Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 258 (2010) 4251–4278

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

On the global existence and wave-breaking criteria for the two-component Camassa–Holm system

Guilong Gui [a,b], Yue Liu [c,*]

[a] *Department of Mathematics, Jiangsu University, Zhenjiang 212013, China*
[b] *Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[c] *Department of Mathematics, University of Texas, Arlington, TX 76019, United States*

Received 9 November 2009; accepted 5 February 2010

Available online 24 February 2010

Communicated by H. Brezis

Abstract

Considered herein is a two-component Camassa–Holm system modeling shallow water waves moving over a linear shear flow. A wave-breaking criterion for strong solutions is determined in the lowest Sobolev space H^s, $s > \frac{3}{2}$ by using the localization analysis in the transport equation theory. Moreover, an improved result of global solutions with only a nonzero initial profile of the free surface component of the system is established in this Sobolev space H^s.

Keywords: Global solutions; Transport equation theory; Two-component Camassa–Holm system; Wave breaking

1. Introduction

We consider here the coupled two-component Camassa–Holm shallow water system [12,23, 30,31], namely,

$$\begin{cases} m_t + um_x + 2u_x m - Au_x + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: glgui@amss.ac.cn (G. Gui), yliu@uta.edu (Y. Liu).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.02.008

where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The parameter $A > 0$ characterizes a linear underlying shear flow so that (1.1) models wave-current interactions (see the discussions in [15,25,26] and see also [4,24]). All of those are measured in dimensionless units. Recently, Ivanov [23] gave a rigorous justification of the derivation of the system (1.1) which is a valid approximation to the governing equations for water waves in the shallow water regime with nonzero constant vorticity, where the nonzero vorticity case arises for example in situations with underlying shear flow [24].

Set $g(x) = \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$. Then $(I - \partial_x^2)^{-1} f = g * f$ for $f \in L^2(\mathbb{R})$, where $*$ denotes the spatial convolution. Let $\eta := \rho - 1$, (1.1) can be rewritten as a quasi-linear nonlocal evolution system of the type

$$\begin{cases} u_t + uu_x = -\partial_x g * \left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta\right), & t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.2}$$

or equivalently,

$$\begin{cases} u_t + uu_x + \partial_x P = 0, & t > 0,\ x \in \mathbb{R}, \\ -\partial_x^2 P + P = u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta, & t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

For $A = \rho = 0$ in (1.1), one obtains the classical Camassa–Holm model [5], whose relevance for water waves was established in [10,27]. The system (1.1) is formally integrable [19,23,31] as it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ, that is,

$$\Psi_{xx} = \left(-\zeta^2\rho^2 + \zeta\left(m - \frac{A}{2}\right) + \frac{1}{4}\right)\Psi,$$
$$\Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi$$

and has a bi-Hamiltonian structure corresponding to the Hamiltonian

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} \left(mu + (\rho - 1)^2\right) dx$$

with $m = u - u_{xx}$ and the Hamiltonian

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(u(\rho - 1)^2 + 2u(\rho - 1) + u^3 + uu_x^2 - Au^2\right) dx.$$

There are two Casimirs, i.e. $\int \rho - 1$ and $\int m$ with boundary conditions are taken as $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$.

The system (1.1) without vorticity, i.e. $A = 0$, was also rigorously justified by Constantin and Ivanov [12] to approximate the governing equations for shallow water waves. M. Chen, S. Liu and Y. Zhang [8] established a reciprocal transformation between the two-component Camassa–Holm system and the first negative flow of the AKNS hierarchy. More recently, Holm, Nraigh and Tronci [22] proposed a modified two-component Camassa–Holm system which possesses singular solutions in component ρ. Mathematical properties of (1.1) with $A = 0$ have been also studied further in many works. For example, Escher, Lechtenfeld and Yin [18] investigated local well-posedness for the two-component Camassa–Holm system with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s \geqslant 2$ and derived some precise blow-up scenarios for strong solutions to the system. Constantin and Ivanov [12] provided some conditions of wave breaking and small global solutions. Gui and Liu [21] recently obtained results of local well-posedness in the Besov spaces (especially in the Sobolev space $H^s \times H^{s-1}$ with $s > \frac{3}{2}$) and wave breaking for certain initial profiles. More recently, Guan and Yin [20] studied global existence and blow-up phenomena for the system (1.2) with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s \geqslant \frac{5}{2}$.

It is known that different from the Korteweg–de Vries (KdV) equation, the Camassa–Holm (CH) equation has a remarkable property, that is, the presence of breaking waves [5,11], which means, the solution remains bounded while its slope becomes unbounded in finite time [9,11]. After wave breaking the solutions of the CH equation can be continued uniquely as either global conservative [2] or global dissipative solutions [3]. The goal of the present paper is to investigate whether or not the two-component Camassa–Holm system has the similar wave-breaking phenomena as the classical Camassa–Holm equation in a lower Sobolev space $H^s \times H^{s-1}$ for $s > \frac{3}{2}$. In other words, whether or not both of two components u and ρ of the solution remain bounded while their slopes become unbounded in finite time.

As we know, a crucial ingredient to obtain wave breaking in finite time or global solution for the CH equation is the following invariant property [9].

$$m\big(t, q(t,x)\big)q_x^2(t,x) = m_0(x), \quad (t,x) \in [0,T) \times \mathbb{R},$$

where $m(t,x) = u(t,x) - u_{xx}(t,x)$ and the function $q \in C^1$ is an increasing diffeomorphism of $\mathbb{R}$ and satisfies the following differential equation,

$$\begin{cases} \dfrac{\partial q}{\partial t} = u(t,q), & 0 < t < T, \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases}$$

This is related to the geodesic equation which is on the diffeomorphism group of the circle [14] or on the Bott–Virasoro group [13,28,29]. Without such a nice invariant property of the CH equation, the issue of whether or not particular initial data of the two-component Camassa–Holm system generate a global solution or wave breaking is more subtle.

Our work is motivated in the study of nonlinear models, especially of the transport equation, that is,

$$\begin{cases} \partial_t f + v\partial_x f = g, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ f|_{t=0} = f_0. \end{cases}$$

It is well known that most of estimates are available when v has enough regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as v belongs to $L^1_{loc}(\mathbb{R}^+; Lip)$.

We give the following remark to explain the meaning of the lowest Sobolev space corresponding to the system (1.1) or (1.2).

Remark 1.1. We say $H^s \times H^{s-1}$ with $s > \frac{3}{2}$ is the lowest Sobolev space for the two-component Camassa–Holm system based on the following facts.

(a) For every Sobolev index $s \leqslant \frac{3}{2}$, the Sobolev space H^s cannot be embedded in the Lipschitz space (*Lip*), which is the lowest condition for preserving the regularity of the strong solution to the two-component Camassa–Holm system according to the localized analysis in the transport equation theory.

(b) Without the effect of linear dispersion, i.e. $A = 0$, the system (1.1) has peakon solitons of the form, $u(x,t) = ce^{-|x-ct|}$, $c \neq 0$ with $\rho \equiv 0$ as the solution of the Camassa–Holm equation. It is noted that the peakon soliton $ce^{-|x-ct|}$ is the weak solution in the Sobolev space H^s only for $s < \frac{3}{2}$.

(c) Following the proof of Proposition 4 in [17], one can see that (1.1) is not locally well posed in $B^{\frac{3}{2}}_{2,\infty}$ in the following sense.

There exists a global solution $u \in L^\infty(\mathbb{R}^+; B^{\frac{3}{2}}_{2,\infty})$ and $\rho \equiv 0$ to (1.1) such that for any positive $T > 0$ and $\epsilon > 0$ there exists a solution $v \in L^\infty(0,T; B^{\frac{3}{2}}_{2,\infty})$ and $\rho \equiv 0$ with

$$\big\|u(0) - v(0)\big\|_{B^{\frac{3}{2}}_{2,\infty}} \leqslant \epsilon \quad \text{and} \quad \|u - v\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty})} \geqslant 1.$$

Therefore, the exponent $s = \frac{3}{2}$ is critical in the range of Besov spaces $B^s_{2,r}$ for $r \in [1,\infty]$.

Inspired by [12], we use the properties of invariance of the component ρ associated to a transport equation with more delicate localization analysis in the transport equation theory to derive a new wave-breaking criterion for solutions for the system (1.1) in the lowest Sobolev spaces $H^s \times H^{s-1}$ with $s > \frac{3}{2}$. In this case, due to the Hamiltonian H_1, the horizontal velocity component u is uniformly bounded by the Sobolev imbedding of H^1 into L^∞. It is shown that the slope of u is bounded below, then the slope of the component ρ cannot break in finite time. This implies that the wave breaking of the solution is determined only by the slope of the component u of the solution definitely. Note in [12,18] that the wave breaking in finite time is determined by either the slope of the first component u or the slope of the second component ρ in the Sobolev space $H^s \times H^{s-1}$ with $s \geqslant \frac{5}{2}$. It is, however, established in Theorem 4.1 and Theorem 4.2 that the wave breaking in finite time only depends on the slope of the first component u in the Sobolev space $H^s \times H^{s-1}$ with $s > \frac{3}{2}$. In other words, the wave breaking in the first component u must occur before that in the second component ρ in finite time.

On the other hand, we find a sufficient condition for global solutions which determined only by a nonzero initial profile of the free surface component ρ of the system in $H^s \times H^{s-1}$ with $s > \frac{3}{2}$. This can be done because the slope of the component u can be controlled by the component ρ in finite time provided the sign of ρ does not change. These of improved results of global solutions and wave breaking indeed reveal more important features of wave propagation to the system (1.1).

Our main results of the present paper are Theorem 4.1 (Wave-breaking criterion), Theorem 4.2 (Precise wave-breaking criterion) and Theorem 5.1 (Global solution).

The remainder of the paper is organized as follows. In Section 2, we recall some basic facts on the Littlewood–Paley theory, which the localization technique is constantly used in the whole paper. Section 3 is devoted to the transport equation theory, where Theorem 3.2 is specially interesting to the system (1.2). Using the transport equation theory in the Besov spaces, two wave-breaking criteria to solutions in the lowest Sobolev space $H^s \times H^{s-1}$ with $s > \frac{3}{2}$ are demonstrated in Section 4. Finally, a result of global existence of solution in the lowest Sobolev space $H^s \times H^{s-1}$ with $s > \frac{3}{2}$ is obtained in the last section, Section 5.

Notation. Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B; $a \lesssim b$ means that there is a uniform constant C that may be different on different lines, such that $a \leqslant Cb$. We denote $(c_j)_{j\in\mathbf{N}}$ (or $(c_j(t))_{j\in\mathbf{N}}$) to be a sequence in ℓ^2 with norm 1. All of different positive constants might be denoted by the uniform constant C which may depend only on initial data.

