = \ell - \mathrm{Ind}[X], \quad \bar{R}^*\bar{R} = r - \mathrm{Ind}[X].$$

(2) *If X is of finite index, R and $\bar{R}$ belong to ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$. So the right Hilbert bimodule $\bar{X}$ is a conjugate of X in ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$. Their right adjoint operators are given by*

$$\bar{R}^* x \otimes \overline{x'} = {}_A(x|x'), \quad R^* \bar{x} \otimes x' = (x|x')_B.$$

Proof. By Proposition 2.16 the left (right) action on the right (left) Hilbert C^*-bimodule X is nondegenerate, therefore the right Hilbert C^*-bimodules X and $\bar{X}$ are objects of ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{H}^{w}_{\mathscr{A}}$. We claim that, under the natural identifications of $X\otimes^r_B\bar{X}$ with $\mathscr{K}(\iota_A, X\otimes^r_B\bar{X}_A)$ and of $\bar{X}\otimes^r_A X$ with $\mathscr{K}(\iota_B, \bar{X}\otimes^r_A X_B)$, the nets $\bar{R}_\mu := \sum_{y\in u_\mu} y\otimes\bar{y}$ and $R_\nu := \sum_{z\in v_\nu} \bar{z}\otimes z$ converge strongly in the universal *-functor to operators $\bar{R}$ and R which do not depend on the choice of the bases. It suffices to show that the first net is strongly Cauchy, as, replacing X with $\bar{X}$, $\mu\to u_\mu$ changes to $\nu\to v_\nu$. Now by Proposition 2.19, $\sum_{y\in u_\mu} {}_A(y|y)$ is a positive, increasing, norm bounded net, and it is strongly convergent in A'' to $r-\mathrm{Ind}[X]$. Since for $\mu<\mu'$, $\sum_{y\in u_{\mu'}}\theta^r_{y,y} - \sum_{y\in u_\mu}\theta^r_{y,y}$ is a positive contraction, we have

$$\left(\sum_{y\in u_{\mu'}} y\otimes\bar{y} - \sum_{y\in u_\mu} y\otimes\bar{y}\Big| \sum_{y\in u_{\mu'}} y\otimes\bar{y} - \sum_{y\in u_\mu} y\otimes\bar{y}\right)_A$$

$$= F_X\left(\left(\sum_{y\in u_{\mu'}}\theta^r_{y,y} - \sum_{y\in u_\mu}\theta^r_{y,y}\right)^2\right) \leqslant F_X\left(\sum_{y\in u_{\mu'}}\theta^r_{y,y} - \sum_{y\in u_\mu}\theta^r_{y,y}\right)$$

$$= \sum_{y\in u_{\mu'}} {}_A(y|y) - \sum_{y\in u_\mu} {}_A(y|y).$$

Therefore the net $\bar{R}_\mu\in\mathscr{K}(\iota_A, X\otimes\bar{X}_A)$ is strongly convergent on a dense subspace of the corresponding universal Hilbert space. We show that this net is norm bounded. We have, for $a\in A$,

$$\|(\bar{R}_\mu(a)|\bar{R}_\mu(a))_A\| = \left\|\left(U\left(\sum_{y\in u_\mu}(y\otimes\bar{y})a\right)\Big|U\left(\sum_{y\in u_\mu}(y\otimes\bar{y})a\right)\right)_A\right\|$$

$$= \left\|F\left(\left(\sum_{y\in u_\mu}\theta^r_{y,y}\phi(a)\right)^*\left(\sum_{y\in u_\mu}\theta^r_{y,y}\phi(a)\right)\right)\right\|$$

$$\leqslant \|(r-\mathrm{Ind}[X])a^*a\|,$$

where U is the biunitary map defined in Proposition 2.13(3). Hence

$$\|\bar{R}_\mu\| \leqslant \left(\sup_{a\neq 0}\frac{\|(r-\mathrm{Ind}[X])a^*a\|}{\|a^*a\|}\right)^{1/2} = \|r-\mathrm{Ind}[X]\|^{1/2}.$$

It follows that $\bar{R}_\mu$ is strongly convergent to an operator $\bar{R}\in(\iota_A, X\otimes\bar{X})\subset\mathscr{H}^w_{\mathscr{A}}$ with $\|\bar{R}\|\leqslant\|r-\mathrm{Ind}[X]\|^{1/2}$. Similarly we define a map $R\in(\iota_B, \bar{X}\otimes X)\subset\mathscr{H}^w_{\mathscr{A}}$ as the strong limit of $\sum_{z\in v_\nu}(\bar{z}\otimes z)$ such that $\|R\|\leqslant\|\ell-\mathrm{Ind}[X]\|^{1/2}$. In order to show that $\bar{R}$ is independent on the basis, we compute its Hilbert space adjoint. Let $\omega : B\in\mathscr{A}\to\omega_B$ be a choice of states of the C^*-algebras of $\mathscr{A}$, and let $\mathscr{F}_\omega : \mathscr{H}_{\mathscr{A}}\to\mathscr{H}$ be the associated cyclic *-functor to the category of Hilbert spaces. For $x, x' \in X$, $a \in A$,

$$
\begin{aligned}
(a, \mathscr{F}_\omega(\bar{R}_\mu)^* x \otimes \overline{x'})_{\omega_A} &= \sum_{y \in u_\mu} (y \otimes \bar{y}a, x \otimes \overline{x'})_{\omega_A} \\
&= \sum_{y \in u_\mu} \omega_A((\bar{y}a|(y|x)_B \overline{x'})_A) = \sum_{y \in u_\mu} \omega_A({}_A(a^* y|x'(x|y)_B)) \\
&= \omega_A\left({}_A\left(a^* \sum_{y \in u_\mu} y(y|x)_B, x'\right)\right).
\end{aligned}
$$

Therefore $\mathscr{F}_\omega((\bar{R}_\mu)^* x \otimes \overline{x'})$ converges weakly to ${}_A(x|x')$, regarded as an element of the Hilbert space $\mathscr{F}_\omega(\iota_A)$. It follows that $\bar{R}^*$, and hence $\bar{R}$, is independent of the generalized right basis. On the other hand the net $\bar{R}_\mu$, regarded as a net in the Hilbert space $\mathscr{F}_\omega(X \otimes \bar{X})$, has norm bounded above by $(r - I[X])^{1/2}$, therefore

$$
\begin{aligned}
||\bar{R}|| = ||\bar{R}^*|| &\geqslant (r - I[X])^{-1/2} \left\| \bar{R}^* \left(\sum_{y \in u_\mu} y \otimes \bar{y} \right) \right\| \\
&= (r - I[X])^{-1/2} \left\| \sum_{y \in u_\mu} {}_A(y|y) \right\|,
\end{aligned}
$$

which shows that $||R|| = (r - I[X])^{1/2}$.

Let now $U \in M(A)$ be a unitary. For any generalized right basis u_μ, $\mu \to \{Uy, y \in u_\mu\}$ is still a generalized right basis, so $U\bar{R}U^* = \bar{R}$ by independence of the operator $\bar{R}$ on the basis. Hence $\bar{R} \in {}_A\mathscr{H}^w_{\mathscr{A}}$.

We show that R and $\bar{R}$ solve the conjugate equations. For $x \in X$, $b \in B$, we have, in the Hilbert space associated to X under the universal *-functor:

$$
\begin{aligned}
\bar{R}^* \otimes I_X \circ I_X \otimes R(xb) &= \bar{R}^* \otimes I_X \left(x \otimes \lim_\nu \sum_{z \in v_\nu} (\bar{z} \otimes z) b \right) \\
&= \lim_\nu \sum_{z \in v_\nu} {}_A(x|z) z b = xb.
\end{aligned}
$$

Since $X \otimes_B \iota_B$ identifies with X via the map $x \otimes b \mapsto xb$, we obtain the conjugate equation $\bar{R}^* \otimes I_X \circ I_X \otimes R = I_X$ in ${}_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$. Similarly we have $R^* \otimes I_Y \circ I_Y \otimes \bar{R} = I_Y$.

