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Numerical ranges of reducible companion matrices ☆

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ABSTRACT

In this paper, we show that a reducible companion matrix is completely determined by its numerical range, that is, if two reducible companion matrices have the same numerical range, then they must equal to each other. We also obtain a criterion for a reducible companion matrix to have an elliptic numerical range, put more precisely, we show that the numerical range of an n-by-n reducible companion matrix C is an elliptic disc if and only if C is unitarily equivalent to $A \oplus B$, where $A \in M_{n-2}$, $B \in M_2$ with $\sigma(B) = \{a\omega_1, a\omega_2\}$, $\omega_1^n = \omega_2^n = 1$, $\omega_1 \neq \omega_2$, and $|a| \geqslant \left(|\omega_1 + \omega_2| + \sqrt{|\omega_1 + \omega_2|^2 + 4(1 + 2\cos(\pi/n))}\right) \Big/ 2$.

1. Introduction

Let M_n be the algebra of n-by-n complex matrices. For any matrix B, $\sigma(B)$ denotes the set of its eigenvalues, $r(B) = \max\{|z| : z \in \sigma(A)\}$ denotes the *spectral radius* of B and the *numerical range* of B is the subset

$$W(B) = \{\langle Bx, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

of the plane. Properties of the numerical range can be found in [10, Chapter 1].

For any complex polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, there is associated an n-by-n matrix

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$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix},$$

called the *companion matrix* of p and denoted by $C(p)$. A familiar special case is the (nilpotent) *Jordan block* J_n when all the a_j's are zero. Such a matrix has the property that its minimal polynomial and characteristic polynomial are both equal to p. Hence companion matrices are nonderogatory and, in particular, are such that every eigenvalue has geometric multiplicity one. They arise as the building blocks in the *rational form* of general matrices: every square matrix is similar to a direct sum $C(p_1) \oplus \cdots \oplus C(p_k)$ of companion matrices with p_{j+1} dividing p_j for all j.

In this paper, we study the numerical ranges of companion matrices. For 2-by-2 companion matrices, the numerical range provides the complete information: *if A and B are* 2*-by-*2 *companion matrices, then* $A = B$ *if and only if* $W(A) = W(B)$. This is the consequence of the fact that 2-by-2 matrices with equal numerical ranges are unitarily equivalent. Unfortunately, the same cannot be said about companion matrices of size three. In [9, Example 2.1], the authors gave an example of two distinct 3-by-3 companion matrices whose numerical ranges are the same elliptic disc. The aim of this paper is to prove that two n-by-n reducible companion matrices have the same numerical range, then they must equal to each other. Recall that a matrix is *reducible* if it is unitarily equivalent to the direct sum of two other matrices. In [9], the authors give a criterion in terms of the eigenvalues for a companion matrix to be reducible. It roughly says that a companion matrix is reducible when its eigenvalues are "equally distributed" on at most two circles with center at the origin and radii reciprocal to each other.

Proposition 1.1 [9]. *An n-by-n ($n \geqslant 2$) companion matrix A is reducible if and only if its eigenvalues are of the form: $a\omega_n^{j_1}, \ldots, a\omega_n^{j_p}, (1/\bar{a})\omega_n^{j_{p+1}}, \ldots, (1/\bar{a})\omega_n^{j_n}$, where $a \neq 0$, ω_n denotes the nth primitive root of* 1, $1 \leqslant p \leqslant n-1$, *and $\{j_1, \ldots, j_p\}$ and $\{j_{p+1}, \ldots, j_n\}$ form a partition of $\{0, 1, \ldots, n-1\}$. In this case, A is unitarily equivalent to a direct sum $A_1 \oplus A_2$ with $\sigma(A_1) = \{a\omega_n^{j_1}, \ldots, a\omega_n^{j_p}\}$ and $\sigma(A_2) = \{(1/\bar{a})\omega_n^{j_{p+1}}, \ldots, (1/\bar{a})\omega_n^{j_n}\}$. In particular, every reducible companion matrix is invertible.*