2. Littlewood–Paley analysis

For convenience of the reader, we shall recall some basic facts on the Littlewood–Paley theory, one may check [1,6,7,16,32] for more details.

Proposition 2.1 *(Littlewood–Paley decomposition). (See [6].) Let* $\mathcal{B} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d,\ |\xi| \leqslant \frac{4}{3}\}$ *and* $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. *There exist two radial functions* $\chi \in C_c^\infty(\mathcal{B})$ *and* $\varphi \in C_c^\infty(\mathcal{C})$ *such that*

$$\chi(\xi) + \sum_{q\geqslant 0} \varphi\big(2^{-q}\xi\big) = 1, \quad \forall \xi \in \mathbb{R}^d,$$

$$\big|q - q'\big| \geqslant 2 \quad \Rightarrow \quad \operatorname{Supp}\varphi\big(2^{-q}\cdot\big) \cap \operatorname{Supp}\varphi\big(2^{-q'}\cdot\big) = \emptyset,$$

$$q \geqslant 1 \quad \Rightarrow \quad \operatorname{Supp}\chi(\cdot) \cap \operatorname{Supp}\varphi\big(2^{-q}\cdot\big) = \emptyset,$$

and

$$\frac{1}{3} \leqslant \chi(\xi)^2 + \sum_{q\geqslant 0} \varphi\big(2^{-q}\xi\big)^2 \leqslant 1, \quad \forall \xi \in \mathbb{R}^d.$$

Let $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$ and $\tilde{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\chi$. Then the dyadic operators Δ_q and S_q can be defined as follows

$$\Delta_q f \stackrel{\text{def}}{=} \varphi\big(2^{-q}D\big) f = 2^{qd} \int_{\mathbb{R}^d} h\big(2^q y\big) f(x-y)\, dy, \quad \text{for } q \geqslant 0,$$

$$S_q f \stackrel{\text{def}}{=} \chi\big(2^{-q}D\big) f = \sum_{-1\leqslant k\leqslant q-1} \Delta_q f = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}\big(2^q y\big) f(x-y)\, dy,$$

$$\Delta_{-1} f \stackrel{\text{def}}{=} S_0 f \quad \text{and} \quad \Delta_q f \stackrel{\text{def}}{=} 0 \quad \text{for } q \leqslant -2. \tag{2.1}$$

Lemma 2.1 *(Bernstein's inequality). (See [6].) Let $\mathcal{B}$ be a ball with center 0 in $\mathbb{R}^d$ and $\mathcal{C}$ a ring with center 0 in $\mathbb{R}^d$. A constant C exists so that, for any positive real number λ, any nonnegative* *integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (a,b) with $b \geqslant a \geqslant 1$, there hold*

$$\operatorname{Supp}\hat{u} \subset \lambda\mathcal{B} \quad\Rightarrow\quad \sup_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{L^b} \leqslant C^{k+1}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},$$

$$\operatorname{Supp}\hat{u} \subset \lambda\mathcal{C} \quad\Rightarrow\quad C^{-1-k}\lambda^k\|u\|_{L^a} \leqslant \sup_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{L^a} \leqslant C^{1+k}\lambda^k\|u\|_{L^a},$$

$$\operatorname{Supp}\hat{u} \subset \lambda\mathcal{C} \quad\Rightarrow\quad \left\|\sigma(D)u\right\|_{L^b} \leqslant C_{\sigma,m}\lambda^{m+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}, \tag{2.2}$$

for any function $u \in L^a$.

Definition 2.1 *(Besov spaces).* (See [6].) Let $s \in \mathbb{R}$, $1 \leqslant p,r \leqslant \infty$. The inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r}\left(\mathbb{R}^d\right) \overset{\text{def}}{=} \left\{f \in \mathcal{S}'\left(\mathbb{R}^d\right);\ \|f\|_{B^s_{p,r}} < \infty\right\},$$

where

$$\|f\|_{B^s_{p,r}} \overset{\text{def}}{=} \begin{cases} (\sum_{q\in\mathbb{Z}} 2^{qsr}\|\Delta_q f\|^r_{L^p})^{\frac{1}{r}}, & \text{for } r<\infty,\\ \sup_{q\in\mathbb{Z}} 2^{qs}\|\Delta_q f\|_{L^p}, & \text{for } r=\infty. \end{cases}$$

If $s=\infty$, $B^\infty_{p,r} \overset{\text{def}}{=} \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Remark 2.1.

(i) $f \in B^s_{p,r}$ if and only if there exist a constant C and a sequence $(c_q)_{q\in\mathbf{N}\cup\{-1\}}$ in ℓ^r with norm 1 satisfying

$$\|\Delta_q f\|_{L^p} \leqslant Cc_q 2^{-qs}. \tag{2.3}$$

(ii) For $s \in \mathbb{R}$, $p=r=2$, the Besov space $B^s_{p,r}$ coincides with the Sobolev space H^s.

Proposition 2.2 *(Gagliardo–Nirenberg inequality). For $s > \frac{1}{2}$, the following statement holds*:

$$\|f\|_{L^\infty} \leqslant C\left(1+\|f\|_{B^0_{\infty,\infty}}\log\left(e+\|f\|_{H^s}\right)\right),$$

where the constant $C = C(s)$ is independent of f.

The proof of this proposition is trivial, which can be found in [6], and we omit it.

The following proposition is devoted to dealing with the pseudo-differential operator $\partial_x(1-\partial_x^2)^{-1}$ (or $\partial_x g*$).

Proposition 2.3. *(See [6].) Let $m \in \mathbb{R}$ and f be an S^m-multiplier (that is, $f: \mathbb{R}^d \to \mathbb{R}$ is smooth and satisfies that for all multi-index α, there exists a constant C_α such that $\forall\xi \in \mathbb{R}^d$, $|\partial^\alpha f(\xi)| \leqslant C_\alpha(1+|\xi|)^{m-|\alpha|}$). Then for all $s \in \mathbb{R}$ and $1 \leqslant p,r \leqslant \infty$, the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

In this paper, we are going to use Bony's decomposition which consists of writing

$$uv = T_u v + T_v u + R(u, v), \tag{2.4}$$

where

$$T_u v = \sum_{q \geqslant -1} S_{q-1} u \Delta_q v \quad \text{and} \quad R(u, v) = \sum_{|q-q'| \leqslant 1} \Delta_q u \Delta_{q'} v = \sum_{q \geqslant -1} \Delta_q u \widetilde{\Delta}_q v,$$

where $\widetilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}$.

Proposition 2.4 *(1-D Moser-type estimates). The following estimates hold.*

(i) *For $s \geqslant 0$,*

$$\|fg\|_{H^s(\mathbb{R})} \leqslant C\big(\|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}\big). \tag{2.5}$$

(ii) *For $s > 0$,*

$$\|f\partial_x g\|_{H^s(\mathbb{R})} \leqslant C\big(\|f\|_{H^{s+1}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|\partial_x g\|_{H^s(\mathbb{R})}\big). \tag{2.6}$$

(iii) *For $s_1 \leqslant \frac{1}{2}$, $s_2 > \frac{1}{2}$ and $s_1 + s_2 > 0$,*

$$\|fg\|_{H^{s_1}(\mathbb{R})} \leqslant C\|f\|_{H^{s_1}(\mathbb{R})}\|g\|_{H^{s_2}(\mathbb{R})}, \tag{2.7}$$

where C's are constants independent of f and g.

Proof. The proof of this lemma is rather classical, and similar estimates can be found in [6]. (2.5) is a standard Moser-type estimate, and (2.7) was used in [16] and [21]. For completeness, we present the detailed proof of (2.6) here. Thanks to Bony's decomposition (2.4), we decompose $f\partial_x g$ as follows:

$$f\partial_x g = T_f \partial_x g + T_{\partial_x g} f + R(f, \partial_x g).$$

Thanks to Bernstein's inequalities (2.2), we have

$$\begin{aligned}\big\|\Delta_q(T_f\partial_x g)\big\|_{L^2} &\leqslant \sum_{|q-q'|\leqslant 5}\big\|\Delta_q(S_{q'-1}f\Delta_{q'}\partial_x g)\big\|_{L^2} \leqslant \sum_{|q-q'|\leqslant 5}\|S_{q'-1}f\|_{L^\infty}\|\Delta_{q'}\partial_x g\|_{L^2}\\ &\leqslant C\|f\|_{L^\infty}\sum_{|q-q'|\leqslant 5}c_{q'}2^{-sq'}\|\partial_x g\|_{H^s} \leqslant Cc_q 2^{-sq}\|f\|_{L^\infty}\|\partial_x g\|_{H^s}\end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
\big\|\Delta_q(T_{\partial_x g} f)\big\|_{L^2} &\leqslant \sum_{|q-q'|\leqslant 5} \big\|\Delta_q(S_{q'-1}\partial_x g \Delta_{q'} f)\big\|_{L^2} \leqslant \sum_{|q-q'|\leqslant 5} \|S_{q'-1}\partial_x g\|_{L^\infty} \|\Delta_{q'} f\|_{L^2} \\
&\leqslant C\|f\|_{H^{s+1}}\|g\|_{L^\infty} \sum_{|q-q'|\leqslant 5} c_{q'} 2^{-(s+1)q'} 2^{q'} \\
&\leqslant C c_q 2^{-sq}\|f\|_{H^{s+1}}\|g\|_{L^\infty}.
\end{aligned} \tag{2.9}
$$

While for $s>0$, using Bernstein's inequalities (2.2) again and Young's inequality, we get

$$
\begin{aligned}
\big\|\Delta_q R(f,\partial_x g)\big\|_{L^2} &\leqslant \sum_{q'\geqslant q-5} \big\|\Delta_q(\Delta_{q'} f \widetilde{\Delta}_{q'}\partial_x g)\big\|_{L^2} \leqslant \sum_{q'\geqslant q-5} \|\widetilde{\Delta}_{q'}\partial_x g\|_{L^\infty}\|\Delta_{q'} f\|_{L^2} \\
&\leqslant C \sum_{q'=-1,\, q'\geqslant q-5} \|\widetilde{\Delta}_{q'} g\|_{L^\infty}\|\Delta_{q'} f\|_{L^2} \\
&\quad + C \sum_{q'\geqslant 0,\, q'\geqslant q-5} \|\widetilde{\Delta}_{q'} g\|_{L^\infty}\|\Delta_{q'}\partial_x f\|_{L^2} \\
&\leqslant C c_q 2^{-sq}\|f\|_{H^{s+1}}\|g\|_{L^\infty},
\end{aligned}
$$

which, together with (2.8), (2.9) and (2.3), completes the proof of (2.6). □

3. Transport equation theory

To study the well-posedness problem of the system (1.2), we need the following theorem on the transport equation (especially taking the space dimension $d=1$), which has been used in [21].