For any $a \in \mathscr{F}(\iota_A)$ we have

$$
\bar{R}^* \bar{R}(a) = \lim_\mu \bar{R}^* \left(\sum_{y \in u_\mu} y \otimes \bar{y} a \right) = \lim_\mu \sum_{y \in u_\mu} {}_A(y|a^* y) = \lim_\mu \sum_{y \in u_\mu} {}_A(y|y) a,
$$

so $\bar{R}^* \bar{R} = r - \mathrm{Ind}[X]$ and $R^* R = \ell - \mathrm{Ind}[X]$ as well.

We show that $\bar{R}^*(\theta^r_{x,z} \otimes I_{\bar{X}})\bar{R} = {}_A(x|z)$ (the similar equation relative to R will follow replacing X with $\bar{X}$). For $a \in A$,

$$\begin{aligned}\bar{R}^*(\theta^r_{x,z} \otimes I_Y)\bar{R}(a) &= \bar{R}^*(\theta^r_{x,z} \otimes I_Y) \lim_\mu \left(\sum_{y \in u_\mu} (y \otimes \bar{y})a \right) \\ &= \lim_\mu \bar{R}^* \left(\sum_{y \in u_\mu} \theta^r_{x,z}(y) \otimes \overline{a^* y} \right) = \lim_\mu \sum_{y \in u_\mu} {}_A(\theta^r_{x,z}(y)|a^* y) \\ &= \lim_\mu \sum_{y \in u_\mu} {}_A(x(z|y)_B|y)a = {}_A\left(x \middle| \lim_\mu \sum_{y \in u_\mu} y(y|z)_B \right) a = {}_A(x|z)a.\end{aligned}$$

(2) In the case where X is of finite index, the net $\bar{R}_\mu(a)$ converges in norm for all $a \in A$, therefore R is actually mapping A to $X \otimes \bar{X}$. Furthermore $\bar{R}$ is right adjointable, in fact its adjoint $\bar{R}^* : X \otimes_B \bar{X} \to A$ is defined by $\bar{R}^*(x \otimes \overline{x'}) = {}_A(x|x')$:

$$\begin{aligned}(\bar{R}(a)|x \otimes \bar{x}')_A &= \lim_\mu \sum_{y \in u_\mu} a^*(y \otimes \bar{y}|x \otimes \bar{x}')_A \\ &= \lim_\mu \sum_{y \in u_\mu} a^*(\bar{y}|(y|x)_B \bar{x}')_A = \lim_\mu \sum_{y \in u_\mu} a^*{}_A(y|x'(y|x)^*_B) \\ &= a^*{}_A\left(\lim_\mu \sum_{y \in u_\mu} y(y|x)_B \middle| x' \right) = a^*{}_A(x|x').\end{aligned}$$

A first consequence of the previous theorem is the fact that the left Hilbert bimodule structure on a finite index bimodule is unique up to equivalence. □

Corollary 4.5. *Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule of finite index. Any other left inner product on the underlying right Hilbert bimodule ${}_AX_B$ making it into a finite index, bi-Hilbertian bimodule is of the form*

$$ {}_A(x|y)' = {}_A(Qx|y), \quad x, y \in X, $$

where Q is a positive invertible element of $\mathscr{L}_B({}_AX)$.

Proof. Consider another left inner product ${}_A(\cdot|\cdot)'$ making X into a bi-Hilbertian, finite index C^*-bimodule. Let X' denote the left Hilbert bimodule structure over X with inner product ${}_A(\cdot|\cdot)'$. By part (2) of Theorem 4.4, we can find another solution $(\overline{X'}, R', \bar{R}')$ to the conjugate equations such that ${}_A(x|x')' = \overline{R'}^*(\theta^r_{x,x'} \otimes I_{\overline{X'}})\overline{R'}$. By uniqueness of the conjugate object (cf [LR]) there is an invertible $U \in {}_B\mathscr{L}_A(\bar{X}, \overline{X'})$ such that $\bar{R}' = I_X \otimes U \circ \bar{R}$. Therefore ${}_A(x|x')' = \bar{R}^*(\theta^r_{x,x'} \otimes U^*U)\bar{R}$. We just need to plug in the fact that $\bar{R} = \lim_\mu \sum_{y \in u_\mu} y \otimes \bar{y}$ in the pointwise norm convergence topology and choose $Q := J_X^{-1} U^* U J_X$, with $J_X : X_B \to {}_B\bar{X}$ the natural conjugation map.

4.4. On the equality ${}_A\mathscr{L}(X_B) = \mathscr{L}_B({}_AX)$ for finite index bimodules

Let ${}_AX_B$ be a bi-Hilbertian C^*-bimodule. We can consider the C^*-algebra ${}_A\mathscr{L}(X_B)$ of right adjointable maps commuting with the left action, but also the C^*-algebra $\mathscr{L}_B({}_AX)$ of left adjointable maps commuting with the right action. If A and B are unital, and X is finitely generated, as a right and left module, any bimodule map on X is right adjointable and left adjointable, therefore ${}_A\mathscr{L}(X_B) = \mathscr{L}_B({}_AX) = {}_A\mathrm{End}_B(X)$. More generally, under which conditions ${}_A\mathscr{L}(X_B) = \mathscr{L}_B({}_AX)$ as algebras? The following result provides an answer.

Corollary 4.6. *If ${}_AX_B$ is a bi-Hilbertian bimodule of finite index, any element of ${}_A\mathscr{L}(X_B)$ is adjointable with respect to the left inner product, and therefore it belongs to $\mathscr{L}_B({}_AX)$. Similarly, any element of $\mathscr{L}_B({}_AX)$ is adjointable with respect to the right inner product. Therefore ${}_A\mathscr{L}(X_B) = \mathscr{L}_B({}_AX)$ as algebras.*

Proof. Let R and $\bar{R}$ be the solution to the conjugate equations arising from the left and right inner products as in the proof of the previous theorem. By Frobenius reciprocity there is a linear isomorphism from ${}_A\mathscr{L}(X_B)$ to ${}_B\mathscr{L}(\iota_B, \bar{X}\otimes X_B)$ given by $T\to I_{\bar{X}}\otimes T\circ R$ and an antilinear isomorphism from ${}_B\mathscr{L}(\iota_B, \bar{X}\otimes X_B)$ to ${}_B\mathscr{L}(\bar{X}_A)$ given by $S\to S^*\otimes I_{\bar{X}}\circ I_{\bar{X}}\otimes\bar{R}$ (see [LR]). A straightforward computation shows that the composition of these maps is the map $T\in{}_A\mathscr{L}(X_B)\to JTJ^{-1}\in{}_B\mathscr{L}(\bar{X}_A)$ where $J: {}_AX\to\bar{X}_A$ is the conjugation map. Therefore the map $T\in{}_A\mathscr{L}(X_B)\to T\in\mathscr{L}_B({}_AX)$ is a linear multiplicative isomorphism. $\square$

4.5. Computing the left index element of X

Lemma 4.7. *Let $X = {}_AX_B$ and $Y = {}_BY_A$ be nondegenerate right Hilbert C^*-bimodules, conjugate of each other as objects of ${}_{\mathscr{A}}\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$, and let $(R, \bar{R})$ be a solution of the corresponding conjugate equations. Let us regard $\mathscr{K}(X_B)$ as a C^*-subalgebra of the intertwiner space $(X, X)\simeq\mathscr{K}(X_B)''$ of $\mathscr{H}^{\mathrm{w}}_{\mathscr{A}}$. Then the map*

$$T\in\mathscr{K}(X_B)\to(\bar{R}^*\circ T\otimes I_Y\circ\bar{R})\otimes I_X\otimes R^*R - T\in\mathscr{K}(X_B)''$$

is completely positive.

