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Existence and Uniqueness of Solutions to Degenerate Semilinear Parabolic Equations

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An elementary proof for the existence of solutions to semilinear degenerate parabolic equations is given. The assumptions are such that solutions to approximating parabolic problems are a priori bounded in L_∞. Then an Ascoli–Arzela argument is used. Under more restrictive conditions on the coefficients, uniqueness of the solutions is shown separately and a regularity result is provided.

1. INTRODUCTION

The main concern of this paper is to provide an elementary proof for the existence of a unique solution to the equation

$$\begin{aligned} \dot{u}(x,t) - \operatorname{div}(a(x)\operatorname{grad} u(x,t)) = f(u(x,t)) \qquad & \text{in} \quad \Omega \times [0, T^+] \\ u(x,t) = 0 \qquad & \text{on} \quad \partial\Omega \times [0, T^+] \\ u(x,0) = u_0(x) \qquad & \text{in} \quad \Omega, \end{aligned}$$

on a smooth bounded subset Ω of $\mathbf{R}^N$ and where T^+ is an arbitrary positive number. The coefficient matrix $a(x)$ is assumed to be a positive definite symmetric $N \times N$ matrix, but its smallest eigenvalue might converge to zero as x approaches the boundary of the domain. This makes the above a degenerate parabolic problem which cannot be solved by the standard methods. The main idea of our proof is to approach the solution with solutions of regularized parabolic problems. We use an Ascoli–Arzela argument. Thus we have to show that the approximating solutions are uniformly bounded and continuous in certain function spaces. To achieve this we assume a geometric condition on the function f which implies an L_∞-bound for the solutions. To solve the approximating problem and to improve the estimates we use the theory of linear semigroups. The variation-of-constants formula makes it very simple to prove that the solutions

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are equicontinuous in the natural energy space which is a weighted Sobolev space $H_a^1(\Omega)$. It is worth observing that our construction is independent of the dimension N of the domain Ω.

The linear version of the above problem (i.e., $f(u) = cu$) was considered by Fichera [2]. He developed a method to solve elliptic–parabolic problems. His proof is based on the Riesz representation theorem used on a weighted Sobolev space. His results were extended by Oleĭnik and Radkevič [11]. Their result is based on elliptic regularization; i.e., they subtract $\varepsilon(\ddot{u} + \Delta u)$ from the left hand side of the equation and then let ε converge to zero. Both methods [2, 11] apply to the more general class of linear second order problems with nonnegative characteristic form.

The linear case of the above problem is also a special case of the results of Showalter [13]. He used the theory of analytic semigroups on a weighted Sobolev space. His results are more general since he can also solve problems with an additional factor $c(x)$ in front of the $\dot{u}$-term. Our existence result applies to a wider class of functions a since a does not have to degenerate with a given order but need only be bounded from below (see (1), (12) and [13, Theorem 3]). Nevertheless, the present result is a natural extension of [13] to the semilinear case, but the methods of proof are independent.

In a recent series of papers Goldstein and Lin [3–6] solved similar quasilinear problems. Their functions a may depend on x and ∇u. This makes the problem considerably more difficult to solve and our method certainly does not solve their problem. Their main tool is the Crandall–Liggett theorem on nonlinear semigroups. As a consequence the function f has to be monotone. Our assumption (8) on f is much weaker. Their description of the solution space is not as precise as ours since it involves the abstract domain of definition of a nonlinear operator. The proof of our main result does not use the obvious semigroup approach since this would impose a strong restriction on the possible nonlinear functions f. We could not find a simple proof for L_∞ a priori estimates which would enlarge the class of nonlinearities that can be handled. But we do list the result of this method in the last section.

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2. Assumptions and Definitions

Let Ω be a bounded open subset of $\mathbf{R}^N$ with smooth boundary Γ. Let $a(x) = [a_{i,j}(x)]_{1 \leqslant i,j \leqslant N}$ be a smooth function on Ω, which extends continuously to the closure of Ω, such that there is a $c_1 > 0$ with

$$\underline{a}(x) \geqslant c_1 d(x)^\alpha \qquad \text{with } 0 < \alpha < 1 \tag{1}$$

in a neighborhood of Γ where $d(x) := \operatorname{dist}(x, \Gamma)$ and

$$\begin{aligned}\underline{a}(x) &= \min\left\{\sum_{i,j} a_{i,j}(x)\,\xi_i\xi_j \mid \|\xi\| = 1\right\},\\ \bar{a}(x) &= \max\left\{\sum_{i,j} a_{i,j}(x)\,\xi_i\xi_j \mid \|\xi\| = 1\right\}.\end{aligned} \tag{2}$$

We assume the existence of a sequence of open sets Ω_n with smooth boundaries such that

$$\begin{aligned}&\Omega_n \subset \Omega_{n+1} \subset \Omega \qquad \bigcup_{n\in\mathbf{N}} \Omega_n = \Omega\\ &\lim_{n\to\infty} \sup\{\underline{a}(x) \mid x \in \Omega\setminus\Omega_n\} = 0.\end{aligned} \tag{3}$$

We also assume that there are functions a_n defined on Ω such that

$$\begin{aligned}&a_n \mid \Omega_n = a, \qquad \underline{a}_n(x) \geqslant \frac{1}{n}\\ &a_n(x)\,\xi\cdot\xi \geqslant a(x)\,\xi\cdot\xi \qquad \forall x\in\Omega \qquad \forall\xi\in\mathbf{R}^N\\ &\lim_{n\to\infty} \sup\{\|a_n(x) - a(x)\| \mid x\in\Omega\} = 0.\end{aligned} \tag{4}$$

The above assumptions are such that the coefficient matrix a might degenerate only on a part of Γ. The character of a might even change as we move along Γ. Degeneracy at isolated points is also included.