Theorem 3.1. *(See [16].) Suppose that $s>-\frac{d}{2}$. Let v be a vector field such that ∇v belongs to $L^1([0,T];H^{s-1})$ if $s>1+\frac{d}{2}$ or to $L^1([0,T];H^{\frac{d}{2}}\cap L^\infty)$ otherwise. Suppose also that $f_0\in H^s$, $F\in L^1([0,T];H^s)$ and that $f\in L^\infty([0,T];H^s)\cap C([0,T];\mathcal{S}')$ solves the d-dimensional linear transport equations*

$$
(T)\qquad \begin{cases} \partial_t f + v\cdot\nabla f = F, \\ f|_{t=0} = f_0. \end{cases}
$$

Then $f\in C([0,T];H^s)$. More precisely, there exists a constant C depending only on s, p and d, and such that the following statements hold:

(1) *If $s\neq 1+\frac{d}{2}$,*

$$
\|f\|_{H^s}\leqslant \|f_0\|_{H^s} + \int_0^t \big\|F(\tau)\big\|_{H^s}\,d\tau + C\int_0^t V'(\tau)\big\|f(\tau)\big\|_{H^s}\,d\tau, \tag{3.1}
$$

or hence,

$$\|f\|_{H^s} \leqslant e^{CV(t)}\left(\|f_0\|_{H^s} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{H^s}\,d\tau\right) \tag{3.2}$$

with $V(t)=\int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}}\cap L^\infty}\,d\tau$ *if* $s<1+\frac{d}{2}$ *and* $V(t)=\int_0^t \|\nabla v(\tau)\|_{H^{s-1}}\,d\tau$ *else.*

(2) *If* $f=v$*, then for all* $s>0$*, the estimates* (3.1) *and* (3.2) *hold with* $V(t)=\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\,d\tau$.

The following theorem (Theorem 3.2) is crucial to prove wave-breaking criterion (Theorem 4.1 in Section 4). Compared with Theorem 3.1, the following theorem is also specially interesting to the regularity propagation of the solution to the second equation of the two-component Camassa–Holm system (1.2) (where $\rho-1=\eta\neq u$), since only one derivative of u is involved in $V(t)$ in (3.3) below. It is noted that the estimate (3.3) is quite different from (3.1) in Theorem 3.1, because there is $(1+\frac{1}{2})$-order derivative of u involved. This then makes the problem more difficult to deal with. The proof actually needs more delicate localization analysis in details.

Theorem 3.2. *Let* $0<\sigma<1$. *Suppose that* $f_0\in H^\sigma$, $g\in L^1([0,T];H^\sigma)$, $v,\partial_x v\in L^1([0,T];L^\infty)$ *and that* $f\in L^\infty([0,T];H^\sigma)\cap C([0,T];\mathcal{S}')$ *solves the* 1*-dimensional linear transport equation*

$$(T)\qquad \begin{cases} \partial_t f + v\partial_x f = g,\\ f|_{t=0}=f_0. \end{cases}$$

Then $f\in C([0,T];H^\sigma)$. *More precisely, there exists a constant* C *depending only on* σ *and such that the following statement holds*:

$$\|f(t)\|_{H^\sigma} \leqslant \|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma}\,d\tau + C\int_0^t \|f(\tau)\|_{H^\sigma} V'(\tau)\,d\tau \tag{3.3}$$

or hence,

$$\|f(t)\|_{H^\sigma} \leqslant e^{CV(t)}\left(\|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma}\,d\tau\right)$$

with $V(t)=\int_0^t(\|v(\tau)\|_{L^\infty}+\|\partial_x v(\tau)\|_{L^\infty})\,d\tau$.

Proof. The proof of this theorem is motivated by the one of Theorem 3.1 (see [16]). Applying the localization operator Δ_q to the transport equation (T), we transform the transport equation (T) along the flow of v, in the following equation (T_q) on $\Delta_q f$, which is a transport equation along the flow of $S_q v$

$$(T_q)\qquad \begin{cases} \partial_t \Delta_q f + S_q v\partial_x \Delta_q f = \Delta_q g - R_q,\\ \Delta_q f|_{t=0}=\Delta_q f_0, \end{cases}$$

where $R_q=R_q(v,f):=\Delta_q(v\partial_x f)-v\Delta_q\partial_x f+(v-S_q v)\Delta_q\partial_x f$.

To deal with R_q, we need to use the following lemma, which we admit for the time being.

Lemma 3.1. *For all* $0 < \sigma < 1$,

$$\left\| R_q(t) \right\|_{L^2} \lesssim c_q(t) 2^{-q\sigma} \left\| f(t) \right\|_{H^\sigma} \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \right)$$

with $c_q(t) \in l^2$ *and* $\|c_q(t)\|_{l^2} \equiv 1$.

With Lemma 3.1 in hand, we can continue the proof of Theorem 3.2. Taking the inner product between the first equation of (T_q) and $\Delta_q f$ in L^2, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q f\|_{L^2}^2 &= -\frac{1}{2} \int_{\mathbf{R}} S_q v \partial_x |\Delta_q f|^2 + (\Delta_q g | \Delta_q f)_{L^2} - \left(R_q(v, f) | \Delta_q f \right)_{L^2} \\ &\leqslant \|\partial_x S_q v\|_{L^\infty} \|\Delta_q f\|_{L^2}^2 + \left(\|R_q\|_{L^2} + \|\Delta_q g\|_{L^2} \right) \|\Delta_q f\|_{L^2}, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q f\|_{L^2}^2 &\leqslant \left(\|\partial_x v\|_{L^\infty} \|\Delta_q f\|_{L^2} + \|R_q\|_{L^2} + \|\Delta_q g\|_{L^2} \right) \|\Delta_q f\|_{L^2} \\ &\leqslant C c_q(t) 2^{-q\sigma} \left(\left\| f(t) \right\|_{H^\sigma} \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \right) + \left\| g(t) \right\|_{H^\sigma} \right) \|\Delta_q f\|_{L^2}. \end{aligned}$$

Therefore, one has

$$\left\| \Delta_q f(t) \right\|_{L^2} \leqslant \|\Delta_q f_0\|_{L^2} + C \int_0^t c_q(\tau) 2^{-q\sigma} \left(\left\| f(\tau) \right\|_{H^\sigma} \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \right) + \left\| g(\tau) \right\|_{H^\sigma} \right) d\tau. \tag{3.4}$$

Multiplying (3.4) by $2^{q\sigma}$, then taking the l^2 norm over q and applying Minkowski's inequality, we reach

$$\left\| f(t) \right\|_{H^\sigma} \leqslant \|f_0\|_{H^\sigma} + C \int_0^t \left\| g(\tau) \right\|_{H^\sigma} d\tau + C \int_0^t \left\| f(r) \right\|_{H^\sigma} \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \right) d\tau.$$

This ends the proof of Theorem 3.2. □

We now are in a position to prove Lemma 3.1.

Proof of Lemma 3.1. Firstly, using Bony's decomposition, we decompose the term R_q as follows

$$\begin{aligned} R_q &= \Delta_q(v\partial_x f) - v\Delta_q\partial_x f + (v - S_q v)\Delta_q\partial_x f \\ &= [\Delta_q; T_v]\partial_x f + \Delta_q T_{\partial_x f} v - T_{\Delta_q\partial_x f} v + \Delta_q R(v, \partial_x f) \\ &\quad - R(v, \Delta_q\partial_x f) + (v - S_q v)\Delta_q\partial_x f := \sum_{i=1}^{6} R_q^i. \end{aligned} \tag{3.5}$$

For $R_q^1 = \sum_{|q-q'|\leqslant 5}[\Delta_q; S_{q'-1}v]\partial_x\Delta_{q'} f$, thanks to (2.1), one has

$$[\Delta_q;\ S_{q'-1}v]\partial_x\Delta_{q'} f = 2^q \int_{\mathbb{R}} h\big(2^q(x-y)\big)\big[S_{q'-1}v(y) - S_{q'-1}v(x)\big]\partial_x\Delta_{q'} f(y)\,dy.$$

Hence, (2.2) applied ensures

$$\big\|R_q^1(t)\big\|_{L^2} \lesssim \sum_{|q'-q|\leqslant 5} \|\partial_x S_{q'-1}v\|_{L^\infty} 2^{-q}\|\partial_x\Delta_{q'} f\|_{L^2} \lesssim c_q(t)2^{-q\sigma}\big\|f(t)\big\|_{H^\sigma}\|\partial_x v\|_{L^\infty}. \tag{3.6}$$

For $R_q^2 = \Delta_q T_{\partial_x f} v = \Delta_q \sum_{q'\geqslant 0,\,|q-q'|\leqslant 5} S_{q'-1}\partial_x f\,\Delta_{q'} v$, (2.2) applied again implies

$$\begin{aligned} \big\|R_q^2(t)\big\|_{L^2} &\lesssim \sum_{q'\geqslant 0,\,|q'-q|\leqslant 5} \|S_{q'-1}\partial_x f\|_{L^2}\|\Delta_{q'} v\|_{L^\infty} \\ &\lesssim \sum_{q'\geqslant 0,\,|q'-q|\leqslant 5}\ \sum_{k\leqslant q'-2} \|\Delta_k\partial_x f\|_{L^2} 2^{-q'}\|\Delta_{q'}\partial_x v\|_{L^\infty} \\ &\lesssim \sum_{q'\geqslant 0,\,|q'-q|\leqslant 5}\ \sum_{k\leqslant q'-2} 2^k\|\Delta_k f\|_{L^2} 2^{-q'}\|\Delta_{q'}\partial_x v\|_{L^\infty}, \end{aligned}$$

which yields that

$$\begin{aligned} \big\|R_q^2(t)\big\|_{L^2} &\lesssim \sum_{q'\geqslant 0,\,|q'-q|\leqslant 5}\ \sum_{k\leqslant q'-2} 2^k c_k 2^{-k\sigma}\|f\|_{H^\sigma} 2^{-q'}\|\Delta_{q'}\partial_x v\|_{L^\infty} \\ &\lesssim \sum_{q'\geqslant 0,\,|q'-q|\leqslant 5}\ \sum_{k\leqslant q'-2} 2^{(k-q')(1-\sigma)} c_k\|f\|_{H^\sigma} 2^{-q\sigma}\|\partial_x v\|_{L^\infty} \\ &\lesssim c_q(t)2^{-q\sigma}\|f\|_{H^\sigma}\|\partial_x v\|_{L^\infty}, \end{aligned} \tag{3.7}$$

where we used the assumption $\sigma < 1$.