Proof. For all $n\in\mathbb{N}$ and any positive $T = (T_{ij})\in M_n(\mathscr{K}(X_B))$,

$$\begin{aligned}T &= (\bar{R}^*\otimes I_X\circ I_X\otimes RT_{ij}I_X\otimes R^*\circ\bar{R}\otimes I_X)_{i,j}\\ &= (\bar{R}^*\otimes I_X(T_{ij}\otimes RR^*)\bar{R}\otimes I_X)_{i,j}\leqslant((\bar{R}^*\circ T_{ij}\otimes I_Y\circ\bar{R})\otimes I_X\otimes(R^*R))_{i,j}\end{aligned}$$

since $RR^*\leqslant I_{Y\otimes X}\otimes(R^*R)$ by Lemma 2.7 in [LR]. $\square$

Combining the previous lemma with the main theorem of [FK], yields the following result.

Theorem 4.8. *Let $_AX_B$ be a bi–Hilbertian bimodule of finite right numerical index, and let $F : \mathscr{K}(X_B) \to A$ be the positive A–A bimodule map constructed in Corollary* 2.11. *Then X is also of finite left numerical index. Denoting by ϕ and ψ the left and right actions of A and B on X respectively, and by q the support projection of the left index element in B'', $\ell - \mathrm{Ind}[X]$ is the smallest central element c of qB'' for which the map $\psi_0(c)\phi \circ F - \mathrm{id} : \mathscr{K}(X_B) \to \mathscr{K}(X_B)''$ is completely positive. Here ψ_0 denotes the extension to $Z(B'')$ of the right action of $Z(B)$ on X defined in Lemma* 2.18.

Proof. We claim that X is of finite left numerical index if and only if there exists a positive real c for which $c\phi \circ F - \mathrm{id} : \mathscr{K}(X_B) \to \mathscr{L}(X_B)$ is completely positive. We show the claim. If X_B has finite left numerical index, we can construct a solution R, $\bar{R}$, $\bar{X}$ to the conjugate equations as in the proof of Theorem 4.4. We have proved there that $R^*R = \ell - \mathrm{Ind}[X]$ and that for $T \in \mathscr{K}(X_B)$, $\bar{R}^*(T \otimes I_{\bar{X}})\bar{R} = F(T)$. So, recalling the definition of tensor products between operators in $_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$, with $\mathscr{A} = \{A, B\}$, we see that

$$I_X \otimes R^*R = \psi_0(\ell - \mathrm{Ind}[X])$$

and

$$(\bar{R}^*(T \otimes I_{\bar{X}})\bar{R}) \otimes I_X = \phi \circ F(T), \quad T \in \mathscr{K}(X_B).$$

Inserting these data in the conclusion of Lemma 4.7, we deduce that $\psi_0(\ell - \mathrm{Ind}[X])\phi \circ F - \mathrm{id}$ is completely positive, as a map from $\mathscr{K}(X_B)$ to $\mathscr{K}(X_B)''$. Therefore, with $c = ||\ell - \mathrm{Ind}[X]||$, $c\phi \circ F - \mathrm{id} : \mathscr{K}(X_B) \to \mathscr{L}(X_B)$ is completely positive. Conversely, if for some positive real c, $c\phi \circ F - \mathrm{id}$ is completely positive on $\mathscr{K}(X_B)$, for $n \in \mathbb{N}$ and for $x_1, \dots, x_n \in X$,

$$\left|\left|\sum_1^n (x_i|x_i)_B\right|\right| = ||(\theta^r_{x_i,x_j})_{ij}|| \leqslant c||(\phi(_A(x_i|x_j)))_{ij}|| = c\left|\left|\sum_1^n \theta^\ell_{x_i,x_i}\right|\right|,$$

so X is of finite left numerical index. On the other hand in Proposition 2.19 we have constructed a surjective conditional expectation $\phi'' \circ E'' : \mathscr{K}(X_B)'' \to \phi''(A'')$ normalizing $\phi''F''$, which does satisfy $\mu||\phi''E''(T)|| \geqslant ||T||$ for some positive real μ and all $T \in \mathscr{K}(X_B)''^+$. By the main result of [FK], $c\phi''E'' - \mathrm{id}$ is completely positive for some positive real c, and therefore $c||\phi''(r - \mathrm{Ind}[X])^{-1}||\phi''F'' - \mathrm{id} : \mathscr{K}(X_B)'' \to \mathscr{K}(X_B)''$ is completely positive. Restricting this map to $\mathscr{K}(X_B)$ and combining with the claim, shows that X is of finite left numerical index.

Let now $v \to v_v$ be a generalized left basis of X. Choosing $n = |v|$, $T = (\theta^r_{z_i,z_j}) \in M_n(\mathscr{K}(X))^+$, we see that, if c is any central element of qB'' for which

$T \in \mathscr{K}(X) \to \psi_0(c)\phi F(T) - T \in \mathscr{K}(X)''$ is completely positive then

$$(\psi_0(c)\phi({}_A(z_i|z_j))_{i,j} = (\psi_0(c)\phi F(\theta^r_{z_i,z_j}))_{i,j} \geqslant (\theta^r_{z_i,z_j})_{i,j},$$

which implies

$$\sum_{i,j} (z_i|{}_A(z_i|z_j)z_j)_B c \geqslant \sum_{i,j} (z_i|z_i(z_j|z_j)_B)_B,$$

or, in other words, $R_v^* R_v c \geqslant (\sum_{z \in v_v} (z|z)_B)^2$. Thus $(\ell - \mathrm{Ind}[X])c \geqslant (\ell - \mathrm{Ind}[X])^2$, so $c \geqslant \ell - \mathrm{Ind}[X]$. $\square$

Corollary 4.9. *Let $A \subset B$ be an inclusion of C^*-algebras, and $E : B \to A$ be a conditional expectation with range A, for which there is $\lambda > 0$ such that $||E(bb^*)|| \geqslant \lambda ||bb^*||$ for all $b \in B$. Let Ind$[E]$ be the index of E defined as in Definition* 2.17. *Then Ind$[E]$ is the smallest central element c of B'' for which $cE - id$ is completely positive.*

Proof. Let ${}_A\bar{X}_B$ be the contragradient of the B–A bimodule X associated to E as part (2) of Proposition 2.12. Clearly $\bar{X}$ is of finite numerical index. Since $\ell - \mathrm{Ind}[\bar{X}] = r - \mathrm{Ind}[X] = \mathrm{Ind}[E]$, and since $\mathscr{K}(\bar{X}_B) = B$ and $F_{\bar{X}} = E$, $\mathrm{Ind}[E]$ is, by Corollary 4.9, the smallest central element c of B'' for which $cE - \mathrm{id}$ is completely positive (recall that $r - \mathrm{Ind}[X]$ is invertible by Corollary 2.20). $\square$

Remark. If $\phi E : \mathscr{K}(X_B) \to \phi(A)$ is the faithful conditional expectation defined in Corollary 2.29 then

$$\mathrm{Ind}[X]\phi \circ E - \mathrm{id}$$

is completely positive on $\mathscr{K}(X_B)$. In fortunate cases where $\phi''(r - \mathrm{Ind}[X])$ is central in $\mathscr{K}(X_B)''$ (e.g. either A is simple, cf. Corollary 2.26, or X arises from a conditional expectation, Proposition 2.12, or $A = B$ is commutative and right action coincides with left action) then $\mathrm{Ind}[X] = \mathrm{Ind}[\phi \circ E]$. This observation thus shows that the index element of a conditional expectation coincides with the index element of the dual conditional expectation.