We use the notations

$$\begin{aligned}Au &= \nabla\cdot(a\,\nabla u)\\ A_n u &= \nabla\cdot(a_n\nabla u)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\|u\|_p &= \|u\|_{L_p(\Omega)}\\ \|u\|_{k,p} &= \|u\|_{W_p^k(\Omega)}\\ \langle u, v\rangle &= \int_\Omega u(x)\,v(x)\,dx\\ \langle a\,\nabla u, \nabla v\rangle &= \int_\Omega \sum_{i,j=1}^N a_{i,j}(x)\,\partial_i u(x)\,\partial_i v(x)\,dx\\ \|u\|^2_{H_a^1(\Omega)} &= \langle a\,\nabla u, \nabla u\rangle.\end{aligned} \tag{6}$$

Let $H_a^1(\Omega)$ be the closure of the smooth functions with support in Ω with respect to the norm $(\|u\|_{H_a^1}^2 + \|u\|_{L_2}^2)^{1/2}$. Observe that $\|u\|_{H_a^1}$ is not a norm on H_a^1 but only a seminorm. This will not cause problems since we will always be able to control the L_2-norm.

It is our intention to solve, on the given time interval $I=[0, T^+]$; the problem

$$\begin{aligned} \dot{u} - Au &= f(u) && \text{in } \Omega \times I \\ u &= 0 && \text{on } \Gamma \times I \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{7}$$

where we assume that the real-valued function f is smooth and that there is a number M such that

$$f(s)\, s < 0 \qquad \forall\, |s| \geqslant M. \tag{8}$$

This assumption will imply an L_∞ a priori bound, which is essential for our purposes. As a consequence we also obtain solutions on time intervals of arbitrary length.

All the results in this paper remain true if we replace (8) by the weaker condition

$$sf(s) \leqslant c(1+s^2) \qquad \forall s \in \mathbf{R} \tag{9}$$

for some $c>0$, but the proofs have to be modified slightly.

We call u a *weak solution* of (7) if all the following conditions are satisfied:

$$u \in C^0(I, L_2(\Omega)) \cap C^0(I, H_a^1(\Omega')) \qquad \text{for any compact subset } \Omega' \text{ of } \Omega$$

$$\|u(t)\|_{H_a^1(\Omega)} + \|u(t)\|_\infty \leqslant c \qquad u(0) = u_0 \qquad u(t)|\Gamma = 0$$

$$\begin{aligned} &\langle \psi(T), u(T)\rangle - \langle \psi(0), u_0\rangle - \int_0^T \langle \dot{\psi}(t), u(t)\rangle\, dt \\ &= \int_0^T -\langle a\nabla\psi(t), \nabla u(t)\rangle + \langle \psi(t), f(u(t))\rangle\, dt \\ &\forall \psi \in C^1(I, L_2(\Omega)) \cap C^0(I, \mathring{W}_2^1(\Omega))\ \forall T \in I. \end{aligned} \tag{10}$$

A generic constant will always be called c and it should be clear from the context on what the constant depends.

3. Main Result

We first state the main existence result for solutions of (7).

THEOREM 1. *Problem* (7) *has for each* $u_0 \in W_2^2(\Omega) \cap L_\infty(\Omega)$ *whose support is compactly contained in* Ω *a global weak solution which is Lipschitz continuous with respect to time with values in* $L_2(\Omega)$.

Comment. The condition of compact support for the initial value can be weakened. If we have a sequence of functions $u_{n,0} \in W_2^2(\Omega)$ with $u_{n,0} \mid \Gamma = 0$ and

$$\| u_{n,0} \|_\infty \leqslant M, \qquad u_{n,0} \to u_0 \text{ in } L^2(\Omega),\ \| A_n u_{n,0} \|_2 \leqslant c \qquad \forall n \in N \quad (11)$$

for some $c > 0$, then we obtain the same result.

If we replace condition (1) by a stronger one which forces the largest eigenvalue $\bar{a}$ of the coefficient matrix a to decay with a given rate as x approaches the boundary we derive a uniqueness result.

THEOREM 2. *If we replace* (1) *by*

$$c_2 d(x)^\alpha \geqslant \bar{a}(x) \geqslant \underline{a}(x) \geqslant c_1 d(x)^\alpha \qquad \textit{with} \quad 0 < \alpha < 1, \tag{12}$$

for constants $c_1, c_2 > 0$, *then the problem* (7) *has, for a given* $u_0 \in L_2(\Omega)$, *at most one weak solution which is Lipschitz continuous with respect to time with values in* $L_2(\Omega)$.

If the coefficient a satisfies (1) (i.e., is bounded from below) and is strictly bounded away from zero close to a connected component of the boundary Γ then the assumption (12) can be dropped for x close to that connected component. This is easily verified by checking the proof.

Using standard results on analytic semigroups we find a simple abstract regularity proof for the solution u.

THEOREM 3. *If we assume the conditions from Theorems* 1 *and* 2 *and* $f(0) = 0$, *then the solution* u *of* (7) *satisfies*

$$u \in C^1((0, T^+), L_2(\Omega)) \cap C^0((0, T^+), H_a^1(\Omega)). \tag{13}$$

A slightly more precise regularity result is given in the proof of this theorem.

In the last section we list the result obtained by the standard semigroup approach to problem (7) and we also show that our approach can be generalized to solve problems with nonselfadjoint differential operators.

4. Some Useful Lemmas

We first consider some properties of solutions of the approximating problem

$$\begin{aligned} \dot{u}_n - A_n u_n &= f(u_n) && \text{in } \Omega \times I \\ u_n &= 0 && \text{on } \Gamma \times I \\ u_n(0) &= u_{n,0} && \text{in } \Omega. \end{aligned} \tag{14}$$

Since A_n is a uniformly elliptic operator in divergence form it is the generator of an analytic contraction semigroup e^{tA_n} on $L_p(\Omega)$ for any $1 < p < \infty$ with dom $A_n = W_p^2(\Omega) \cap \{u \mid \Gamma = 0\}$ (see [12]).

The first lemma shows that the solutions of (14) are uniformly bounded. This is the starting point of all the estimates to come. Its proof would be trivial if the solutions were classical at time zero. We essentially have to show that the L_∞-norm does not blow up as t converges to zero.