Similarly, for $R_q^3 = -T_{\Delta_q\partial_x f} v = -\sum_{q'\geqslant 0,\,q'\geqslant q-5} S_{q'-1}\Delta_q\partial_x f\,\Delta_{q'} v$, we have

$$\begin{aligned} \big\|R_q^3(t)\big\|_{L^2} &\lesssim \sum_{q'\geqslant 0,\,q'\geqslant q-5} \|S_{q'-1}\Delta_q\partial_x f\|_{L^2}\|\Delta_{q'} v\|_{L^\infty} \\ &\lesssim \sum_{q'\geqslant 0,\,q'\geqslant q-5} \|\Delta_q f\|_{L^2} 2^{q-q'}\|\Delta_{q'}\partial_x v\|_{L^\infty} \\ &\lesssim \sum_{q'\geqslant 0,\,q'\geqslant q-5} 2^{q-q'} c_q 2^{-q\sigma}\|f\|_{H^\sigma}\|\partial_x v\|_{L^\infty} \lesssim c_q(t)2^{-q\sigma}\|f\|_{H^\sigma}\|\partial_x v\|_{L^\infty}. \end{aligned} \tag{3.8}$$

Since $R_q^4 = \Delta_q R(v, \partial_x f) = \Delta_q \sum_{q' \geqslant q-5} \Delta_{q'} v \widetilde{\Delta}_{q'} \partial_x f$, we get from (2.2) that

$$\begin{aligned}
\left\| R_q^4(t) \right\|_{L^2} &\lesssim \sum_{q' \geqslant q-5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \partial_x f\|_{L^2} \\
&= \sum_{q'=-1,\, q' \geqslant q-5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \partial_x f\|_{L^2} + \sum_{q' \geqslant 0,\, q' \geqslant q-5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \partial_x f\|_{L^2} \\
&\lesssim \sum_{q'=-1,\, q' \geqslant q-5} \|v\|_{L^\infty} \|\widetilde{\Delta}_{q'} f\|_{L^2} + \sum_{q' \geqslant 0,\, q' \geqslant q-5} \|\Delta_{q'} \partial_x v\|_{L^\infty} \|\widetilde{\Delta}_{q'} f\|_{L^2},
\end{aligned}$$

which gives rise to

$$\begin{aligned}
\left\| R_q^4(t) \right\|_{L^2} &\lesssim \sum_{q'=-1,\, q' \geqslant q-5} c_{q'} 2^{(q-q')\sigma} \|v\|_{L^\infty} 2^{-q\sigma} \|f\|_{H^\sigma} \\
&\quad + \sum_{q' \geqslant 0,\, q' \geqslant q-5} c_{q'} 2^{(q-q')\sigma} \|\partial_x v\|_{L^\infty} 2^{-q\sigma} \|f\|_{H^\sigma} \\
&\lesssim c_q(t) 2^{-q\sigma} \|f\|_{H^\sigma} \big(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \big),
\end{aligned} \tag{3.9}$$

where we used the assumption $\sigma > 0$.

While for $R_q^5 = R(v, \Delta_q \partial_x f) = \sum_{|q-q'| \leqslant 5} \Delta_{q'} v \widetilde{\Delta}_{q'} \Delta_q \partial_x f$, we have

$$\begin{aligned}
\left\| R_q^5(t) \right\|_{L^2} &\lesssim \sum_{|q-q'| \leqslant 5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \Delta_q \partial_x f\|_{L^2} \\
&= \sum_{q'=-1,\, |q-q'| \leqslant 5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \Delta_q \partial_x f\|_{L^2} \\
&\quad + \sum_{q' \geqslant 0,\, |q-q'| \leqslant 5} \|\Delta_{q'} v\|_{L^\infty} \|\widetilde{\Delta}_{q'} \Delta_q \partial_x f\|_{L^2},
\end{aligned}$$

from which and (2.2), we get

$$\begin{aligned}
\left\| R_q^5(t) \right\|_{L^2} &\lesssim \sum_{q'=-1,\, |q-q'| \leqslant 5} 2^{-q'\sigma} c_{q'} 2^{(q-q')\sigma} \|v\|_{L^\infty} 2^{-q\sigma} \|f\|_{H^\sigma} \\
&\quad + \sum_{q' \geqslant 0,\, |q-q'| \leqslant 5} c_{q'} 2^{(q-q')\sigma} \|\partial_x v\|_{L^\infty} 2^{-q\sigma} \|f\|_{H^\sigma} \\
&\lesssim c_q(t) 2^{-q\sigma} \|f\|_{H^\sigma} \big(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} \big).
\end{aligned} \tag{3.10}$$

Finally, for $R_q^6 = (v - S_q v) \Delta_q \partial_x f$,

$$\begin{aligned}
\left\| R_q^6(t) \right\|_{L^2} &\lesssim \sum_{q' \geqslant q} \|\Delta_{q'} v\|_{L^\infty} \|\Delta_q \partial_x f\|_{L^2} \\
&= \sum_{q'=-1,\, q' \geqslant q} \|\Delta_{q'} v\|_{L^\infty} \|\Delta_q \partial_x f\|_{L^2} + \sum_{q' \geqslant 0,\, q' \geqslant q} \|\Delta_{q'} v\|_{L^\infty} \|\Delta_q \partial_x f\|_{L^2},
\end{aligned}$$

from which and (2.2), we reach

$$\begin{aligned}\left\|R_q^6(t)\right\|_{L^2} &\lesssim \sum_{q'=-1,\, q'\geqslant q} \|v\|_{L^\infty}\|\Delta_q f\|_{L^2} + \sum_{q'\geqslant 0,\, q'\geqslant q} 2^{q-q'}\|\Delta_{q'}\partial_x v\|_{L^\infty}\|\Delta_q f\|_{L^2} \\ &\lesssim c_q(t)2^{-q\sigma}\|f\|_{H^\sigma}\left(\|v\|_{L^\infty}+\|\partial_x v\|_{L^\infty}\right),\end{aligned}$$

which together with (3.5)–(3.10) completes the proof of Lemma 3.1. □

4. Wave-breaking criteria

Let us first state the following local well-posedness result of (1.2), which was obtained in [21] (up to a slight modification).

Lemma 4.1. *Suppose that* $\mathbf{u}_0 = (u_0, \eta_0) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$. *Then there exist* $T = T(\|\mathbf{u}_0\|_{H^s\times H^{s-1}}) > 0$ *and a unique solution* $\mathbf{u} = (u, \eta) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$ *of* (1.2) *with* $\mathbf{u}(0) = \mathbf{u}_0$. *Moreover, the solution* $\mathbf{u}$ *depends continuously on the initial value* $\mathbf{u}_0$ *and the maximal time of existence* $T > 0$ *is independent of* s. *In addition, the Hamiltonian*

$$H = H(u,\eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right) dx \tag{4.1}$$

is independent of the existence time T.

With Lemma 4.1 in hand, we establish the associated Lagrangian scale of (1.2) the initial-value problem

$$\begin{cases} \dfrac{\partial q}{\partial t} = u(t,q), & 0 < t < T, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.2}$$

where $u \in C([0,T), H^s)$ is the first component of the solution (u,η) of (1.2) with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s > \frac{3}{2}$, and $T > 0$ being the maximal time of existence. A direct calculation also yields $q_{tx}(t,x) = u_x(t, q(t,x))q_x(t,x)$. Hence for $t > 0$, $x \in \mathbb{R}$, we have

$$q_x(t,x) = e^{\int_0^t u_x(\tau, q(\tau,x))\,d\tau} > 0,$$

which implies that $q(t,\cdot): \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0,T)$. This is inferred that the L^∞ norm of any function $u(t,\cdot) \in L^\infty(\mathbb{R})$, $t \in [0,T)$ is preserved under the family of diffeomorphisms $q(t,\cdot)$ with $t \in [0,T)$, that is,

$$\left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} = \left\|u\left(t, q(t,\cdot)\right)\right\|_{L^\infty(\mathbb{R})}, \quad t \in [0,T). \tag{4.3}$$

Similarly, one gets

$$\sup_{x\in\mathbb{R}} u(t,x) = \sup_{x\in\mathbb{R}} u\left(t, q(t,x)\right).$$

The following wave-breaking criterion shows that the wave breaking only depends on the slope of u but not the slope of ρ. This improves the wave-breaking criterion in [21] and [20], where the slopes of both components u and ρ must be considered. The proof of the following result strongly depends on Theorem 3.2 on the localization analysis for the transport equation.

Theorem 4.1. *Let $\mathbf{u}_0 = (u_0, \eta_0) \in H^s \times H^{s-1}$ be as in Lemma 4.1 with $s > \frac{3}{2}$ and $\mathbf{u} = (u, \eta)$ being the corresponding solution to (1.2). Assume $T^*_{\mathbf{u}_0} > 0$ is the maximal time of existence. Then*

$$T^\star_{\mathbf{u}_0} < \infty \quad \Rightarrow \quad \int_0^{T^\star_{\mathbf{u}_0}} \left\| \partial_x u(\tau) \right\|_{L^\infty} d\tau = \infty.$$

Proof. We shall prove this theorem by an inductive argument with respect to the index s. To this end, let us first give a control on $\|\eta(t)\|_{L^\infty}$.

In fact, applying the maximal principle to the transport equation about ρ,

$$\rho_t + u\rho_x + \rho u_x = 0,$$

we have

$$\left\|\rho(t)\right\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} + C \int_0^t \|\partial_x u\|_{L^\infty} \|\rho\|_{L^\infty} d\tau.$$

A simple application of Gronwall's inequality implies

$$\left\|\rho(t)\right\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} e^{C\int_0^t \|\partial_x u\|_{L^\infty} d\tau},$$

which gives rise to

$$\left\|\eta(t)\right\|_{L^\infty} \leqslant \left\|\rho(t)\right\|_{L^\infty} + 1 \leqslant 1 + \left(1 + \|\eta_0\|_{L^\infty}\right) e^{C\int_0^t \|\partial_x u\|_{L^\infty} d\tau}. \tag{4.4}$$

Now let us concentrate our attentions to the proof of Theorem 4.1. This can be achieved as follows.