4.6. From conjugation to finite index

Let X be a bi-Hilbertian bimodule with a conjugate in ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$. Our next aim is to construct a left inner product on X making it into a bi-Hilbertian bimodule of finite index.

Lemma 4.10. *Let ${}_BY_A$ be a conjugate object of ${}_AX_B$ in the tensor 2-C^*-category ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$, and let R and $\bar{R}$ be a pair of intertwiners solving the conjugate equations, in the sense of Definition* 4.3. *There exist unique positive semidefinite left inner products on X and*

Y such that

$$_A(x|a^*x') = \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(a) \in A \quad \textit{for } a \in A, x, x' \in X,$$

$$_B(y|b^*y') = R^*(\theta^r_{y,y'} \otimes I_X)R(b) \in B \quad \textit{for } b \in B, y, y' \in Y.$$

Proof. For $x, x' \in X$ we define an element $_A(x|x') \in M(A) = \mathscr{L}(\imath_A)$ by

$$_A(x|x')(a) = \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(a) \in A \quad \text{for } a \in A.$$

Then $x, x' \mapsto {}_A(x|x')$ defines a continuous sesquilinear form on X with values in $M(A)$. We claim that $_A(x|x') \in A$. Let $\{u_i\}_i$ be a selfadjoint approximate unit of A with $||u_i|| \leqslant 1$. We show that $\{\bar{R}(\theta^r_{x,x'} \otimes I_Y)R(u_i)\}_i$ is a norm Cauchy net. First we assume that x' is of the form $y = a^*x''$ for some $a \in A$ and $x'' \in X$. Since

$$\bar{R}^*(\theta^r_{x,a^*x''} \otimes I_Y)\bar{R}(u_i) = \bar{R}^*(\theta^r_{x,x''} \otimes I_Y)\bar{R}(au_i),$$

and $au_i \to a$ in norm, $\{\bar{R}^*(\theta^r_{x,a^*x''} \otimes I_Y)\bar{R}(u_i)\}_i$ is a Cauchy net in norm. For a general element $x' \in X$, we choose $\tilde{x} \in AX$ sufficiently close to x' (this being possible as left A-action is nondegenerate), so one easily obtains

$$\begin{aligned}
&||\bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_i) - \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_j)|| \\
&\quad \leqslant 2||\bar{R}||^2||x||||x' - \tilde{x}|| + ||\bar{R}^*(\theta^r_{x,\tilde{x}} \otimes I_Y)\bar{R}(u_i) - \bar{R}^*(\theta^r_{x,\tilde{x}} \otimes I_Y)\bar{R}(u_j)||.
\end{aligned}$$

Thus $\{\bar{R}^*(\theta_{x,x'} \otimes I)\bar{R}(u_i)\}_i$is still a Cauchy net in A.

For $a \in A$, we have

$$\bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_i)a = \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_i a) \to \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(a)$$

in norm. This shows that the limit of the Cauchy net

$$\{\bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_i)\}_i$$

in A coincides with $_A(x|x') \in A$ and does not depend on the choice of approximate unit $\{u_i\}_i$.

It is easy to see that $(x, x') \mapsto {}_A(x|x')$ is left A-linear and right conjugate A-linear. Since for $b, c \in A$ and $x, x' \in X$,

$$\begin{aligned}
((_A(x|x'))^*b)^*c &= b^*\bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(c) \\
&= b^* \lim_i \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(u_i)c = (_A(x'|x)b)^*c,
\end{aligned}$$

we have ${}_A(x|x')^* = {}_A(x'|x)$. Since

$$({}_A(x|x)(a)|a)_A = (\bar{R}^*(\theta_{x,x} \otimes I)\bar{R}(a))|a)_A = ((\theta_{x,x} \otimes I)\bar{R}(a))|\bar{R}(a))_A \geqslant 0$$

in the canonical Hilbert A-module $\iota_A = A_A$ with $(a|b)_A = a^*b$, we have ${}_A(x|x) \geqslant 0$. Now exchanging the roles of X and Y, and of R and $\bar{R}$, and applying this argument to Y, we deduce the existence of a left inner product on Y as well.

We next show that X and Y become left Hilbert modules. To do so, we construct isomorphisms with the contragradient left Hilbert bimodules $\bar{Y}$ and $\bar{X}$ respectively. For a A–B bimodule X, we shall denote by $J_X : X \to \bar{X}$ the map associating $\bar{x}$ to x, for any $x \in X$. Clearly, $J_X(ax) = J_X(x)a^*$ and $J_X(xb) = b^*J_X(x)$ for $a \in A$, $b \in B$, $x \in X$. $\square$

Lemma 4.11. *Let X and Y be conjugate objects of ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$, and let us endow them with left inner products defined, as in the previous lemma, by a pair of intertwiners R and $\bar{R}$ solving the conjugate equations. Then there exist natural bimodule isomorphisms $U : \bar{Y} \to X$ and $V : \bar{X} \to Y$ from the contragradient bimodules, which preserve the corresponding left and right inner products and satisfy*

$$VJ_X = (UJ_Y)^{-1}.$$

In particular, X and Y become bi-Hilbertian C^*-bimodules.

Proof. For $y \in Y$ we define $l_y : X \to Y \otimes_A X$ by $l_y(x) = y \otimes x$. Then we have $l_y^*(y' \otimes x) = (y|y')_A x$. We notice that the set $\{a^*xb | a \in A,\ x \in X,\ b \in B\}$ is total in X since, by assumption, left action is nondegenerate.

By the first conjugate equation

$$a^*xb = (\bar{R}^* \otimes I_X)(I_X \otimes R)(a^*xb) = (\bar{R}^* \otimes I_X)(a^*x \otimes R(b)).$$

For $\tilde{a} \in A$, $x' \in X$

$$\begin{aligned}
((\bar{R}^* \otimes I_X)(a^*x \otimes R(b))|\tilde{a}x')_B &= (a^*x \otimes R(b)|\bar{R}(\tilde{a}) \otimes x')_B \\
&= (x \otimes R(b)|a\bar{R}(\tilde{a}) \otimes x')_B = (x \otimes R(b)|\bar{R}(a) \otimes \tilde{a}x')_B \\
&= (R(b)|l_x^*(\bar{R}(a)) \otimes \tilde{a}x')_B = (R(b)|l_{l_x^*(\bar{R}(a))}(\tilde{a}x'))_B \\
&= (l^*_{l_x^*(\bar{R}(a))}(R(b))|\tilde{a}x')_B.
\end{aligned}$$

Thus for $a \in A$, $b \in B$ and $x \in X$,

$$a^*xb = l^*_{l_x^*(\bar{R}(a))}(R(b)) \in X.$$

This shows that $\{l_y^*(R(b))|y\in Y, b\in B\}$ is total in X. Similarly, for $a\in A$, $b\in B$ and $y\in Y$,

$$b^*ya = l^*_{l_y^*(R(b))}(\bar{R}(a))\in Y.$$

We next show that for y, $y'\in Y$, b, $b'\in B$

$${}_A(l_{y'}^*R(b')|l_y^*R(b)) = {}_A(\overline{y'}b'|\bar{y}b),$$

where the left-hand side is computed with respect to the new inner product on X introduced in Lemma 4.10 and the right-hand side with respect to the left inner product on $\bar{Y}$ defined in the paragraph following Definition 2.8. We start from the right-hand side. For a, $a'\in A$,

$$\begin{aligned}
{}_A(a'\overline{y'}b'|a\bar{y}b) &= (b'^*y'a'^*|b^*ya^*)_A = (l^*_{l^*_{y'}R(b')}\bar{R}(a'^*)|l^*_{l^*_yR(b)}\bar{R}(a^*))_A\\
&= (l_{l^*_yR(b)}l^*_{l^*_{y'}R(b')}\bar{R}(a'^*)|\bar{R}(a^*))_A = ((\theta^r_{l^*_yR(b),l^*_{y'}R(b')}\otimes I_Y)\bar{R}(a'^*)|\bar{R}(a^*))_A\\
&= (\bar{R}^*(\theta^r_{l^*_yR(b),l^*_{y'}R(b')}\otimes I_Y)\bar{R}(a'^*)|a^*)_A = ({}_A(l^*_yR(b)|l^*_{y'}R(b'))a'^*|a^*)_A\\
&= ({}_A(l^*_yR(b)|l^*_{y'}R(b'))a'^*)^*a^* = a'{}_A(l^*_{y'}R(b')|l^*_yR(b))a^*.
\end{aligned}$$