Lemma 1. *For all $t \in [0, T]$ and for all $n \in \mathbf{N}$ we have $\|u_n(t)\|_\infty < M + 1$.*

Proof. We first modify the function f for arguments x with $|x| \geqslant 2M$ such that f is globally Lipschitz continuous and (8) remains valid. Using $q = \mathrm{vol}(\Omega)$ we derive easily $\|u_{n,0}\|_p \leqslant q^{1/p} M$. From the well known variation-of-constants formula we obtain

$$u_n(t) = e^{tA_n} u_{n,0} + \int_0^t e^{(t-\tau)A_n} f(u_n(\tau))\, d\tau. \tag{15}$$

Using the Lipschitz constant of f and the contraction property of the semigroup we deduce

$$\|u_n(t)\|_p \leqslant q^{1/p} M + c \int_0^t 1 + \|u_n(\tau)\|_p \, d\tau. \tag{16}$$

Gronwall's inequality shows that for p big enough $\|u_n(t)\|_p \leqslant M + 1$ for all $0 \leqslant t \leqslant t_0$, where t_0 does not depend on n and p. This implies $\|u_n(t)\|_\infty \leqslant M + 1$ for a positive t. Since we have classical solutions for $t > 0$ the condition (8) leads us now to the same estimate for all positive t by a simple calculus argument. Observing that the above modification of f does not influence the solutions, we have the desired result. ∎

If we want to prove Lemma 1 using the assumption (9) then we have to subtract a term $\hat{c}u$ on both sides of the equation and look at the modified nonlinearity

$$\hat{f}(s) = f(s) - \hat{c}s \tag{17}$$

which satisfies property (8) if we only choose $\hat{c}$ big enough. The modified differential operators A_n will not generate contraction semigroups on L_p but the exponential growth rate does not depend on p. This is sufficient to prove the required results.

Now we want to show that $\dot{u}_n$ is equibounded in $L_2(\Omega)$. For this purpose we set

$$v_n(t) = \dot{u}_n(t), \qquad v_{n,0} = \dot{u}_{n,0} = A_n u_{n,0} + f(u_{n,0}). \tag{18}$$

From

$$v_n(t) = e^{tA_n} v_{n,0} + \int_0^t e^{(t-\tau)A_n} f'(u_n(\tau))\, v_n(\tau)\, d\tau \tag{19}$$

one obtains

$$\begin{aligned} \| v_n(t)\|_2 &\leqslant \| v_{n,0}\|_2 + \int_0^t c\, \| v_n(\tau)\|_2\, d\tau \\ \| v_n(t)\|_2 &\leqslant \| v_{n,0}\|_2\, e^{ct}. \end{aligned} \tag{20}$$

Now the inequality $\| v_{n,0}\|_2 \leqslant \| A_n u_{n,0}\|_2$ implies

$$\| v_n(t)\|_2 \leqslant c, \tag{21}$$

where c does not depend on n. Observe that this implies that the functions u_n are equicontinuous in $L_2(\Omega)$, i.e.,

$$\| u_n(t+h) - u_n(t)\|_2 \leqslant ch \qquad \forall t \in I, \forall n \in \mathbf{N}, \tag{22}$$

where the constant c does not depend on n and t. The L_∞ a priori bound and Hölder's inequality imply that u_n is equicontinuous in any L_p where $p < \infty$.

Using a bootstrapping argument we now prove that u_n is equicontinuous in $H_a^1(\Omega)$. Using (4), Green's formula, the differential equation (14), and the estimate (21) we deduce

$$\begin{aligned} \| u_n(t+h) - u_n(t)\|^2_{H_a^1(\Omega)} &\leqslant \| u_n(t+h) - u_n(t)\|^2_{H^1_{a_n}(\Omega)} \\ &= \langle a_n \nabla(u_n(t+h) - u_n(t)), \nabla(u_n(t+h) - u_n(t))\rangle \\ &= \langle -A_n(u_n(t+h) - u_n(t)), (u_n(t+h) - u_n(t))\rangle \\ &= \langle \dot{u}_n(t) - \dot{u}_n(t+h) - f(u_n(t+h)) + f(u_n(t)), u_n(t+h) - u_n(t)\rangle \\ &\leqslant (\| \dot{u}_n(t+h) - \dot{u}_n(t)\|_2 + \| f(u_n(t+h)) - f(u_n(t))\|_2) \\ &\qquad \| u_n(t+h) - u_n(t)\|_2 \leqslant c\, \| u_n(t+h) - u_n(t)\|_2. \end{aligned} \tag{23}$$

Thus (22) implies

$$\| u_n(t+h) - u_n(t)\|_{H_a^1(\Omega)} \leqslant ch^{1/2} \qquad \forall t \in I. \tag{24}$$

A similar argument shows that

$$\|u_n(t)\|_{H_a^1(\Omega)} \leqslant \|u_n(t)\|_{H_{a_n}^1(\Omega)} \leqslant c. \tag{25}$$

Thus we have the following basic results which turn out to be important for our method of proof.

LEMMA 2. *The sequence $\{u_n\}$ of solutions of (14) is equicontinuous in $H_a^1(\Omega)$ and $L_p(\Omega)$ for $1 \leqslant p < \infty$ and equibounded in $H_a^1(\Omega)$. $\dot{u}_n$ is equibounded in $L_2(\Omega)$.*

The next lemma is a compactness result for the weighted Sobolev space $H_a^1(\Omega)$. A similar but stronger result is given in [10, Theorem 4.11]. Since our proof is much simpler than the one in [10] and we obtain the result needed for our purposes we give the proof.