Step 1. For $s \in (\frac{3}{2}, 2)$, applying Theorem 3.2 to the transport equation with respect to η,

$$\eta_t + u\eta_x + \eta u_x + u_x = 0, \tag{4.5}$$

we have (for every $1 < s < 2$, indeed)

$$\left\|\eta(t)\right\|_{H^{s-1}} \leqslant \|\eta_0\|_{H^{s-1}} + C \int_0^t \|\eta\partial_x u + \partial_x u\|_{H^{s-1}} d\tau + C \int_0^t \|\eta\|_{H^{s-1}} \left(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\right) d\tau.$$

Thanks to the Moser-type estimate (2.5), one has

$$\|\eta\partial_x u+\partial_x u\|_{H^{s-1}}\leqslant\|\partial_x u\|_{H^{s-1}}+C\big(\|\partial_x u\|_{H^{s-1}}\|\eta\|_{L^\infty}+\|\eta\|_{H^{s-1}}\|\partial_x u\|_{L^\infty}\big).\tag{4.6}$$

Therefore, we have

$$\begin{aligned}\big\|\eta(t)\big\|_{H^{s-1}}\leqslant{}&\|\eta_0\|_{H^{s-1}}+C\int_0^t\big\|\partial_x u(\tau)\big\|_{H^{s-1}}\big(1+\big\|\eta(\tau)\big\|_{L^\infty}\big)\,d\tau\\&+C\int_0^t\big\|\eta(\tau)\big\|_{H^{s-1}}\big(\big\|u(\tau)\big\|_{L^\infty}+\big\|\partial_x u(\tau)\big\|_{L^\infty}\big)\,d\tau.\end{aligned}\tag{4.7}$$

On the other hand, Theorem 3.1 applied to the equation about u,

$$u_t+uu_x+\partial_x g*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\eta^2+\eta-Au\right)=0,$$

implies (for every $s>1$, indeed)

$$\begin{aligned}\big\|u(t)\big\|_{H^s}\leqslant{}&\|u_0\|_{H^s}+C\int_0^t\left\|\partial_x g*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\eta^2+\eta-Au\right)(\tau)\right\|_{H^s}d\tau\\&+C\int_0^t\big\|u(\tau)\big\|_{H^s}\big\|\partial_x u(\tau)\big\|_{L^\infty}\,d\tau.\end{aligned}$$

Thanks to the Moser-type estimate (2.5) and Proposition 2.3, one has

$$\begin{aligned}\left\|\partial_x g*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\eta^2+\eta-Au\right)\right\|_{H^s}&\leqslant C\left\|u^2+\frac{1}{2}u_x^2+\frac{1}{2}\eta^2+\eta-Au\right\|_{H^{s-1}}\\&\leqslant C\big(\|u\|_{H^{s-1}}\|u\|_{L^\infty}+\|\partial_x u\|_{H^{s-1}}\|\partial_x u\|_{L^\infty}\\&\quad+\|\eta\|_{H^{s-1}}\|\eta\|_{L^\infty}+\|u\|_{H^{s-1}}+\|\eta\|_{H^{s-1}}\big).\end{aligned}$$

From this, we reach

$$\begin{aligned}\big\|u(t)\big\|_{H^s}\leqslant{}&\|u_0\|_{H^s}+C\int_0^t\big\|u(\tau)\big\|_{H^s}\big(\big\|u(\tau)\big\|_{L^\infty}+\big\|\partial_x u(\tau)\big\|_{L^\infty}+1\big)\,d\tau\\&+C\int_0^t\big\|\eta(\tau)\big\|_{H^{s-1}}\big(\big\|\eta(\tau)\big\|_{L^\infty}+1\big)\,d\tau,\end{aligned}\tag{4.8}$$

which together with (4.7) ensures that

$$\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \leqslant \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}} + C\int_0^t \left(\left\|\eta(\tau)\right\|_{H^{s-1}} + \left\|u(\tau)\right\|_{H^s}\right)$$
$$\times \left(\left\|u(\tau)\right\|_{L^\infty} + \left\|\partial_x u(\tau)\right\|_{L^\infty} + \left\|\eta(\tau)\right\|_{L^\infty} + 1\right) d\tau. \tag{4.9}$$

Thanks to the Gronwall's inequality again, one can see

$$\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \leqslant \left(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\right) e^{C\int_0^t (\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|\eta\|_{L^\infty} + 1)\, d\tau}. \tag{4.10}$$

Using the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ (for $s > \frac{1}{2}$), we get from (4.1) that

$$\left\|u(t)\right\|_{L^\infty} \leqslant C\left(\|u_0\|_{H^1} + \|\eta_0\|_{L^2}\right), \tag{4.11}$$

which together with (4.4) and (4.10) implies that

$$\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \leqslant \left(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\right) e^{C_1(t+1)\exp\{\int_0^t C\|\partial_x u(\tau)\|_{L^\infty}\, d\tau\}}, \tag{4.12}$$

where $C_1 = C_1(\|u_0\|_{H^1}, \|\eta_0\|_{L^2}, \|\eta_0\|_{L^\infty})$.

Therefore, if the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$ satisfies $\int_0^{T^\star_{\mathbf{u}_0}} \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, we obtain from (4.12) that

$$\lim\sup_{t \to T^\star_{\mathbf{u}_0}} \left(\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}}\right) < \infty \tag{4.13}$$

contradicts the assumption on the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$. This completes the proof of Theorem 4.1 for $s \in (\frac{3}{2}, 2)$.

Step 2. For $s \in [2, \frac{5}{2})$, applying Theorem 3.1 to the transport equation (4.5), we have

$$\left\|\eta(t)\right\|_{H^{s-1}} \leqslant \|\eta_0\|_{H^{s-1}} + C\int_0^t \left\|(\eta\partial_x u + \partial_x u)(\tau)\right\|_{H^{s-1}} d\tau + C\int_0^t \|\eta\|_{H^{s-1}} \|\partial_x u\|_{L^\infty \cap H^{\frac{1}{2}}} d\tau.$$

(4.6) applied implies that

$$\left\|\eta(t)\right\|_{H^{s-1}} \leqslant \|\eta_0\|_{H^{s-1}} + C\int_0^t \|\partial_x u\|_{H^{s-1}}\left(1 + \|\eta\|_{L^\infty}\right) d\tau + C\int_0^t \|\eta\|_{H^{s-1}} \|\partial_x u\|_{L^\infty \cap H^{\frac{1}{2}}} d\tau,$$

which together with (4.8) yields

$$\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \leqslant \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}$$
$$+ C\int_0^t \left(\left\|\eta(\tau)\right\|_{H^{s-1}} + \left\|u(\tau)\right\|_{H^s}\right)\left(\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \left\|\eta(\tau)\right\|_{L^\infty} + 1\right) d\tau$$

with $0 < \epsilon_0 < \frac{1}{2}$, where we used the fact $H^{\frac{1}{2}+\epsilon_0} \hookrightarrow L^\infty \cap H^{\frac{1}{2}}$. Gronwall's inequality applied gives that

$$\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}} \leqslant \left(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\right) e^{C\int_0^t (\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \|\eta(\tau)\|_{L^\infty} + 1)\, d\tau}. \tag{4.14}$$

Therefore, thanks to the uniqueness of solution in Lemma 4.1, (4.1) and (4.13), we get that: if the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$ satisfies $\int_0^{T^\star_{\mathbf{u}_0}} \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, then (4.14) implies that

$$\limsup_{t \to T^\star_{\mathbf{u}_0}} \left(\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}}\right) < \infty$$

contradicts the assumption on the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$. This completes the proof of Theorem 4.1 for $2 \leqslant s < \frac{5}{2}$.

Step 3. For $2 < s < 3$, by differentiating once (4.5) with respect to x, we have

$$\partial_t \eta_x + u\partial_x(\eta_x) + 2u_x\eta_x + \eta u_{xx} + u_{xx} = 0. \tag{4.15}$$

Theorem 3.2 applied to (4.15) implies that

$$\begin{aligned} \|\eta_x(t)\|_{H^{s-2}} \leqslant{}& \|\eta_{0x}\|_{H^{s-2}} + C\int_0^t \left\|(2\eta_x u_x + \eta\partial_x u_x + \partial_{xx}u)(\tau)\right\|_{H^{s-2}}\, d\tau \\ &+ C\int_0^t \|\eta_x(\tau)\|_{H^{s-2}} \left(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty}\right) d\tau \\ \leqslant{}& \|\eta_{0x}\|_{H^{s-2}} + C\int_0^t \left(\|\eta\|_{H^{s-1}} + \|u\|_{H^s}\right)\left(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|\eta\|_{L^\infty} + 1\right) d\tau, \end{aligned} \tag{4.16}$$

where we used the following Moser-type estimates (from (2.6)):

$$\|\eta_x u_x\|_{H^{s-2}} \leqslant C\left(\|\partial_x u\|_{H^{s-1}} \|\eta\|_{L^\infty} + \|\partial_x \eta\|_{H^{s-2}} \|u_x\|_{L^\infty}\right)$$

and

$$\|\eta \partial_x u_x\|_{H^{s-2}} \leqslant C\left(\|\eta\|_{H^{s-1}} \|\partial_x u\|_{L^\infty} + \|u_{xx}\|_{H^{s-2}} \|\eta\|_{L^\infty}\right).$$

(4.16), together with (4.8) and (4.7) (where $s-1$ is replaced by $s-2$), implies that

$$\begin{aligned} \|\eta(t)\|_{H^{s-1}} + \|u(t)\|_{H^s} \leqslant{}& \|\eta_0\|_{H^{s-1}} + \|u_0\|_{H^s} + C\int_0^t \left(\|\eta(\tau)\|_{H^{s-1}} + \|u(\tau)\|_{H^s}\right) \\ &\times \left(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\eta(\tau)\|_{L^\infty} + 1\right) d\tau. \end{aligned}$$

Gronwall's inequality applied again gives (4.10). Hence, using arguments as in Step 1, it completes the proof of Theorem 4.1 for $2 < s < 3$.