Therefore $U:\bar{y}b\in\bar{Y}\mapsto l_y^*R(b)\in X$ is well defined and extends to a left A-linear map from $\bar{Y}$ to X preserving left A-valued inner product. Since the right A-valued inner product of Y is definite, the left A-valued inner product on X constructed in Lemma 4.10 is also definite. Clearly this map is also right B-linear. Since $\bar{Y}$ is a left Hilbert bimodule, so is X with respect to its left inner product. Since this left inner product is continuous with respect to the right one, X is bi-Hilbertian by general Banach space theory.

Similarly, $V:\bar{x}a\in\bar{X}\mapsto l_x^*\bar{R}(a)\in Y$ extends to a B–A linear map preserving the left inner product from $\bar{X}$ to Y and making Y into a left Hilbert bimodule. Now UJ_Y takes b^*y to $l_y^*R(b)$ and VJ_X takes a^*x to $l_x^*\bar{R}(a)$. Therefore UJ_YVJ_X takes a^*xb to

$$UJ_Y(l^*_{xb}\bar{R}(a)) = UJ_Y(b^*l^*_x\bar{R}(a)) = l^*_{l^*_x\bar{R}(a)}R(b)$$

which we have already shown to coincide with a^*xb. One similarly obtains: $VJ_XUJ_Y = I_Y$. Since U preserves the left inner products, $J_XUJ_Y = V^{-1}$, and therefore V, preserves the right inner products. For the same reason, U preserves the right inner products as well.

Lemma 4.12. *Let X be an object of ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ with a conjugate object Y in ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$, and let us make X and Y into bi-Hilbertian C^*-bimodules with left inner products defined, as in Lemma* 4.10, *by a solution $(R,\ \bar{R})$ of the conjugate equations. Let us identify Y, as a bi-Hilbertian C^*-bimodule, with $\bar{X}$ via the biunitary map $V:\bar{X}\to Y$ defined in*

Lemma 4.11. *Then for any* $x, x' \in X$,

$$\bar{R}^*(x \otimes \bar{x}') = {}_A(x|x') \quad \textit{and} \quad R^*(\bar{x} \otimes x') = (x|x')_B.$$

Proof. We shall show that

$$\bar{R}^*(x \otimes V\overline{x'}) = {}_A(x|x') \quad \text{and} \quad R^*(V\bar{x} \otimes x') = (x|x')_B.$$

The first equation follows from

$$\bar{R}^*(x \otimes V(\overline{x'}a)) = \bar{R}^*(x \otimes l^*_{x'}\bar{R}(a)) = \bar{R}^*(\theta^r_{x,x'} \otimes I_Y)\bar{R}(a) = {}_A(x|x')a.$$

Similarly, we have

$$R^*(y \otimes U\bar{y}')) = {}_B(y|y'),$$

where the operator U is still defined in Lemma 4.11. Now writing $y = V\bar{x}$ and $y' = V\overline{x'}$, and using the relation $UJ_YVJ_X = I_X$ obtained in Lemma 4.11, gives

$$R^*V\bar{x} \otimes x' = {}_B(V\bar{x}|V\overline{x'}) = (x|x')_B. \qquad \square$$

We are now ready to prove a converse of part (2) of Theorem 4.4.

Theorem 4.13. *Let X be a right Hilbert A–B C^*-bimodule with a nondegenerate left action. If X has a conjugate object in the 2-C^*-category ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ of nondegenerate right Hilbert bimodules, it can be given a left A-valued inner product making it into a finite index bi-Hilbertian C^*-bimodule. More precisely, any solution $(Y, R, \bar{R})$ to the conjugate equations in ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$ induces a left inner product defining a finite index bi-Hilbertian structure on X by*

$$_A(x|ax') = \bar{R}^*\theta^r_{x,x'} \otimes I_Y\bar{R}(a^*), \quad x, x' \in X, a \in A. \tag{4.1}$$

It turns out that Y is biunitarily equivalent, as a bi-Hilbertian bimodule, to $\bar{X}$ and, under this identification, the intertwiners R and $\bar{R}$ are defined by

$$\bar{R}^*x \otimes \overline{x'} = {}_A(x|x'), \quad R^*\bar{x} \otimes x' = (x|x')_B. \tag{4.2}$$

Proof. Suppose that X has a conjugate Y defined by intertwiners R and $\bar{R}$. So far we have proved that a solution R, $\bar{R}$ of the conjugate equations induces a bi-Hilbertian structure on X (Lemma 4.11). Also, we have been able to identify Y biunitarily with $\bar{X}$ (via the map V defined in Lemma 4.11) with R and $\bar{R}$ acting as in Lemma 4.12. Since $\theta^r_{x,x}$ is positive, we have

$$||R||^{-2}\theta^r_{x,x} \leqslant \bar{R}^*(\theta^r_{x,x} \otimes I)\bar{R} = {}_A(x|x),$$

by Lemma 4.7. Therefore for any $x \in X$, $||(x|x)_B|| = ||\theta^r_{x,x}|| \leqslant ||R||^2||_A(x|x)||$. On the other hand, since $\bar{R}$ is bounded, for any $x_1, \dots, x_n \in X$,

$$\left|\left|\sum_{i=1}^{n} {}_A(x_i|x_i)\right|\right| = \left|\left|\bar{R}^*\left(\sum_{i=1}^{n} \theta^r_{x_i,x_i} \otimes I\right)\bar{R}\right|\right| \leqslant ||\bar{R}||^2\left|\left|\sum_{i=1}^{n} \theta^r_{x_i,x_i}\right|\right|,$$

and this shows that X is of finite right numerical index. Since the corresponding map $F : \mathscr{K}(X_B) \to A$ is given by $F(T) = \bar{R}^* T \otimes 1_{\bar{X}} \bar{R}$, $r - \mathrm{Ind}[X] = \bar{R}^*\bar{R} \in M(A)$. Therefore, taking into account Theorem 2.22, all the assumptions of Definition 2.23 are satisfied, and this shows that X is of finite right index. Similarly, $Y = \bar{X}$ is of finite right index, i.e. X is of finite left index as well, and therefore of finite index. □

Remark. The arguments of the proof show that the minimal dimension of a bimodule is the infimum of the square roots of the numerical indices.

4.7. A characterization of strong Morita equivalences

We next characterize strong Morita equivalences among general Hilbert C^*-bimodules as those objects with minimal dimension (or numerical index) equal to 1.

Corollary 4.14. *For a right Hilbert C^*-bimodule ${}_AX_B$ the following properties are equivalent.*

(1) *X_B is full and it can be given a full left inner product making it into finite index bi-Hilbertian bimodule with respect to which $I[X] = 1$,*
(2) *X is an object of the category of nondegenerate full right Hilbert C^*-bimodules with a conjugate such that* $\dim X = 1$,
(3) *X can be given a left inner product making it into finite index Hilbert bimodule with $r - Ind[X] = I_A$ and $\ell - \mathrm{Ind}[X] = I_B$,*
(4) *X can be given a left inner product making it into a strong Morita equivalence bimodule from A to B.*

Proof. (1) ⇒ (2): This implication follows from the previous theorem.