LEMMA 3. *If a set of functions is bounded in the $H_a^1(\Omega)$-norm and the $L_2(\Omega)$-norm then the set is relatively compact in $L_p(\Omega)$ for $1 \leqslant p < 2/(1+\alpha)$.*

Proof. The proof is based on a sufficient condition for a set to be pre-compact in $L_p(\Omega)$ due to Kolmogorov and Frechet (see [7, Theorem 2.5.2]). Let v be a function in the above set and let Ω' be a compactly contained smooth subset of Ω. We trivially have

$$\|v\|_{L_p(\Omega\setminus\Omega')} \leqslant \|v\|_2 \operatorname{vol}(\Omega\setminus\Omega')^{1/p-1/2}. \tag{26}$$

The assumption on p implies $\alpha p/(2-p) < 1$. Since $\underline{a}(x) \geqslant c_1 d(x)^\alpha$ if x is close to Γ we know that

$$\int_\Omega \underline{a}(x)^{-p/(2-p)}\, dx < \infty. \tag{27}$$

For a smooth function v and a vector h with $2|h| \leqslant \operatorname{dist}(\Omega', \Gamma)$ Hölder's inequality leads to

$$\begin{aligned}
&\int_{\Omega'} |v(x+h) - v(x)|^p\, dx \\
&\leqslant \int_{\Omega'} \int_0^1 |\nabla v(x+th)|^p\, dt\, |h|^p\, dx \\
&\leqslant |h|^p \int_0^1 \left[\left(\int_{\Omega'} \underline{a}(x+th)^{-p/(2-p)}\, dx \right)^{1-p/2} \right. \\
&\qquad \left. \left(\int_{\Omega'} \underline{a}(x+th) |\nabla v(x+th)|^2\, dx \right)^{p/2} \right] dt \\
&\leqslant |h|^p\, c\, \|v\|_{H_a^1(\Omega)}^p.
\end{aligned} \tag{28}$$

A convolution argument shows that the inequality (28) is correct for the given functions v. Kolmogorov's result now gives the desired convergent subsequence. ∎

Since the coefficients of A degenerate only on the boundary Γ we can find interior estimates for the solutions u_n.

LEMMA 4. *For each $m \in \mathbf{N}$ there is a constant $c(m)$ such that*

$$\|u_n(t)\|_{W_2^2(\Omega_m)} \leqslant c(m) \qquad \forall n \in \mathbf{N},\ t \in I. \tag{29}$$

Proof. For the sake of short notation we omit the argument t in this proof. We only have to look at n such that $A_n \mid \Omega_{2m} = A \mid \Omega_{2m}$. Pick a function ψ with

$$\psi \in C^\infty(\Omega, \mathbf{R}), \quad \psi \mid \Omega_m = 1, \qquad \operatorname{supp} \psi \subset \Omega_{2m}. \tag{30}$$

On Ω_{2m} the product rule implies

$$A(\psi u_n) = \psi A u_n + 2\nabla\psi(a\nabla u_n) + u_n A\psi. \tag{31}$$

Using the differential equation (14) we deduce

$$\begin{aligned} A(\psi u_n) &= \psi \dot{u}_n - \psi f(u_n) + 2\nabla\psi(a\nabla u_n) + u_n A\psi \qquad && \text{in } \Omega_{2m} \\ \psi u_n &= 0 && \text{on } \partial\Omega_{2m}. \end{aligned} \tag{32}$$

Because of the equiboundedness of u_n in $H_a^1(\Omega)$ and (21) the right hand side of this elliptic equation is bounded in $L_2(\Omega_{2m})$ uniformly with respect to t and n. $A \mid \Omega_{2m}$ is uniformly elliptic on Ω_{2m} and the constant of ellipticity depends only on m. Thus standard elliptic theory implies

$$\|\psi u_n\|_{W_2^2(\Omega_{2m})} \leqslant c(m), \tag{33}$$

which immediately gives the desired result. ∎

The above lemma gives us control on the behavior of the solutions in the interior of the domain Ω. If we modify the weight function a in $H_a^1(\Omega)$ slightly we obtain also some information near the boundary.

LEMMA 5. *Let $\hat{a}(x)$ be a positive definite smooth modification of the matrix valued function $a(x)$ such that*

$$\hat{a}(x) \leqslant c\underline{a}(x)^{1+\hat{\alpha}} \qquad \textit{for some } \hat{\alpha} > 0 \tag{34}$$

in a neighborhood of Γ. If a sequence of functions satisfies

$$\|u_i\|_{H_a^1(\Omega)} \leqslant c, \qquad \|u_i\|_\infty \leqslant c, \qquad \|u_i\|_{W_2^2(\Omega_m)} \leqslant c(m), \tag{35}$$

then there exist a $u \in H^1_{\hat{a}}(\Omega)$ *and a subsequence (again denoted by* $\{u_i\}$*) such that*

$$u_i \to u \qquad in\ H^1_{\hat{a}}(\Omega). \tag{36}$$

Proof. The usual Sobolev imbedding and a diagonal sequence argument give a subsequence $\{u_i\}$ which converges to u in $L_2(\Omega)$ and $H^1_a(\Omega_m)$ for each m. In the next section we show by an independent proof that $u \in H^1_a(\Omega)$. Thus we have $u \in H^1_{\hat{a}}(\Omega)$. For m big enough we calculate

$$\begin{aligned} \|u-u_i\|^2_{H^1_{\hat{a}}(\Omega)} \leqslant & \int_{\Omega_m} \hat{a}(x)|\nabla(u(x)-u_i(x))|^2\,dx \\ & + \int_{\Omega\setminus\Omega_m} \underline{a}(x)^{\hat{\alpha}}\, \underline{a}(x)|\nabla(u(x)-u_i(x))|^2\,dx \\ \leqslant & \; c\,\|u-u_i\|^2_{H^1_a(\Omega_m)} \\ & + \sup\{\underline{a}(x)^{\hat{\alpha}} \mid x \in \Omega\setminus\Omega_m\}\,\|u-u_i\|^2_{H^1_a(\Omega\setminus\Omega_m)}. \end{aligned} \tag{37}$$

For a given $\varepsilon > 0$ we choose m such that the second term in the above expression is smaller than ε. Then we determine an i_0 such that the first term is smaller than ε, if only $i \geqslant i_0$. Thus we have the desired result. ▮

5. Proof of Theorem 1

The following result collects the statements in the previous section and constructs the function u which is the solution to the original problem (7).

Proposition 1. *Given* $I = [0, T^+]$, *there exists a subsequence of* $\{u_i\}$ *convergent in* $E = C^0(I, L_p(\Omega)) \cap C^0(I, H^1_{\hat{a}}(\Omega)) \cap C^0(I, H^1_a(\Omega_m))$ *towards* $u \in E$, *where* $1 \leqslant p < \infty$ *is arbitrary.*

Proof. By repeated extraction of subsequences the results of the previous section imply that for each $t \in I$ the sequence $\{u_n(t)\}$ has a convergent subsequence and the convergence is in the norms of $L_1(\Omega)$, $H^1_{\hat{a}}(\Omega)$, and $H^1_a(\Omega_m)$.