It follows as corollaries that a companion matrix unitarily equivalent to a direct sum with one unitary summand or with at least three summands must itself be unitary. (cf. [9, Corollaries 1.2 and 1.3]). Moreover, if an n-by-n reducible companion matrix has spectral radius one, then it is unitary and its numerical range is a regular n-sided polygon. Consequently, two n-by-n reducible companion matrices have the same numerical range and one of them has spectral radius one, then they must equal to each other. Therefore, we will restrict our attention to the reducible companion matrices with spectral radius larger than one. Note that if C is an n-by-n reducible companion matrix, by Proposition 1.1, the following are equivalent: (a) C is not unitary; (b) $r(C) > 1$; (c) C is unitarily equivalent to a direct sum $A \oplus B$, where $A \in M_k$ ($1 \leqslant k \leqslant n-1$) with $r(A) < 1$ and $B \in M_{n-k}$ with $r(B) = 1/r(A) > 1$. In this case, we call C the *nonunitary reducible companion matrix*. In Section 2, we first prove that if C is a nonunitary companion matrix, then C is unitarily equivalent to a direct sum $A \oplus B$ with $\text{rank}(I_k - A^*A) = \text{rank}(I_{n-k} - B^*B) = 1$. It is known that the numerical range $W(C)$ is equal to the convex hull of $W(A) \cup W(B)$. Therefore, we make a detailed study of the numerical ranges of n-by-n matrices T with $\text{rank}(I_n - T^*T) = 1$. We obtain that such matrix T is determined by its numerical range up to unitary equivalence, is irreducible and cyclic, and the boundary of its numerical range is an algebraic curve which contains no line segment. If, in addition, $r(T) > 1$, then the homogeneous polynomial $p_T(x, y, z) \equiv \det(x(T + T^*)/2 + y(T - T^*)/(2i) + zI_n)$ is irreducible. Finally, we use the preceding results to prove that two reducible companion matrices have the same numerical range, then they must equal to each other. On the other hand, we also give a sufficient and necessary condition in terms of eigenvalues for a reducible companion matrix has elliptic numerical range, more precisely, we show that if C is an n-by-n reducible companion matrix, then $W(C)$ is an elliptical disc if and only if C is unitarily equivalent to $A \oplus B$, where $A \in M_{n-2}, B \in M_2$ with $\sigma(B) = \{a\omega_1, a\omega_2\}, \omega_1^n = \omega_2^n = 1$, $\omega_1 \neq \omega_2$, and $|a| \geqslant \left(|\omega_1 + \omega_2| + \sqrt{|\omega_1 + \omega_2|^2 + 4(1 + 2\cos(\pi/n))}\right) \Big/ 2$.

2. Main results

We start with the properties of the direct summands of a nonunitary reducible companion matrix. For abbreviation, we write $d_T = \text{rank}(I_n - T^*T)$ for an n-by-n matrix T.

Theorem 2.1. *If C is an n-by-n nonunitary reducible companion matrix. Then C is unitarily equivalent to a direct sum $A \oplus B$ with $d_A = d_B = 1$.*

To prove Theorem 2.1, we need the following lemma.

Lemma 2.2. *Let C be an n-by-n matrix with no eigenvalue on the unit circle and A be a restriction of C. If $d_C = 1$, then $d_A = 1$.*

Proof. Since A is a restriction of C, there exists a unitary $U \in M_n$ such that

$$U^*CU = \begin{bmatrix} A & * \\ 0 & * \end{bmatrix},$$

where $A \in M_k$ for some k, $1 \leqslant k \leqslant n$. Moreover, a simple computation yields that

$$U^*(I_n - C^*C)U = I_n - (U^*CU)^*(U^*CU) = \begin{bmatrix} I_k - A^*A & * \\ * & * \end{bmatrix}.$$

Since C has no eigenvalue on the unit circle and $\sigma(A) \subseteq \sigma(C)$, hence A is not unitary and

$$0 < \text{rank}(I_k - A^*A) \leqslant \text{rank}(U^*(I_n - C^*C)U) = \text{rank}(I_n - C^*C) = 1,$$

which show that $\text{rank}(I_k - A^*A) = 1$, completing the proof. $\square$

Proof of Theorem 2.1. By Proposition 1.1, we may assume that $\sigma(A) = \{a\omega^{j_1}, \ldots, a\omega^{j_k}\}$ and $\sigma(B) = \{(1/\bar{a})\omega^{j_{k+1}}, \ldots, (1/\bar{a})\omega^{j_n}\}$, where $0 < |a| < 1$ and ω denotes the nth primitive root of 1. For each $i = 1, \ldots, k$, let $x_i = (1, aw^{j_i}, (aw^{j_i})^2, \ldots, (aw^{j_i})^{n-1})^T \in \mathbb{C}^n$ be the eigenvector of C corresponding to the eigenvalue aw^{j_i}. Let H be the subspace of $\mathbb{C}^n$ generated by $\{x_1, \ldots, x_k\}$. From the proof of [9, Theorem 1.1], A is the restriction $C|_H$ of C on H. Consider the n-by-n companion matrix

$$C_A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a^n & 0 & \ldots & 0 \end{bmatrix}.$$