Step 4. For $s = k \in \mathbf{N}$, $k \geqslant 3$, by differentiating (4.5) $k-2$ times with respect to x, we have

$$\partial_t \partial_x^{k-2}\eta + u\partial_x\big(\partial_x^{k-2}\eta\big) + \sum_{\ell_1+\ell_2=k-3,\, \ell_1,\ell_2\geqslant 0} C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta + \eta\partial_x\big(\partial_x^{k-2}u\big) + \partial_x^{k-1}u = 0. \tag{4.17}$$

Applying Theorem 3.1 to the transport equation (4.17), we have

$$\begin{aligned}
&\big\|\partial_x^{k-2}\eta(t)\big\|_{H^1} \\
&\quad\leqslant \big\|\partial_x^{k-2}\eta_0\big\|_{H^1} + C\int_0^t \big\|\partial_x^{k-2}\eta(\tau)\big\|_{H^1}\big\|\partial_x u(\tau)\big\|_{L^\infty\cap H^{\frac{1}{2}}}\,d\tau \\
&\qquad + C\int_0^t \bigg\|\bigg(\sum_{\ell_1+\ell_2=k-3,\, \ell_1,\ell_2\geqslant 0} C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta + \eta\partial_x\big(\partial_x^{k-2}u\big) + \partial_x^{k-1}u\bigg)(\tau)\bigg\|_{H^1}\,d\tau.
\end{aligned}$$

Since H^1 is an algebra, we have

$$\big\|\eta\partial_x\big(\partial_x^{k-2}u\big)\big\|_{H^1} \leqslant C\|\eta\|_{H^1}\big\|\partial_x^{k-1}u\big\|_{H^1} \leqslant C\|\eta\|_{H^1}\|u\|_{H^s}$$

and

$$\begin{aligned}
&\bigg\|\sum_{\ell_1+\ell_2=k-3,\, \ell_1,\ell_2\geqslant 0} C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta\bigg\|_{H^1} \\
&\quad\leqslant C\sum_{\ell_1+\ell_2=k-3,\, \ell_1,\ell_2\geqslant 0} C_{\ell_1,\ell_2}\big\|\partial_x^{\ell_1+1}u\big\|_{H^1}\big\|\partial_x^{\ell_2+1}\eta\big\|_{H^1} \leqslant C\|u\|_{H^{s-1}}\|\eta\|_{H^{s-1}}.
\end{aligned}$$

Hence,

$$\big\|\partial_x^{k-2}\eta(t)\big\|_{H^1} \leqslant \big\|\partial_x^{k-2}\eta_0\big\|_{H^1} + C\int_0^t \big(\|\eta\|_{H^{s-1}} + \|u\|_{H^s}\big)\big(\|u\|_{H^{s-1}} + \|\eta\|_{H^1} + 1\big)\,d\tau. \tag{4.18}$$

(4.18), together with (4.8) and (4.7) (where $s-1$ is replaced by 1), implies that

$$\begin{aligned}
&\big\|\eta(t)\big\|_{H^{s-1}} + \big\|u(t)\big\|_{H^s} \\
&\quad\leqslant \|\eta_0\|_{H^{s-1}} + \|u_0\|_{H^s} + C\int_0^t \big(\big\|\eta(\tau)\big\|_{H^{s-1}} + \big\|u(\tau)\big\|_{H^s}\big)\big(\big\|u(\tau)\big\|_{H^{s-1}} + \big\|\eta(\tau)\big\|_{H^1} + 1\big)\,d\tau.
\end{aligned}$$

Gronwall's inequality applied yields that

$$\left\|u(t)\right\|_{H^s}+\left\|\eta(t)\right\|_{H^{s-1}}\leqslant\left(\|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}\right)e^{C\int_0^t(\|u\|_{H^{s-1}}+\|\eta\|_{H^1}+1)\,d\tau}. \tag{4.19}$$

Therefore, if the maximal existence time $T^\star_{\mathbf{u}_0}<\infty$ satisfies $\int_0^{T^\star_{\mathbf{u}_0}}\|\partial_x u(\tau)\|_{L^\infty}\,d\tau<\infty$, thanks to the uniqueness of solution in Lemma 4.1, we get that

$$\left\|u(t)\right\|_{H^{s-1}}+\left\|\eta(t)\right\|_{H^1}$$

is uniformly bounded by the induction assumption, which together with (4.19) implies

$$\lim\sup_{t\to T^\star_{\mathbf{u}_0}}\left(\left\|u(t)\right\|_{H^s}+\left\|\eta(t)\right\|_{H^{s-1}}\right)<\infty.$$

This leads to a contradiction.

Step 5. For $k<s<k+1$ with $k\in\mathbf{N}$, $k\geqslant 3$, by differentiating (4.5) $k-1$ times with respect to x, we have

$$\partial_t\partial_x^{k-1}\eta+u\partial_x\left(\partial_x^{k-1}\eta\right)+\sum_{\ell_1+\ell_2=k-2,\,\ell_1,\ell_2\geqslant 0}C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta+\eta\partial_x\left(\partial_x^{k-1}u\right)+\partial_x^k u=0.$$

Theorem 3.2 applied again implies that

$$\begin{aligned}
&\left\|\partial_x^{k-1}\eta(t)\right\|_{H^{s-k}}\\
&\quad\leqslant\left\|\partial_x^{k-1}\eta_0\right\|_{H^{s-k}}+C\int_0^t\left\|\partial_x^{k-1}\eta(\tau)\right\|_{H^{s-k}}\left(\left\|u(\tau)\right\|_{L^\infty}+\left\|\partial_x u(\tau)\right\|_{L^\infty}\right)d\tau\\
&\qquad+C\int_0^t\left\|\left(\sum_{\ell_1+\ell_2=k-2,\,\ell_1,\ell_2\geqslant 0}C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta+\eta\partial_x\left(\partial_x^{k-1}u\right)+\partial_x^k u\right)(\tau)\right\|_{H^{s-k}}d\tau.
\end{aligned}$$

Using the Moser-type estimate (2.6) and the Sobolev embedding inequality, we have for $\forall 0<\epsilon_0<\frac{1}{2}$

$$\begin{aligned}
\left\|\eta\partial_x\left(\partial_x^{k-1}u\right)\right\|_{H^{s-k}}&\leqslant C\left(\|\eta\|_{L^\infty}\left\|\partial_x^k u\right\|_{H^{s-k}}+\|\eta\|_{H^{s-k+1}}\left\|\partial_x^{k-1}u\right\|_{L^\infty}\right)\\
&\leqslant C\left(\|\eta\|_{L^\infty}\|u\|_{H^s}+\|\eta\|_{H^{s-k+1}}\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sum_{\ell_1+\ell_2=k-2,\,\ell_1,\ell_2\geqslant 0}C_{\ell_1,\ell_2}\partial_x^{\ell_1+1}u\partial_x^{\ell_2+1}\eta\right\|_{H^{s-k}}\\
&\quad\leqslant C\sum_{\ell_1+\ell_2=k-2,\,\ell_1,\ell_2\geqslant 0}C_{\ell_1,\ell_2}\left(\left\|\partial_x^{\ell_1+1}u\right\|_{L^\infty}\left\|\partial_x^{\ell_2+1}\eta\right\|_{H^{s-k}}+\left\|\partial_x^{\ell_2}\eta\right\|_{L^\infty}\left\|\partial_x^{\ell_1+1}u\right\|_{H^{s-k+1}}\right)\\
&\quad\leqslant C\left(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\|\eta\|_{H^{s-1}}+\|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}}\|u\|_{H^s}\right).
\end{aligned}$$

Hence,

$$\left\|\partial_x^{k-1}\eta(t)\right\|_{H^{s-k}} \leqslant \left\|\partial_x^{k-1}\eta_0\right\|_{H^{s-k}} + C\int_0^t \left(\left\|\eta(\tau)\right\|_{H^{s-1}} + \left\|u(\tau)\right\|_{H^s}\right) \times \left(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1\right) d\tau. \tag{4.20}$$

(4.20), together with (4.8) and (4.7) (where $s-1$ is replaced by $s-k$), implies that

$$\left\|\eta(t)\right\|_{H^{s-1}} + \left\|u(t)\right\|_{H^s} \leqslant \|\eta_0\|_{H^{s-1}} + \|u_0\|_{H^s} + C\int_0^t \left(\left\|\eta(\tau)\right\|_{H^{s-1}} + \left\|u(\tau)\right\|_{H^s}\right) \times \left(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1\right) d\tau.$$

Applying Gronwall's inequality then gives that

$$\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \leqslant \left(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\right) e^{C\int_0^t (\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1)\, d\tau}.$$

In consequence, if the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$ satisfies $\int_0^{T^\star_{\mathbf{u}_0}} \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, thanks to the uniqueness of solution in Lemma 4.1, then we get that

$$\left\|u(t)\right\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \left\|\eta(t)\right\|_{H^{k-\frac{3}{2}+\epsilon_0}}$$

is uniformly bounded by the induction assumption, which implies

$$\limsup_{t\to T^\star_{\mathbf{u}_0}} \left(\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}}\right) < \infty,$$

which leads to a contradiction.

Therefore, from Step 1 to Step 5, we complete the proof of Theorem 4.1. □

Theorem 4.2. *Let (u_0, η_0) be as in Lemma 4.1 with $s > \frac{3}{2}$ and $\mathbf{u} = (u, \eta)$ being the corresponding solution to (1.2). Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t\to T^\star_{\mathbf{u}_0}} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty. \tag{4.21}$$

Lemma 4.2. *Let (u, η) (with $\eta := \rho - 1$) be the solution of (1.2) with initial value $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, and T the maximal existence time. If there is $M_1 \geqslant 0$ such that*

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} u_x(t,x) \geqslant -M_1, \tag{4.22}$$

then

$$\left\|\rho(t,\cdot)\right\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} e^{M_1 t}, \tag{4.23}$$

$$\sup_{x\in\mathbb{R}} u_x(t,x) \leqslant \|u_{0,x}\|_{L^\infty} + C_1 + \|\rho_0\|_{L^\infty} e^{M_1 t} \tag{4.24}$$

hold for $t \in [0,T)$, *with*

$$C_1 = \left(\frac{1+A^2}{2}\right)^{\frac{1}{2}} \left\|(u_0, \rho_0 - 1)\right\|_{H^1\times L^2}, \tag{4.25}$$

and C *a positive constant depending only on* A, M_1 *and the norm* $\|(u_0,\rho_0-1)\|_{H^s\times H^{s-1}}$.