We know that $|u_n(t,x)| \leqslant M+1$. This implies $\|u(t)\|_\infty \leqslant M+1$. Using Hölder's inequality we obtain

$$\begin{aligned} \|u_n(t)-u(t)\|_p &\leqslant \|u_n(t)-u(t)\|_1^\theta\, \|u_n(t)-u(t)\|_\infty^{1-\theta} \\ &\leqslant \|u_n(t)-u(t)\|_1^\theta\,(2M+2)^{1-\theta}, \end{aligned} \tag{38}$$

where $1/p = \theta$. This implies convergence of the same subsequence in any L_p

space. Now a simple application of the Banach space version of the Ascoli–Arzela lemma (e.g., [9]) leads to the claimed result. ∎

The next lemma shows that $u(t)$ is in $H_a^1(\Omega)$ and it is uniformly bounded.

LEMMA 6. *For each $t \in I$ we have $u(t) \in H_a^1(\Omega)$ and there is a constant c such that*

$$\| u(t) \|_{H_a^1(\Omega)} \leqslant c. \tag{39}$$

Proof. Let χ_m be the characteristic function of the set Ω_m and let

$$\psi_m(x) = \chi_m(x)\, a(x)\, \nabla u(x) \cdot \nabla u(x), \qquad \psi(x) = a(x)\, \nabla u(x) \cdot \nabla u(x). \tag{40}$$

The sequence $\{\psi_m\}$ is monotone increasing and converges almost everywhere to ψ. Observing that

$$\lim_{n \to \infty} \int_{\Omega_m} a(x)\, \nabla u_n(x) \cdot \nabla u_n(x)\, dx = \int_{\Omega} \psi_m(x)\, dx \tag{41}$$

and recalling $\| u_n(t) \|_{H_a^1(\Omega)} \leqslant c$ we can apply Lebesgue's monotone convergence theorem to show that

$$\int_\Omega \psi(x)\, dx \leqslant c. \tag{42}$$

This proves the lemma. ∎

Since u_n converges uniformly to u in $L_2(\Omega)$ it is obvious that (22) implies the Lipschitz continuity of u in $L_2(\Omega)$. The Dirichlet boundary condition in (7) is satisfied since the trace operator is continuous from $H_a^1(\Omega)$ to $L_2(\Gamma)$. This is a consequence of a simple imbedding and a trace result quoted in [8, Theorem 9.14].

It remains to be shown that u is, in fact, a weak solution of (7). For this purpose we consider a fixed test function

$$\psi \in C^1(I, L_2(\Omega)) \cap C^0(I, \mathring{W}_2^1(\Omega)). \tag{43}$$

Since u_n is a classical solution of (14) we have for $T \in I$

$$\begin{aligned} &\langle \psi(T), u_n(T) \rangle - \langle \psi(0), u_{n,0} \rangle - \int_0^T \langle \dot{\psi}(t), u_n(t) \rangle\, dt \\ &\quad = \int_0^T - \langle a_n \nabla \psi(t), \nabla u_n(t) \rangle + \langle \psi(t), f(u_n(t)) \rangle\, dt. \end{aligned} \tag{44}$$

Using Proposition 1 it is obvious that the first line in the above expression converges to the same expression, where u_n is replaced by u if n converges to infinity. Similarly

$$\langle \psi(t), f(u_n(t)) \rangle \to \langle \psi(t), f(u(t)) \rangle \tag{45}$$

uniformly with respect to $t \in [0, T]$.

The only problem is to show that

$$\langle a_n \nabla\psi(t), \nabla u_n(t) \rangle \to \langle a \nabla\psi(t), \nabla u(t) \rangle. \tag{46}$$

For a given $\varepsilon > 0$ one deduces by elementary calculations that

$$\begin{aligned}
&| \langle a_n \nabla u_n - a \nabla u, \nabla\psi \rangle | \\
&\leqslant \int_\Omega |a_n - a| \, |\nabla u_n| \, |\nabla\psi| \, dx + | \langle a \nabla(u_n - u), \nabla\psi \rangle | \\
&\leqslant c \, \| u_n \|_{H^1_{a_n}(\Omega)} \sup\{ |a_n(x) - a(x)| \mid x \in \Omega \}^{1/2} \, \| \psi \|_{1,2} \\
&\quad + \left| \int_{\Omega_m} a \nabla(u_n - u) \cdot \nabla\psi \, dx \right| \\
&\quad + \left| \int_{\Omega \setminus \Omega_m} a \nabla(u_n - u) \cdot \nabla\psi \, dx \right| \\
&\leqslant c \, \| u_n \|_{H^1_{a_n}(\Omega)} \sup\{ |a_n(x) - a(x)| \mid x \in \Omega \setminus \Omega_n \} \, \| \psi \|_{1,2} \\
&\quad + c \, \| u_n - u \|_{H^1_a(\Omega_m)} \, \| \psi \|_{1,2} \\
&\quad + (\| u_n \|_{H^1_\alpha(\Omega)} + \| u \|_{H^1_a(\Omega)}) \sup \{ |a(x)| \mid x \in \Omega \setminus \Omega_m \}^{1/2} \, \| \psi \|_{1,2} \\
&\leqslant \varepsilon \, \| \psi \|_{1,2},
\end{aligned} \tag{47}$$

if only n is large enough. To achieve the last inequality one has to use (3), (25), and Proposition 1. First we can choose m such that the third term is small enough and then make n big enough so that the first two terms are small. This implies that u is a weak solution of (7).

A close inspection of the above calculations indicates that one could choose test functions ψ in a bigger function space than indicated in the definition of a weak solution. But the improvement would be marginal. The above proof does not allow us to choose u as a test function.

6. Proof of Theorem 2

The uniqueness proof does not depend on the construction of the solution. We use only the definition of a weak solution and the growth condition (12) on the coefficient a. Suppose we have two solutions v and w of (7) with the same initial value. Then $u := v - w$ would be a weak solution with $u_0 = 0$ and $f(u)$ replaced by $F(t) := f(v(t)) - f(w(t))$. We have to show that u is identically zero. The obvious way to achieve this would be to choose u as a test function. Unfortunately this is not directly possible and we have use an approximation procedure.