(2) ⇒ (4): Let R and $\bar{R}$ satisfy the conjugate equations with $||R||||\bar{R}|| < \sqrt{2}$. Since X and its conjugate are full, the left and right indices of X must be invertible by Corollary 2.28, and therefore so are R^*R and $\bar{R}^*\bar{R}$. The operators $S := R(R^*R)^{-1/2}$ and $\bar{S} := \bar{R}(\bar{R}^*\bar{R})^{-1/2}$ are isometries whose ranges generate, as in [LR], projections satisfying the Jones relations with parameter β, where $\beta^{-1} = (||R||||\bar{R}||)^2 < 2$, thus $\beta = 1$ by Jones fundamental result [J]. This shows that the numerical index $I[X]$ of X with respect to the original right inner product and the left inner product induced by this pair, is 1. Let $\phi E : \mathscr{K}(X_B) \to \phi(A)$ denote the conditional expectation defined in

Corollary 2.29. Since, by Corollary 4.9, $I[X]\phi E(T) \geqslant T$ for any positive T in $\mathscr{K}(X_B)$, and since $\phi E(\phi E(T) - T) = 0$, ϕE, being faithful, must be the identity map. Defining a new left inner product on X by:

$$_A(x|y)' := (\bar{R}^* \bar{R})^{-1} {}_A(x|y) = \theta^r_{x,y},$$

makes X into a strong Morita equivalence bimodule.

(4) $\Rightarrow$ (3): It is easy to show that a strong Morita equivalence bimodule is full as a left as well as a right Hilbert module and has index 1. In fact, in this case $A = \mathscr{K}(X_B)$ and one has a bi-Hilbertian structure given by $_A(x|x') = \theta^r_{x,x'}$, for $x, x' \in X$. Let $\{u_\mu\}_\mu$ be a generalized right basis for X. Since $_A(x|x') = \theta^r_{x,x'}$, for $x, x' \in X$, we have

$$r - \mathrm{Ind}[X] = \lim_\mu \sum_{y \in u_\mu} {}_A(y|y) = \lim_\mu \sum_{y \in u_\mu} \theta^r_{y,y} = I_A.$$

One similarly shows that $\ell - \mathrm{Ind}[X] = I_B$.

(3) $\Rightarrow$ (1): This implication is obvious. $\square$

5. Tensoring finite index bimodules

Let A, B and C be σ-unital C^*-algebras, X a right Hilbert A–B bimodule and Y a right Hilbert B–C bimodule. If X and Y are of finite index, is $X \otimes^r_B Y$ still of finite index? We have shown in Proposition 2.13 that if $_AX_B$ is of finite right numerical index and $_BY_C$ is of finite left numerical index, the seminorms of $X \odot_B Y$ arising from the left and right inner products are equivalent, therefore we can form a unique bi-Hilbertian bimodule, $X \otimes_B Y$ completing in any of these seminorms. We now show that this bimodule is of finite index if X and Y are.

Theorem 5.1. *Let A, B and C be C^*-algebras, and $X = {}_AX_B$ and $Y = {}_BY_C$ be bi-Hilbertian C^*-bimodules. If $_AX_B$ and $_BY_C$ have finite index (respectively, finite numerical index), then also $X \otimes_B Y$ has finite index (respectively, finite numerical index) with respect to the bi-Hilbertian structure defined in Section* 2.2.

Proof. Since X and Y are bi-Hilbertian and of finite numerical index $X \otimes_B Y$ is bi-Hilbertian by Proposition 2.13, and therefore left and right actions are nondegenerate by Proposition 2.16. Since X and Y have finite numerical index, the contragradient of the corresponding underlying left Hilbert modules are their respective conjugates, by Theorem 4.4. Namely, there are intertwiners in $_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$, with $\mathscr{A} = \{A, B\}$, $\bar{R}_1 \in (\iota_A, X \otimes_B \bar{X})$, $R_1 \in (\iota_B, \bar{X} \otimes_A \bar{X})$, $\bar{R}_2 \in (\iota_B, Y \otimes_C \bar{Y})$, $R_2 \in (\iota_C, \bar{Y} \otimes_B \bar{Y})$ solving the corresponding conjugate equations. It is shown in [LR] that $\bar{Y} \otimes_B \bar{X}$ is a conjugate of $X \otimes_B Y$ in $_{\mathscr{A}}\mathscr{H}^w_{\mathscr{A}}$. A solution to the conjugate equations is given by operators R and $\bar{R}$ defined in the following way. Consider the map $i(R_1)$ from $\bar{Y} \otimes Y$ to $\bar{Y} \otimes \bar{X} \otimes X \otimes Y$, given by $I_{\bar{Y}} \otimes R_1 \otimes I_Y$, and the map $j(\bar{R}_2)$ from $X \otimes \bar{X}$ to

$X \otimes Y \otimes \bar{Y} \otimes \bar{X}$ given by $I_X \otimes \bar{R}_2 \otimes I_{\bar{X}}$. Define the C–C bimodule homomorphism $R \in {}_C\mathscr{L}(\iota_C, (\bar{Y} \otimes \bar{X} \otimes X \otimes Y))$ by $R = i(R_1) \circ R_2$, and the A–A bimodule homomorphism $\bar{R} \in {}_A\mathscr{L}(\iota_A, X \otimes Y \otimes \bar{Y} \otimes \bar{X}_A)$ by $\bar{R} = j(\bar{R}_2) \circ \bar{R}_1$. One can easily check the following relations:

$${}_A(z|z') := \bar{R}^*(\theta^r_{z,z'} \otimes 1_{\bar{Y} \otimes \bar{X}})\bar{R},$$

$$(z|z')_C := R^*(\theta^r_{\bar{z},\overline{z'}} \otimes 1_{X \otimes Y})R.$$

Here $z \in X \otimes Y \to \bar{z} \in \bar{Y} \otimes \bar{X}$ is the map taking the simple tensor $x \otimes y$ to $\bar{y} \otimes \bar{x}$. (This map is a well defined, A–C antilinear and bi-antiunitary with respect to the corresponding bi-Hilbertian structures.) For $z_1, \dots, z_n \in X \otimes_B Y$,

$$\left|\left|\sum{}_A(z_i|z_i)\right|\right| = \left|\left|\bar{R}^*\left(\sum_1^n \theta^r_{z_i,z_i}\right) \otimes 1_{\bar{Y} \otimes \bar{X}}\bar{R}\right|\right| \leqslant ||\bar{R}||^2 \left|\left|\sum_1^n \theta^r_{z_i,z_i}\right|\right|,$$

therefore $X \otimes Y$ has finite right numerical index. With a similar argument, $X \otimes Y$ has finite left numerical index. If X and Y have finite index, R and $\bar{R}$ are intertwiners of the C^*-category ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{A}}$, by part (2) of Theorem 4.4, so $X \otimes_B Y$ has finite index by Theorem 4.13. $\square$

6. Examples

In this section we discuss examples of Hilbert C^* bimodules of finite index with countable bases.

6.1. Finite index bimodules generating Cuntz–Krieger algebras

In the next example we construct a Hilbert C^*-bimodule of finite index which generates a countably generated Cuntz–Krieger algebra, see [KPRR] and [KPW2].

Let Σ be a countable set, and let $G = (G(i,j))_{i,j \in \Sigma}$ be an infinite matrix with entries in $\{0, 1\}$. We shall assume that no row and no column of G is identically zero. We associate to the matrix G the directed graph $\mathscr{G} = (\Sigma, E, s, r)$, where Σ is the set of vertices and $E = \{(i,j) \in \Sigma \times \Sigma | G(i,j) = 1\}$ is the set of edges. For an edge $\gamma = (i,j) \in E$, the source $s(\gamma)$ is i and the range $r(\gamma)$ is j. We assume that $\mathscr{G}$ is locally finite, that is, for any $j \in \Sigma$, $\{i \in \Sigma | G(i,j) = 1\}$ and, for any $i \in \Sigma$, $\{j \in \Sigma | G(i,j) = 1\}$ are finite.