Proof. By Lemma 4.1 and a simple density argument, it is needed only to show the desired results are valid when $s \geqslant 3$. So in the sequel of this section $s = 3$ is taken for simplicity of notation. Differentiating both sides of the first equation of (1.2) with respect to x and using the identity $-\partial_x^2 g * f = f - g * f$ lead to

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 = A\partial_x g * u_x + u^2 + \frac{1}{2}\rho^2 - g * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{4.26}$$

Given $x \in \mathbb{R}$, let

$$M(t) = u_x\big(t, q(t,x)\big), \qquad \gamma(t) = \rho\big(t, q(t,x)\big), \tag{4.27}$$

$t \in [0,T)$, with $q(t,x)$ determined in (4.2). Using these notations, Eq. (4.26) and the second one of (1.2) can be rewritten, respectively, as

$$\begin{aligned} M'(t) &= -\frac{1}{2}M^2 + \frac{1}{2}\gamma^2 + f\big(t, q(t,x)\big),\\ \gamma'(t) &= -\gamma M, \end{aligned} \tag{4.28}$$

for $t \in [0,T)$, where the notation $'$ denotes the derivative with respect to t and f represents the function

$$f = A\partial_x g * u_x + u^2 - g * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{4.29}$$

Hence, we have

$$\begin{aligned} f &= Ag_x * u_x + u^2 - g * \left(u^2 + \frac{1}{2}u_x^2\right) - \frac{1}{2}g * 1 - g * (\rho - 1) - \frac{1}{2}g * (\rho-1)^2 \\ &\leqslant |A||g_x * u_x| + \frac{1}{2}u^2 - \frac{1}{2} + \left|g * (\rho - 1)\right|, \end{aligned}$$

with the help of $g * (u^2 + \frac{1}{2}u_x^2) \geqslant \frac{1}{2}u^2$ (cf. [9]). Applying Young's inequality and $g(x) = \frac{1}{2}e^{-|x|}$ leads to

$$|A||g_x * u_x|(t,x) \leqslant |A|\|g_x\|_{L^2}\|u_x\|_{L^2} \leqslant \frac{1}{2}|A|\|u_x\|_{L^2} \leqslant \frac{1}{4} + \frac{1}{4}A^2\|u_x\|_{L^2}^2, \tag{4.30}$$

$$\big|g*(\rho-1)\big|(t,x) \leqslant \|g\|_{L^2}\|\rho-1\|_{L^2} \leqslant \frac{1}{2}\|\rho-1\|_{L^2} \leqslant \frac{1}{4} + \frac{1}{4}\|\rho-1\|_{L^2}^2, \tag{4.31}$$

for $(t,x) \in [0,T) \times \mathbb{R}$. On the other hand, the continuous embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ gives (cf. [11], for example, for the best embedding constant)

$$2u^2(t,x) \leqslant \int_{\mathbb{R}} u^2 + u_x^2 \leqslant \int_{\mathbb{R}} u^2 + u_x^2 + (\rho-1)^2 = \int_{\mathbb{R}} u_0^2 + u_{0x}^2 + (\rho_0-1)^2, \tag{4.32}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$, where we used the fact that $H(u,\rho-1)$ is the conservation law of the system (1.2) in the last identity. Combining (4.30), (4.31), and (4.32) together gives

$$f \leqslant \frac{1}{4}\big(1+A^2\big)\big\|(u,\rho-1)\big\|_{H^1\times L^2}^2 = \frac{1}{4}\big(1+A^2\big)\big\|(u_0,\rho_0-1)\big\|_{H^1\times L^2}^2 = \frac{1}{2}C_1^2, \tag{4.33}$$

where the conservation law $H(u,\eta) = \int_{\mathbb{R}} u^2 + u_x^2 + \eta^2$ of (1.2) was used again in the second identity and C_1 was introduced in (4.25). Similarly, we have

$$\begin{aligned} -f &\leqslant |A||g_x * u_x| + g*\left(u^2 + \frac{1}{2}u_x^2\right) + \frac{1}{2} + \big|g*(\rho-1)\big| + \frac{1}{2}g*(\rho-1)^2 \\ &\leqslant \frac{1}{4} + \frac{1}{4}A^2\|u_x\|_{L^2}^2 + \frac{1}{2}\left\|u^2 + \frac{1}{2}u_x^2\right\|_{L^1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\|\rho-1\|_{L^2}^2 + \frac{1}{4}\|\rho-1\|_{L^2}^2, \end{aligned}$$

where we used the estimate $g*(u^2+\frac{1}{2}u_x^2)(t,x) \leqslant \|g\|_{L^\infty}\|u^2+\frac{1}{2}u_x^2\|_{L^1} \leqslant \frac{1}{2}\|u\|_{H^1}^2$. Therefore, we get

$$-f \leqslant 1 + \frac{1+A^2}{2}\big\|(u,\rho-1)\big\|_{H^1\times L^2}^2 \leqslant 1 + \frac{1+A^2}{2}\big\|(u_0,\rho_0-1)\big\|_{H^1\times L^2}^2 \leqslant 1 + C_1^2. \tag{4.34}$$

In view of the definition of $M(t)$ in (4.27), the assumption (4.22) is now expressed as, for each $x \in \mathbb{R}$,

$$M(t) \geqslant -M_1, \quad \text{for } t \in [0,T).$$

In view of this condition, it then follows from the second equation of (4.28) that, for each $x \in \mathbb{R}$,

$$\big|\rho\big(t,q(t,x)\big)\big| = \big|\gamma(t)\big| = \big|\gamma(0)\big|e^{\int_0^t -M(\tau)\,d\tau} \leqslant \|\rho_0\|_{L^\infty}e^{M_1 t}, \tag{4.35}$$

for $t \in [0,T)$. Hence combining this with (4.3) leads to (4.23).

Given any $x \in \mathbb{R}$, let us define

$$P(t) = M(t) - \|u_{0,x}\|_{L^\infty} - C_1 - \|\rho_0\|_{L^\infty}e^{M_1 t},$$

with $M(t) = u_x(t,q(t,x))$ and C_1 in (4.25). Observe that $P(t)$ is a C^1-differentiable function in $[0,T)$ and satisfies

$$P(0) = M(0) - \|u_{0,x}\|_{L^\infty} - C_1 - \|\rho_0\|_{L^\infty} \leqslant u_{0,x}(x) - \|u_{0,x}\|_{L^\infty} \leqslant 0.$$

We now claim

$$P(t) \leqslant 0, \quad \text{for all } t \in [0, T). \tag{4.36}$$

Assume the contrary that there is $t_0 \in [0, T)$ such that $P(t_0) > 0$. Let

$$t_1 = \max\bigl\{t < t_0;\ P(t) = 0\bigr\}.$$

Then $P(t_1) = 0$ and $P'(t_1) \geqslant 0$, or equivalently,

$$M(t_1) = \|u_{0,x}\|_{L^\infty} + C_1 + \|\rho_0\|_{L^\infty} e^{M_1 t_1}, \tag{4.37}$$

and

$$M'(t_1) \geqslant M_1 \|\rho_0\|_{L^\infty} e^{M_1 t_1} > 0. \tag{4.38}$$

From (4.32), (4.35), (4.37), and the first equation of (4.28), it follows that

$$\begin{aligned}
M'(t_1) &= -\frac{1}{2} M^2(t_1) + \frac{1}{2}\gamma^2(t_1) + f\bigl(t_1, q(t_1, x)\bigr) \\
&\leqslant -\frac{1}{2}\bigl(\|u_{0,x}\|_{L^\infty} + C_1 + \|\rho_0\|_{L^\infty} e^{M_1 t_1}\bigr)^2 + \frac{1}{2}\|\rho_0\|_{L^\infty}^2 e^{2M_1 t_1} + \frac{1}{2} C_1^2 \\
&\leqslant 0,
\end{aligned}$$

a contradiction to (4.38), so the claim (4.36) is valid. Therefore, the arbitrarily chosen of x and (4.3) imply (4.24). $\square$

Proof of Theorem 4.2. Assume (4.21) is not valid. Then there is some positive number $M_1 > 0$ such that

$$u_x(t, x) \geqslant -M_1$$

holds for $(t, x) \in [0, T) \times \mathbb{R}$. It now follows from (4.24) in Lemma 4.2 that

$$\bigl|u_x(t, x)\bigr| \leqslant C e^{M_1 t},$$

with C a positive constant depending only on A, M_1 and the norm $\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}$. Theorem 4.1 applied implies that the maximal existence time $T^\star_{\mathbf{u}_0} = \infty$, which contradicts the assumption on the maximal existence time $T^\star_{\mathbf{u}_0} < \infty$.

Conversely, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (with $s > \frac{1}{2}$) implies that if (4.21) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 4.2. $\square$

Theorem 4.3. *Assume that the initial value* $(u_0, \eta_0) \in H^s \times H^{s-1}$ *with* $s > \frac{3}{2}$. *Let* $T_{\mathbf{u}_0} > 0$ *be the maximal time of existence for the corresponding solution* (u, η) *to the system* (1.2). *Then we have*

$$T^\star_{u_0} < \infty \quad \Rightarrow \quad \int_0^{T^\star_{u_0}} \left(\left\| \partial_x u(\tau) \right\|_{B^0_{\infty,\infty}} + \left\| \rho(\tau) - 1 \right\|_{B^0_{\infty,\infty}} \right) d\tau = \infty.$$

Proof. We only need to prove this theorem for the case $\frac{3}{2} < s < 2$. For $s \geqslant 2$, the induction argument as in the proof of Theorem 4.1 will complete the proof of Theorem 4.3. Thanks to Proposition 2.2, we have for $s > \frac{3}{2}$,

$$\|\partial_x u\|_{L^\infty} \leqslant C\big(1 + \|\partial_x u\|_{B^0_{\infty,\infty}} \log\big(e + \|\partial_x u\|_{H^{s-1}}\big)\big) \tag{4.39}$$

and

$$\|\eta\|_{L^\infty} \leqslant C\big(1 + \|\eta\|_{B^0_{\infty,\infty}} \log\big(e + \|\eta\|_{H^{s-1}}\big)\big). \tag{4.40}$$

Plugging (4.39) and (4.40) into (4.10), and using the fact (4.11), we get

$$\begin{aligned}
&\left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}} \\
&\quad \leqslant \big(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\big) e^{Ct + C\int_0^t (\|\partial_x u(\tau)\|_{B^0_{\infty,\infty}} + \|\eta(\tau)\|_{B^0_{\infty,\infty}}) \log(e + \|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}})\, d\tau}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\log\big(e + \left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}}\big) \\
&\quad \leqslant \log\big(e + \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\big) + Ct \\
&\qquad + C \int_0^t \big(\left\|\partial_x u(\tau)\right\|_{B^0_{\infty,\infty}} + \left\|\eta(\tau)\right\|_{B^0_{\infty,\infty}}\big) \log\big(e + \left\|u(\tau)\right\|_{H^s} + \left\|\eta(\tau)\right\|_{H^{s-1}}\big)\, d\tau.
\end{aligned}$$

Applying Gronwall's inequality yields

$$\begin{aligned}
&\log\big(e + \left\|u(t)\right\|_{H^s} + \left\|\eta(t)\right\|_{H^{s-1}}\big) \\
&\quad \leqslant e^{C\int_0^t (\|\partial_x u(\tau)\|_{B^0_{\infty,\infty}} + \|\eta(\tau)\|_{B^0_{\infty,\infty}})\, d\tau} \big(\log(e + \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}) + Ct\big).
\end{aligned}$$

Hence, the proof of Theorem 4.3 is complete. $\square$

5. Global existence

In view of the criterion for wave breaking (Theorem 4.1), a sufficient condition of global solutions can be obtained in the following.