LEMMA 7. *There exist open sets*

$$S_n \subset S_{n+1} \qquad \Omega = \bigcup_n S_n \tag{48}$$

and linear operators R_n such that

$$\begin{aligned} R_n &\in \mathscr{L}(H_a^1(\Omega), H_a^1(\Omega)) \cap \mathscr{L}(H_a^1(\Omega), W_2^1(\Omega)) \\ \textit{with} &\quad \|R_n u\|_{H_a^1(\Omega)} \leqslant c\, \|u\|_{H_a^1(\Omega)} \\ \textit{and} &\quad R_n u(x) = u(x) \qquad \forall x \in S_n. \end{aligned} \tag{49}$$

The main point in the above lemma is the uniform boundedness of the linear operators R_n in $H_a^1(\Omega)$. The condition (12) allows us to prove the right estimate. In a concrete example it might be possible to construct the above operators without assumption (12).

Proof. Using local coordinates immediately reduces the problem to the situation where $\Omega \subset \mathbf{R}^{N-1} \times \mathbf{R}^+$ and

$$S_n = \{x = (\hat{x}, x') \in \Omega \text{ with } x' > n^{-1}\}. \tag{50}$$

We have $d(x) = x'$. R_n is now constructed by reflection at the plane $x' = n^{-1}$, i.e.,

$$Ru_n(\hat{x}, n^{-1} + h) := \begin{cases} u(\hat{x}, n^{-1} + h) & \text{if } \quad h \geqslant 0 \\ 4u\left(\hat{x}, n^{-1} - \dfrac{h}{2}\right) - 3u(\hat{x}, n^{-1} - h) & \text{if } \quad -n^{-1} < h < 0. \end{cases} \tag{51}$$

Using the notation $S_n^c := \Omega \setminus S_n$ we compute now

$$\begin{aligned} &\int_{S_n^c} a(x) \nabla R_n u(x) \cdot \nabla R_n u(x)\, dx \\ &\quad \leqslant c_2 \int_{S_n^c} n^{-\alpha}\, |\nabla R_n u(x)|^2\, dx \\ &\quad \leqslant c c_2 n^{-\alpha} \int_{n^{-1} < x' < 2n^{-1}} |\nabla R_n u(x)|^2\, dx \\ &\quad \leqslant c c_2 c_1^{-1} \int_{S_n} a(x)\, \nabla R_n u(x) \cdot \nabla R_n u(x)\, dx. \end{aligned} \tag{52}$$

A similar calculation shows that

$$\| R_n u \|^2_{W^1_2(\Omega)} \leqslant c(n) \| u \|^2_{H^1_a(\Omega)} \tag{53}$$

where $c(n)$ might blow up as n goes to infinity. This proves (49). ∎

The following lemma is an easy consequence of the above.

LEMMA 8. *For all u, $v \in H^1_a(\Omega)$ we have*

$$\langle a\nabla R_n u, \nabla v \rangle \to \langle a\nabla u, \nabla v \rangle \tag{54}$$

as n goes to infinity.

Proof. We have

$$\begin{aligned}\langle a\nabla(u - R_n u), \nabla v \rangle &= \int_{S_n^c} a(x)\, \nabla(u - R_n u) \cdot \nabla v \, dx \\ &\leqslant \| u - R_n u \|_{H^1_a(\Omega)} \left(\int_{S_n^c} a(x) |\nabla v|^2 \, dx \right)^{1/2}.\end{aligned} \tag{55}$$

Using (49) and Lebesgue's bounded convergence theorem we deduce the claimed result. ∎

Note that we did not prove that $R_n u$ converges to u in $H^1_a(\Omega)$.

Since u is only defined for $t \in I$ we extend it continuously to $\mathbf{R}$ by 0 for negative arguments and by $u(T^+)$ for $t > T^+$. To obtain uniqueness of the solution we use the test functions

$$\psi_{n,\delta}(t) := \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} R_n u(\tau)\, d\tau, \tag{56}$$

where $\delta > 0$. Obviously we have $\psi_{n,\delta} \in C^1(I, L_2(\Omega)) \cap C^0(I, H^1_a(\Omega'))$ and

$$\dot{\psi}_{n,\delta}(t) = \frac{1}{2\delta} R_n(u(t+\delta) - u(t-\delta)). \tag{57}$$

The following arguments are formal; e.g., it is not obvious whether the limit $\dot{K}_n$ exists. We omit the formally correct proof and give only the formulas that serve as guidelines for the technical, lengthy calculations that have to be carried out to achieve the desired result, namely (63).

We first want to let δ converge to zero.

$$\begin{aligned}\int_0^T \langle \dot{\psi}_{n,\delta}(t), u(t) \rangle \, dt &= \int_0^T \frac{1}{2\delta} \langle u(t+\delta) - u(t-\delta), u(t) \rangle_{S_n} \, dt \\ &\quad + \int_0^T \int_{S_n^c} \frac{1}{2\delta} R_n(u(t+\delta) - u(t-\delta))\, u(t)\, dx\, dt \\ &\to \frac{1}{2} (\| u(T) \|^2_{L_2(S_n)} - \| u(0) \|^2_{L_2(S_n)}) + K_n,\end{aligned} \tag{58}$$

where

$$K_n := \lim_{\delta \to 0+} \int_0^T \int_{S_n^c} \frac{1}{2\delta} R_n(u(t+\delta) - u(t-\delta))\, u(t)\, dx\, dt$$
$$|K_n| \leqslant \int_0^T cM \operatorname{vol}(S_n^c)^{1/2}\, dt. \tag{59}$$