Let $A = c_0(\Sigma)$ be the C^*-algebra of the functions on Σ vanishing at infinity and let $A_0 = c_{00}(\Sigma)$ be the dense *-subalgebra of functions with finite support. We denote by P_j the projection in A given by $P_j(i) = \delta_{ij}$. Since the set of edges E is a subset of $\Sigma \times \Sigma$, we may regard E as a set-theoretic correspondence. The vector space $X_0 = c_{00}(E)$ of the function on E with finite support is an A–A bimodule by

$$(a \cdot f \cdot b)(i,j) = a(i)f(i,j)b(j)$$

for a, $b \in A$, $f \in X_0$ and $(i,j) \in E$. We define an A-valued inner product on X_0 by

$$(f|g)_A(j) = \sum_{\{i|(i,j)\in E\}} \overline{f(i,j)}g(i,j)$$

for f, $g \in X_0$. X_0 becomes in this way a right pre-Hilbert A-module. We denote by X the completion of X_0. The left A-action on X_0 can be extended to an action $\phi : A \to \mathscr{L}(X_A)$ on X by continuity. Since G is a row finite matrix, $\phi(a) \subset \mathscr{K}(X_A)$. Since no column of G is zero, the range map r is onto. Thus X_A is full.

We shall introduce an A-valued left inner product on X. We need an additional datum. Assume that we are given a nonnegative matrix $T = (T_{ij})_{ij}$ such that $T_{ij} > 0$ if and only if (i,j) is an edge, i.e. $G(i,j) = 1$. We call such a matrix T a *weight matrix* for the graph $\mathscr{G}$. Yonetani suggested that the weight matrix T gives an A-valued left inner product ${}_A(\ |\ )$ on X_0 by

$${}_A(f|g)(i) = \sum_j T_{ij} f(i,j)\overline{g(i,j)}$$

for f, $g \in X_0$. Then we have two associated norms

$${}_A||f|| = \sqrt{\sup_i \sum_j T_{ij}|f(i,j)|^2}$$

and

$$||f||_A = \sqrt{\sup_j \sum_i |f(i,j)|^2}.$$

Definition 6.1. A weight matrix T for the graph $\mathscr{G}$ is called *of finite index* if

$$c_1 := \sup_i \sum_j T_{ij} < \infty, \quad c_2 := \sup_j \sum_i \frac{1}{T_{ij}} < \infty.$$

Example 6.2. Let $\Sigma = \mathbb{N}$ and $G(i,j) = 1$ if $|i-j| = 1$ and $G(i,j) = 0$ if $|i-j| \neq 1$. Consider a weight matrix T defined as follows: $T_{12} = 1$. $T_{ij} = 1/2$ if $|i-j| = 1$ and $(i,j) \neq (1,2)$. $T_{ij} = 0$ if $|i-j| \neq 1$. Then $c_1 = 1$ and $c_2 = 4$. Thus T is of finite index.

Example 6.3. Let $\Sigma = \mathbb{Z}$ and $G(i,j) = 1$ if $|i-j| = 1$ and $G(i,j) = 0$ if $|i-j| \neq 1$. Consider a weight matrix T defined by $T_{ij} = 1/2$ if $|i-j| = 1$ and $T_{ij} = 0$ if $|i-j| \neq 1$. Then $c_1 = 1$ and $c_2 = 4$. Thus T is of finite index.

Example 6.4. The homogeneous tree Tree(n) of degree n is the tree where all vertices have degree n. For example Tree(2) is the graph above with $\Sigma = \mathbb{Z}$ and $G(i,j) = 1$ if $|i-j| = 1$ and $G(i,j) = 0$ if $|i-j| \neq 1$. Tree(4) is the Cayley graph of the free group F_2 with respect to the generators. We define a weight matrix T for Tree(n) by associating the value $1/n$ with each edge. Then $c_1 = 1$ and $c_2 = n^2$. Thus T is of finite index.

Example 6.5. A tree has a weight matrix of finite index if and only if it has bounded degree. In general a locally finite graph has a weight matrix of finite index if and only if both in- and out-degrees are bounded. In fact suppose that the in-degree is unbounded. We may assume that $c_1 < \infty$. For any edge $(i,j) \in E$, $0 < T_{ij} \leqslant \sup_i \sum_j T_{ij} = c_1$. Then we have

$$c_2 = \sup_j \sum_i \frac{1}{T_{ij}} \geqslant \sum_i \frac{1}{T_{ij}} \geqslant \sum_i \frac{1}{c_1}.$$

Since the in-degree is unbounded, the last term goes to ∞. Therefore $c_2 = \infty$. The rest may be similarly shown.

Simple computations show the following:

Lemma 6.6. *Let $T = (T_{ij})_{ij}$ be a weight matrix for a graph $\mathscr{G}$. Then the following are equivalent*:

(1) *T is of finite index.*
(2) *The two norms ${}_A\|\ \|$ and $\|\ \|_A$ on X_0 are equivalent.*

If T is of finite index, we can identify the two completions of X_0 with respect to the two norms above defined. We shall denote by X its completion. The left and right actions of A extend to injective *-homomorphisms $\phi : A \to \mathscr{L}(X_A)$ and $\psi : A \to \mathscr{L}({}_AX)$.

Making use of easy estimates, we get the following theorem.

Theorem 6.7. *In the above situation, if a weight matrix T is of finite index, then X is of finite index and*

$$r-\mathrm{Ind}[X] = \left(\sum_j T_{ij}\right)_i \in \ell^\infty(\Sigma), \quad \ell - \mathrm{Ind}[X] = \left(\sum_i \frac{1}{T_{ij}}\right)_j \in \ell^\infty(\Sigma).$$

Proof. We first show that X is of finite right index. The left A-action on X is included in $\mathscr{K}(X_A)$, as an immediate consequence of the fact that the graph is locally finite. We next show that X is of finite right numerical index (Definition 2.8). By Proposition 2.7, we need to show that for any $f_1, \ldots, f_n \in X_0$ and $g_1, \ldots, g_n \in X_0$,

$$\left\| \sum_{p=1}^n {}_A(f_p|g_p) \right\| \leqslant c_1 \left\| \sum_{p=1}^n \theta^r_{f_p,g_p} \right\|.$$

Now by Lemma 2.1 in [KPW1]

$$\left\|\sum_{p=1}^{n} \theta^r_{f_p,g_p}\right\| = \sup_j \left\|\left(\sum_i \overline{f_p(i,j)}f_q(i,j)\right)^{1/2}_{pq}\left(\sum_i \overline{g_p(i,j)}g_q(i,j)\right)^{1/2}_{pq}\right\|,$$

which implies

$$\begin{aligned}\left\|\sum_{p=1}^{n} {}_A(f_p|g_p)\right\| &\leqslant \sup_i\left(\left(\sum_j T_{ij}\right)\sup_j\left|\sum_{p=1}^{n} f_p(i,j)\overline{g_p(i,j)}\right)\right|\right)\\ &\leqslant c_1 \sup_j \sup_i \|(\overline{f_p(i,j)}f_q(i,j))^{1/2}_{pq}(\overline{g_p(i,j)}g_q(i,j))^{1/2}_{pq}\|\\ &\leqslant c_1 \sup_j\left\|\left(\sum_i \overline{f_p(i,j)}f_q(i,j)\right)^{1/2}_{pq}\left(\sum_i \overline{g_p(i,j)}g_q(i,j)\right)^{1/2}_{pq}\right\|\\ &= c_1\left\|\sum_{p=1}^{n} \theta^r_{f_p,g_p}\right\|.\end{aligned}$$