Theorem 5.1 *(Global solution). Let* $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s > \frac{3}{2}$, *and* $T > 0$ *being the maximal time of existence of the solution* (u, ρ) *to the system* (1.1) *with initial data* (u_0, ρ_0). *If*

$$\inf_{x \in \mathbb{R}} \rho_0(x) > 0, \tag{5.1}$$

then $T = +\infty$, *and the solution* (u, ρ) *is global.*

Remark 5.1. Theorem 5.1 improves the result of the global solutions in [20], where the special case $s = 2$ is required.

To prove Theorem 5.1, we need the following lemma.

Lemma 5.1. *Assume* (u, ρ) *is the local solution of* (1.1) *with the initial value* $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, *and* T *the maximal existence time. If* $\inf_{x \in \mathbb{R}} \rho_0(x) > 0$, *then*

$$\bigl|u_x\bigl(t, q(t,x)\bigr)\bigr|, \bigl|\rho\bigl(t, q(t,x)\bigr)\bigr| \leqslant \frac{1}{|\rho_0(x)|} C_5 e^{C_4 t} \tag{5.2}$$

hold for all $t \in [0, T)$, *with* (*see* (4.25) *for* C_1)

$$\begin{aligned} C_4 &= \frac{3}{2} + C_1^2 = \frac{3}{2} + \frac{1}{2}\bigl(1 + A^2\bigr)\bigl\|(u_0, \rho_0 - 1)\bigr\|_{H^1 \times L^2}^2, \\ C_5 &= 1 + \|u_{0,x}\|_{L^\infty}^2 + \|\rho_0\|_{L^\infty}^2, \end{aligned}$$

and positive constant C *depending only on* A *and the norm* $\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}$.

Proof. In view of the proof of Lemma 4.2, by Lemma 4.1 and a simple density argument, it suffices to show that the desired results are valid when $s \geqslant 3$. So in the sequel of this section $s = 3$ is taken for simplicity of notation. Observe that the system (1.2) leads to the following ordinary differential equations (see Lemma 4.2 for derivation) for a fixed $x \in \mathbb{R}$,

$$M'(t) = -\frac{1}{2}M^2 + \frac{1}{2}\gamma^2 + f\bigl(t, q(t,x)\bigr), \qquad \gamma'(t) = -\gamma M, \tag{5.3}$$

for $t \in [0, T)$ with notation $M(t) = u_x(t, q(t,x))$, $\gamma(t) = \rho(t, q(t,x))$ defined in (4.27) and f in (4.29). The second equation of (5.3) implies that $\gamma(t)$ and $\gamma(0)$ are of the same sign.

For every $x \in \mathbb{R}$ satisfying $\gamma(0) = \rho_0(x) > 0$, define the Lyapunov function (cf. [12]),

$$w(t) := \gamma(0)\gamma(t) + \frac{\gamma(0)}{\gamma(t)}\bigl(1 + M^2(t)\bigr),$$

which is a positive function of $t \in [0, T)$. By (5.3), it yields

$$\begin{aligned}
w'(t) &= \gamma(0)\gamma' - \frac{\gamma(0)}{\gamma^2}\gamma'\left(1+M^2\right) + \frac{2}{\gamma}\gamma(0)MM' \\
&= \frac{2}{\gamma}\gamma(0)M\left(f\left(t, q(t,x)\right) + \frac{1}{2}\right) \\
&\leqslant \frac{\gamma(0)}{\gamma}\left(1+M^2\right)\left(\left|f\left(t, q(t,x)\right)\right| + \frac{1}{2}\right) \leqslant C_4 w(t),
\end{aligned}$$

in $[0, T)$, where $|f| \leqslant 1 + C_1^2$ is derived from (4.33) and (4.34). The preceding differential inequality gives

$$w(t) \leqslant w(0)e^{C_4 t} = C_5 e^{C_4 t}, \quad t \in [0, T) \tag{5.4}$$

with the help of

$$w(0) = \rho_0^2(x) + 1 + u_{0,x}^2(x) \leqslant 1 + \|u_{0,x}\|_{L^\infty}^2 + \|\rho_0\|_{L^\infty}^2 = C_5.$$

Recalling that $\gamma(t)$ and $\gamma(0)$ are of the same sign, the definition of w implies $\gamma(0)\gamma(t) \leqslant w(t)$ and $|\gamma(0)||M(t)| \leqslant w(t)$. By (5.4),

$$\begin{aligned}
\left|u_x\left(t, q(t,x)\right)\right| &= \left|M(t)\right| \leqslant \frac{1}{|\gamma(0)|}w(t) \leqslant \frac{1}{|\rho_0(x)|}C_5 e^{C_4 t}, \\
\left|\rho\left(t, q(t,x)\right)\right| &= \left|\gamma(t)\right| \leqslant \frac{1}{|\gamma(0)|}w(t) \leqslant \frac{1}{|\rho_0(x)|}C_5 e^{C_4 t}
\end{aligned}$$

are valid for $t \in [0, T)$. Thus the conclusions of (5.2) are obtained. □

Proof of Theorem 5.1. Assume the contrary that $T < \infty$ and the solution blows up in finite time. It then transpires from Theorem 4.1 that

$$\int_0^T \left\|u_x(t,x)\right\|_{L^\infty} dt = \infty. \tag{5.5}$$

Note that $\inf_{x\in\mathbb{R}} \rho_0(x) > 0$. By (5.2) in Lemma 5.1, we have

$$\left|u_x(t,x)\right| \leqslant \frac{1}{|\rho_0(x)|}Ce^{Ct} \leqslant \frac{1}{\inf_{x\in\mathbb{R}} \rho_0(x)}Ce^{CT} < \infty$$

for all $(t,x) \in [0,T) \times \mathbb{R}$, a contrary to (5.5). So $T = +\infty$, and the solution (u, ρ) is global. □

Acknowledgments

The work of G. Gui is partly supported by Morningside Center of Mathematics, Chinese Academy of Sciences and the work of Y. Liu is partially supported by the NSF grant DMS-0906099. The authors would like to thank the referees for constructive suggestions and comments.

References

[1] J.M. Bony, Calcul symbolique et propagation des singularités pour les q́uations aux drivées partielles non linéaires, Ann. Sci. École Norm. Sup. (4) 14 (1981) 209–246.
[2] A. Bressan, A. Constantin, Global conservative solutions of the Camassa–Holm equation, Arch. Ration. Mech. Anal. 183 (2007) 215–239.
[3] A. Bressan, A. Constantin, Global dissipative solutions of the Camassa–Holm equation, Appl. Anal. 5 (2007) 1–27.
[4] J.C. Burns, Long waves on running water, Math. Proc. Cambridge Philos. Soc. 49 (1953) 695–706.
[5] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[6] J.Y. Chemin, Localization in Fourier space and Navier–Stokes system, in: Phase Space Analysis of Partial Differential Equations, in: CRM Series, Scuola Norm. Sup., Pisa, 2004, pp. 53–136.
[7] J.Y. Chemin, Perfect Incompressible Fluids, Oxford Univ. Press, New York, 1998.
[8] M. Chen, S. Liu, Y. Zhang, A 2-component generalization of the Camassa–Holm equation and its solutions, Lett. Math. Phys. 75 (2006) 1–15.
[9] A. Constantin, Global existence of solutions and breaking waves for a shallow water equation: a geometric approach, Ann. Inst. Fourier (Grenoble) 50 (2000) 321–362.
[10] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, Arch. Ration. Mech. Anal. 192 (2009) 165–186.
[11] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998) 229–243.
[12] A. Constantin, R.I. Ivanov, On an integrable two-component Camassa–Holm shallow water system, Phys. Lett. A 372 (2008) 7129–7132.
[13] A. Constantin, T. Kappeler, B. Kolev, P. Topalov, On geodesic exponential maps of the Virasoro group, Ann. Global Anal. Geom. 31 (2007) 155–180.
[14] A. Constantin, B. Kolev, Geodesic flow on the diffeomorphism group of the circle, Comment. Math. Helv. 78 (2003) 787–804.
[15] A. Constantin, W.A. Strauss, Exact steady periodic water waves with vorticity, Comm. Pure Appl. Math. 57 (2004) 481–527.
[16] R. Danchin, A few remarks on the Camassa–Holm equation, Differential Integral Equations 14 (2001) 953–988.
[17] R. Danchin, A note on well-posedness for Camassa–Holm equation, J. Differential Equations 192 (2003) 429–444.
[18] J. Escher, O. Lechtenfeld, Z.Y. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation, Discrete Contin. Dyn. Syst. 19 (2007) 493–513.
[19] G. Falqui, On a Camassa–Holm type equation with two dependent variables, J. Phys. A 39 (2006) 327–342.
[20] C. Guan, Z.Y. Yin, Global existence and blow-up phenomena for an integrable two-component Camassa–Holm shallow water system, J. Differential Equations 248 (8) (2010) 2003–2014.
[21] G. Gui, Y. Liu, On the Cauchy problem for the two-component Camassa–Holm system, Math. Z. (2010) doi:10.1007/s00209-009-0660-2.
[22] D.D. Holm, L. Nraigh, C. Tronci, Singular solutions of a modified two-component Camassa–Holm equation, Phys. Rev. E 79 (2009) 016601.
[23] R. Ivanov, Two-component integrable systems modelling shallow water waves: The constant vorticity case, Wave Motion 46 (2009) 389–396.
[24] R.S. Johnson, The Camassa–Holm equation for water waves moving over a shear flow, Fluid Dynam. Res. 33 (2003) 97–111.
[25] R.S. Johnson, Nonlinear gravity waves on the surface of an arbitrary shear flow with variable depth, in: Nonlinear Instability Analysis, in: Adv. Fluid Mech., vol. 12, Comut. Mech. Southampton, 1997, pp. 221–243.
[26] R.S. Johnson, On solutions of the Burns condition, Geophys. Astrophys. Fluid Dyn. 57 (1991) 115–133.

[27] R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, J. Fluid Mech. 455 (2002) 63–82.
[28] B. Kolev, Poisson brackets in hydrodynamics, Discrete Contin. Dyn. Syst. 19 (2007) 555–574.
[29] G. Misiolek, A shallow water equation as a geodesic flow on the Bott–Virasoro group, J. Geom. Phys. 24 (1998) 203–208.
[30] P. Olver, P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E 53 (1996) 1900.
[31] A.B. Shabat, L. Martínez Alonso, On the prolongation of a hierarchy of hydrodynamic chains, in: A.B. Shabat, et al. (Eds.), New Trends Integrability and Partial Solvability, Proc. NATO Advanced Research Workshop, Cadiz, Spain 2002, in: NATO Sci. Ser. II Math. Phys. Chem., Kluwer Academic Publishers, Dordrecht, 2004, pp. 263–280.
[32] H. Triebel, Theory of Function Spaces, Monogr. Math., vol. 78, Birkhäuser Verlag, Basel, 1983.