To verify the above inequality one uses the L_∞-bound and the Lipschitz condition on u. Since $u \in C^0(I, L_2(\Omega)) \cap C^0(I, H_a^1(\Omega'))$ we deduce

$$\int_0^T - \langle a\nabla\psi_{n,\delta}(t), \nabla u(t)\rangle + \langle \psi_{n,\delta}(t), F(t)\rangle\, dt$$
$$\to \int_0^T - \langle a\nabla R_n u(t), \nabla u(t)\rangle + \langle R_n u(t), F(t)\rangle\, dt \tag{60}$$

as δ approaches zero. Since u is a weak solution we obtain

$$\|u(T)\|^2_{L_2(\Omega)} - \|u(0)\|^2_{L_2(\Omega)} - \tfrac{1}{2}(\|u(T)\|^2_{L_2(S_n)} - \|u(0)\|^2_{L_2(S_n)}) + K_n$$
$$= \int_0^T - \langle a\nabla R_n u(t), \nabla u(t)\rangle + \langle R_n u(t), F(t)\rangle\, dt. \tag{61}$$

Now we let n to tend to infinity and we use the above lemma, $u(0) = 0$ and

$$\|F(t)\|_2 = \|f(v(t)) - f(w(t))\|_2 \leqslant c\, \|v(t) - w(t)\|_2 \tag{62}$$

to derive

$$\|u(T)\|_2^2 \leqslant 2c \int_0^T \|u(t)\|_2^2\, dt. \tag{63}$$

This implies that u is vanished identically and the proof of Theorem 2 is complete.

7. Proof of Theorem 3

In this section we use semigroup theory to prove an abstract regularity result for the solution from the previous sections. For this purpose we need the following definition:

$$u \in \operatorname{dom} A \text{ and } Au = g \in L_2(\Omega) \qquad \text{iff}$$
$$u \in H_a^1 \qquad \text{and} \qquad \langle a\nabla u, \nabla v\rangle = -\langle g, v\rangle \qquad \forall v \in H_a^1. \tag{64}$$

Basic results imply that the selfadjoint positive operator A is the generator of an analytic semigroup on $L_2(\Omega)$ and $\operatorname{dom} A^{1/2} = H_a^1$ (e.g., [15, Theorem 2.2.3 and Sect. 3.6]). For a given L_2-valued function F, which depends on the time t, we call v a *mild solution* of

$$\begin{aligned} \dot{v} - Av &= F(t) \\ v(0) &= v_0 \end{aligned} \tag{65}$$

if

$$v(t) = e^{At} v_0 + \int_0^t e^{(t-\tau)A} F(\tau)\, d\tau. \tag{66}$$

The $F(t)$ will be equal to $f(u(t))$, where u is the unique solution from Theorem 2. We use the following well known regularity results on analytic semigroups.

LEMMA 9. *If* $F \in C^0(\mathbf{R}^+, \operatorname{dom} A^{\beta})$ *then we have* $v \in C^0((0, \infty), \operatorname{dom} A^{\beta+\gamma})$ *for any* $\gamma < 1$. *If* $F \in C^0(\mathbf{R}^+, \operatorname{dom} A^{\beta})$ *for some* $\beta > 0$ *then we have* $v \in C^1((0, \infty), L_2(\Omega))$.

Now we have to show that there is a relation between the above definition of a mild solution and the previous concept of a weak solution.

LEMMA 10. *If* $v \in C^0(\mathbf{R}^+, H_a^1) \cap \operatorname{Lip}(\mathbf{R}^+, L_2(\Omega))$ *is a mild solution of* (65) *and* $F \in C^0(\mathbf{R}^+, L_2(\Omega))$ *then* v *is also a weak solution of* (7), *except for the* L_∞*-bound.*

Proof. We apply arguments very similar to the ones used in [14] in the case of hyperbolic problems. Without loss of generality we assume that the operator A has a continuous inverse in L_2. We define

$$w(t) = e^{At} A^{-1} v_0 + \int_0^t e^{(t-\tau)A} A^{-1} F(\tau)\, d\tau - \int_0^t A^{-1} F(\tau)\, d\tau \tag{67}$$

and realize

$$\dot{w}(t) = v(t) \qquad w(0) = A^{-1} v_0 \qquad w \in C^0(\mathbf{R}^+, \operatorname{dom} A). \tag{68}$$

Now we choose a test function $\psi \in C^1(\mathbf{R}, H_a^1)$ and multiply its derivative with the equality $Aw(t) = \dot{w}(t) - \int_0^t F(\tau)\, d\tau$. Integrating the resulting equation with respect to time and subsequent integrations by parts lead us to

$$\begin{aligned} &\langle \psi(T), v(T) \rangle - \langle \psi(0), v_0 \rangle - \int_0^T \langle \dot{\psi}(t), v(t) \rangle\, dt \\ &\quad = \int_0^T - \langle a \nabla \psi(t), \nabla v(t) \rangle + \langle \psi(t), F(t) \rangle\, dt. \end{aligned} \tag{69}$$

Thus v is a weak solution of (7), where $f(u(t))$ is replaced by $F(t)$. ∎

Using $F(t) = f(u(t))$ we easily derive

LEMMA 11.

$$F \in C^0(I, \operatorname{dom} A^{\beta}) \qquad \text{for any } \beta < \tfrac{1}{2}. \tag{70}$$

Proof. Using the L_∞ and H_a^1-bounds on u and $\nabla f(u) = f'(u) \nabla u$ we see that $f(u)$ is bounded in H_a^1. Because of $f(0) = 0$ we have a bound for $f(u)$ in dom $A^{1/2}$. u is Lipschitz continuous with values in L_2 thus $f(u)$ has the same property. A simple interpolation argument proves the desired result. ∎

By redoing the uniqueness proof in the previous section for the right hand side $F(t)$ instead of $f(u(t))$ we observe that only an L_2-bound on F is necessary. Thus the above lemmas imply that the solution v of (65) is equal to the solution u of (7) given in Theorem 2 and

$$u \in C^1((0, T^+), L_2(\Omega)) \cap C^0((0, T^+), \operatorname{dom} A). \tag{71}$$

This proves the result claimed in Theorem 3.

8. SOME REMARKS

The obvious approach to solve (7) would be to look at the integral equation (66) which defines a mild solution. To be able to use this approach we need the following slight extension of the classical Sobolev imbedding.