Similarly,

$$\left\|\sum_{p=1}^{n} \theta^r_{f_p,f_p}\right\| \leqslant c_2\left\|\sum_{p=1}^{n} {}_A(f_p|f_p)\right\|.$$

We next show that X is of finite left index. We denote by Y_0 be the A–A bimodule $c_{00}(E)$ with the following two-sided inner products: For $\hat{f}$, $\hat{g} \in Y_0$,

$${}_A\langle \hat{f}|\hat{g}\rangle(i) = \sum\nolimits_j \hat{f}(i,j)\overline{\hat{g}(i,j)}$$

and

$$\langle \hat{f}|\hat{g}\rangle_A(j) = \sum_i \frac{1}{T_{ij}}\overline{\hat{f}(i,j)}\hat{g}(i,j).$$

We denote by Y its completion. For $f \in Y_0$, define $Uf \in Y_0$ by $(Uf)(i,j) = \sqrt{T_{ij}}f(i,j)$. Then we have ${}_A(f|g) = {}_A\langle Uf|Ug\rangle$ and $(f|g)_A = \langle Uf|Ug\rangle_A$. The map U extends to a surjective isometry $X \to Y$ with respect to the two-sided inner products. Combining the fact with the preceding argument, we see that X is of finite left index. Since $\{\delta_{(i,j)}\}_{(i,j)\in E}$ is a right basis for X,

$$r - \mathrm{Ind}[X] = \sum_{(i,j)} {}_A(\delta_{(i,j)}|\delta_{(i,j)}) = \left(\sum_j T_{ij}\right)_i \in \ell^\infty(\Sigma).$$

Since $\{\frac{1}{\sqrt{T_{ij}}}\delta_{(i,j)}\}_{(i,j)\in E}$ is a left basis for X the formula for $\ell - \mathrm{Ind}[X]$ is obtained similarly. $\square$

6.2. Crossed products of Hilbert C^*-bimodules by locally compact groups

In [K], the first-named author studied continuous crossed products of Hilbert C^*-bimodules by locally compact groups. Let B be a unital C^*-algebra, and A be a C^*-subalgebra of B with the same unit. Let $E : B \to A$ be a conditional expectation of finite index in the sense of [W]. So there exists a finite basis $\{u_1, u_2, \dots, u_n\}$ of B such that $x = \sum_{i=1}^n E(xu_i^*)u_i$ for any $x \in B$. Let $X = {}_AB_B$ be a A–B bimodule with right B-valued inner product $(x|y)_B = x^*y$ and left A-valued inner product ${}_A(x|y) = E(xy^*)$. Then X is a Hilbert $A - B$ bimodule of finite index.

Let G be a second countable locally compact group and α a continuous homomorphism from G to the automorphism group of B such that $E(\alpha_g(b)) = \alpha_g(E(b))$ for every $b \in B$ and every $g \in G$. It can be shown that $A \rtimes_\alpha G$ can be embedded as a C^*-algebra in $B \rtimes_\alpha G$ in a natural way, and it can be shown that there exists a conditional expectation $\tilde{E}$ from $B \rtimes_\alpha G$ to $A \rtimes_\alpha G$ which extends E. Put $Y = B \rtimes_\alpha G$. We define a $A \rtimes_\alpha G$- $B \rtimes_\alpha G$ bimodule structure on Y and a left inner product over $A \rtimes_\alpha G$ and a right inner product over $B \rtimes_\alpha G$ in the obvious way using $\tilde{E}$. Then it can be shown that Y is a countably generated Hilbert C^*-bimodule of finite index (see [K]). The left and right indices of Y are essentially the same as those of X.

6.3. Correspondences

Let Ω be a compact Hausdorff space. Most Hilbert C^*-bimodules over the commutative C^*-algebra $A = C(\Omega)$ naturally arise from set-theoretical correspondences (i.e. closed subsets $\mathscr{C}$ of $\Omega \times \Omega$) similar to the case of commutative von Neumann algebras considered by Connes. We say that a pair $(\mathscr{C}, \mu)$ is a (multiplicity free) topological correspondence on Ω if $\mathscr{C}$ is a (closed) subset of $\Omega \times \Omega$ and $\mu = (\mu^y)_{y\in\Omega}$ is a family of finite regular Borel measure on Ω satisfying the following conditions:

(1) (faithfulness) the support $supp\mu^y$ of the measure μ^y is the y-section $\mathscr{C}^y := \{x \in \Omega | (x, y) \in \mathscr{C}\}$,
(2) (continuity) for any $f \in C(\mathscr{C})$, the map $y \in \Omega \to \int_{\mathscr{C}^y} f(x, y)\, d\mu^y(x) \in \mathbb{C}$ is continuous.

The vector space $X_0 = C(\mathscr{C})$ is an A–A bimodule by

$$(a \cdot f \cdot b)(x, y) = a(x)f(x, y)b(y)$$

for a, $b \in A$, $f \in X_0$ and $(x, y) \in \mathscr{C}$. We define an A-valued inner product on X_0 by

$$(f|g)_A(y) = \int_{\mathscr{C}^y} \overline{f(x, y)}g(x, y)\, d\mu^y(x)$$

for $f, g \in X_0$. Faithfulness and continuity of μ imply that X_0 is a right pre-Hilbert A-module. We denote by X the completion X_0. The left A-action on X_0 can be extended to a *-homomorphism $\phi : A \to \mathscr{L}_A(X_A)$. Thus we obtain a right Hilbert A–A bimodule with right inner products from the correspondence $(\mathscr{C}, \mu)$. See [D] and [KW1] for a more precise treatment.

We usually assume that for any $x \in \Omega$ there exists $y \in \Omega$ with$(x, y) \in \mathscr{C}$. This condition implies that left action ϕ is faithful. We also assume that for any $y \in \Omega$ there exists $x \in \Omega$ with $(x, y) \in \mathscr{C}$. The condition shows that right inner product on X is full. In fact let $\omega(y) = \mu^y(\mathscr{C}^y)$. Then $\omega \in A$ is invertible. For any $a \in A$, put $f(x, y) = a(y)$. Then $(I|f)_A = a\omega$. Hence the right inner product is full.

Example 6.9. Let us assume that projection maps

$$r : (x, y) \in \mathscr{C} \mapsto x \in \Omega \quad \text{and} \quad s : (x, y) \in \mathscr{C} \mapsto y \in \Omega$$

are local homeomorphisms. For any $y \in \Omega$, let μ^y be the counting measure on $\mathscr{C}^y$. Then $(\mathscr{C}, \mu)$ is a topological correspondence on Ω. We shall show X has a finite basis. In fact, since $\mathscr{C}$ is compact, and by our assumption, there exist a finite set $\{(x_1, y_1), \dots (x_n, y_n)\} \subset \mathscr{C}$ and open neighborhoods U_k of (x_k, y_k) for $k = 1, \dots, n$ such that the restrictions of the projection maps r and s to U_k are local homeomorphisms and $\mathscr{C} = \bigcup_{k=1}^n U_k$ is an open covering. Let $\{f_1, \dots, f_n\} \subset C(\mathscr{C})$ be a partition of unity for this open covering. Put $g_k = f_k^{1/2} \geqslant 0$. Then for any $(x_1, y), (x_2, y) \in \mathscr{C}$, we have

$$\sum_{k=1}^n g_k(x_1, y)\overline{g_k(x_2, y)} = \delta_{x_1, x_2}.$$

Using these equalities, for any $h \in C(\mathscr{C})$, we have that $\sum_{k=1}^n g_k(g_k|h)_A = h$. Thus $\{g_1, \dots, g_n\}$ is a finite basis for X.

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