LEMMA 12. *If $p > 1 + \alpha$ then we have for smooth bounded domains Ω the imbedding*

$$W^1_{p,\alpha}(\Omega) \subsetneq L_s(\Omega) \qquad \text{for} \quad \frac{1}{p} \geqslant \frac{1}{s} > \frac{1+\alpha}{p} - \frac{1}{N}, \tag{72}$$

where

$$\|u\|^p_{W^1_{p,\alpha}} = \int_\Omega |u(x)|^p \, dx + \int_\Omega d(x)^\alpha \, |\nabla u(x)|^p \, dx. \tag{73}$$

Proof. For $1 + \alpha < r \leqslant p$ we define $1/r' = 1 - 1/r$, $q = p/r$, and $\beta = \alpha/r$. Now we choose r such that $1/q - 1/N \leqslant 1/s$. Using Hölder's inequality we obtain

$$
\begin{aligned}
\|\nabla u\|_q^q &= \int_\Omega d(x)^{\quad \beta}\, d(x)^\beta\, |\nabla u(x)|^q\, dx \\
&\leqslant \left(\int_\Omega d(x)^{\quad \beta r'}\, dx\right)^{1/r'} \left(\int_\Omega d(x)^{\beta r}\, |\nabla u(x)|^{qr}\, dx\right)^{1/r} \\
&\leqslant c\, \|u\|^q_{W^1_{p,\alpha}}.
\end{aligned}
\tag{74}
$$

The usual Sobolev imbedding implies the desired result. ∎

Now we look at (7) as a semilinear problem whose linear part generates an analytic semigroup, and we obtain the obvious result.

THEOREM 4. *If $u_0 \in H^1_a(\Omega)$, and*

$$
|f'(x)| \leqslant c(1+|x|^{\gamma-1})\ \forall x \in \mathbf{R} \qquad \textit{where} \quad \gamma < \frac{N}{N(1+\alpha)-2}, \tag{75}
$$

then there exists a maximal time $\hat{T} > 0$ such that (7) *has a unique mild solution u with*

$$
u \in C^0([0, \hat{T}), L_2(\Omega)) \cap C^1((0, \hat{T}), L_2(\Omega)) \cap C^0([0, \hat{T}), H^1_a(\Omega)). \tag{76}
$$

If $u_0 \in L_2(\Omega)$ and the real function f is globally Lipschitz continuous then (7) *has a unique global mild solution u with*

$$
u \in C^0([0, \infty), L_2(\Omega)) \cap C^0((0, \infty), H^1_a(\Omega)). \tag{77}
$$

Proof. In the first case we realize that the substitution operator induced by f is locally Lipschitz continuous from H^1_a to L_2. Thus we can use a Banach fixed point argument to obtain the solution in $C^0([0, T_1], H^1_a(\Omega))$ for some small T_1. A continuation argument and well known abstract regularity results prove the result.

In the second case we realize that the substitution operator induced by f is Lipschitz continuous from L_2 to L_2. Thus similar arguments prove the result. ∎

Comment. The above approach is much simpler than the one we used in the previous sections. But please observe that the class of nonlinearities that can be handled is much smaller and we obtain less information about the solutions. E.g., there is no obvious way to prove L_∞ a-priori bounds for the above solutions. This justifies the use of our approach even if it seems to be more complicated at first sight.

Now we want to show that our approach also applies to the nonselfadjoint problem; i.e., we add a term of first order to the equation. Let

$$Bu = \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i}(x). \tag{78}$$

The definition of a weak solution for the equation

$$\begin{aligned} \dot{u} - Au - Bu = f(u) \quad & \text{in} \quad \Omega \times I \\ u = 0 \quad & \text{on} \quad \Gamma \times I \\ u(0) = u_0 \quad & \text{in } \Omega \end{aligned} \tag{79}$$

is very similar to (10); we only add a term

$$\int_0^T \langle \psi(t), B(u(t)) \rangle \, dt \tag{80}$$

to the right hand side. Then we obtain the following extension of Theorem 1.

THEOREM 5. *We use the assumptions of Theorem* 1 *and*

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} b_i(x) \leqslant K \tag{81}$$

for some constant K and

$$b_i(x) \leqslant c d(x)^{\beta} \qquad \text{with } \beta < \begin{cases} \dfrac{\alpha}{2} & \text{if} \quad \alpha \geqslant \dfrac{2}{n} \\ \dfrac{1}{n} & \text{if} \quad \alpha < \dfrac{2}{n}. \end{cases} \tag{82}$$

Then problem (79) *has a global weak solution which is Lipschitz continuous with respect to time with values in* $L_2(\Omega)$.

To verify the correctness of the above statement we only list the changes that have to be made in the proof of Theorem 1.

A result of Amann [1, Corollary 11.2] and assumption (81) imply that $A_n + B$ generate semigroups on L_p whose exponential growth rate does not depend on n and p. Thus we can prove a result similar to Lemma 1.

To modify calculation (23) we use the growth conditions (82), the

Sobolev imbedding (72), and multiple applications of Hölders inequality to derive

$$
\begin{aligned}
|\langle Bv, v\rangle| &\leqslant \sum_i \int_\Omega \left| b_i \frac{\partial v}{\partial x_i} v \right| dx \\
&\leqslant \sum_i \left(\int_\Omega d^\alpha \left(\frac{\partial v}{\partial x_i} \right)^2 dx \right)^{1/2} \left(\int_\Omega d^{-\alpha} b_i^2 v^2 \, dx \right)^{1/2} \\
&\leqslant c \, \|v\|_{H_a^1} \sum_i \left(\int_\Omega d^{-\alpha r} b_i^{2r} \, dx \right)^{1/2r} \|v\|_{L_s} \\
&\leqslant c \, \|v\|_{H_a^1} \|v\|_{L_s} \\
&\leqslant c \, \|v\|_{H_a^1} \|v\|_{L_2}^{\theta} \|v\|_{H_a^1}^{1-\theta}
\end{aligned}
\tag{83}
$$

with $1/r = 1 - 2/s$ and s is given by (72) and $\theta < 1$. Thus we can reprove Lemma 2 for problem (79). The other lemmas remain essentially unchanged and the only minor problem is to show the limit u of the convergent sequence is a weak solution. To achieve this one has to perform an additional calculation similar to (47) only less complicated.

This finishes the outline of the proof of Theorem 5.

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