Then $\sigma(C_A) = \{a\omega^j : 0 \leqslant j \leqslant n-1\}$. We can see that x_i is the common eigenvector of C and C_A corresponding to the common eigenvalue aw^{j_i} for each $i = 1, \ldots, k$. Moreover, for any vector $h \in H$, then $h = \sum_{i=1}^k c_i x_i$ for some scalars c_i. Since

$$Ch = C\left(\sum_{i=1}^k c_i x_i\right) = \sum_{i=1}^k c_i C x_i = \sum_{i=1}^k c_i a w^{j_i} x_i = \sum_{i=1}^k c_i C_A x_i = C_A\left(\sum_{i=1}^k c_i x_i\right) = C_A h,$$

hence $C|_H$ is equal to the restriction $C_A|_H$ of C_A on H. It follows that A is also a restriction of C_A. It is easily check that $\text{rank}(I_n - C_A^*C_A) = 1$, hence $d_A = 1$ follows from Lemma 2.2.

Next, consider the n-by-n companion matrix

$$C_B = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1/\bar{a}^n & 0 & \dots & 0 \end{bmatrix}.$$

Then $\sigma(C_B) = \{(1/\bar{a})\omega^j : 0 \leqslant j \leqslant n-1\}$ and $d_{C_B} = 1$. For each $i = k+1, \dots, n$, let $y_i = (1, (1/\bar{a})w^{j_i}, ((1/\bar{a})w^{j_i})^2, \dots, ((1/\bar{a})w^{j_i})^{n-1})^T \in \mathbb{C}^n$, then y_i is the common eigenvector of C and C_B corresponding to the common eigenvalue $(1/\bar{a})w^{j_i}$. Let N be the subspace of $\mathbb{C}^n$ generated by $\{y_{k+1}, \dots, y_n\}$. From the proof of [9, Theorem 1.1], B is the restriction $C|_N$ of C on N. By a similar argument, $C|_N$ is equal to the restriction $C_B|_N$ of C_B on N, it follows that B is also a restriction of C_B. Since $d_{C_B} = 1$, hence $d_B = 1$ by Lemma 2.2, and we complete the proof. □

An n-by-n complex matrix A is said to be of *class* S_n if (i) the eigenvalues of A are all in the open unit disc $\mathbb{D}$, and (ii) $d_A = 1$. In recent years, properties of the numerical ranges of S_n-matrices have been intensely studied (cf. [4,6,7,13,14]). Among other things, it was obtained that the boundary of the numerical range $W(A)$ of an S_n-matrix A has the $(n+1)$-Poncelet property. This means that there are infinitely many $(n+1)$-gons interscribing between the unit circle $\partial\mathbb{D}$ and the boundary $\partial W(A)$ or, put more precisely, for any point a on $\partial\mathbb{D}$ there is a (unique) $(n+1)$-gon with a as one of its vertices such that all its $n+1$ vertices are in $\partial\mathbb{D}$ and all its $n+1$ edges are tangent to $\partial W(A)$ (cf. [4, Theorem 2.1] or [13, Theorem 1]).

If an S_n-matrix A is invertible, then

$$d_{A^{-1}} = \operatorname{rank}(I_n - (A^{-1})^*(A^{-1})) = \operatorname{rank}((A^{-1})^*(A^*A - I_n)(A^{-1})) = 1,$$

and all eigenvalues of A^{-1} have modulus greater than one. Recall that an n-by-n complex matrix B is said to be of *class* S_n^{-1} if (i) all eigenvalues of B have modulus greater than one, and (ii) $d_B = 1$. It is easily seen that if B is in S_n^{-1}, then B^* and $e^{i\theta}B$ are also in S_n^{-1} for all $\theta \in \mathbb{R}$. On the other hand, by Proposition 1.1 and Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let C be an n-by-n nonunitary reducible companion matrix. Then C is unitarily equivalent to a direct sum $A \oplus B$ with $A \in S_k$ and $B \in S_{n-k}^{-1}$, $1 \leqslant k \leqslant n-1$.*

In [6], the authors give a matrix representation for operators in S_n. Here we also give a matrix representation for operators in S_n^{-1}. Its proof is essentially the same as the one for [6, Corollary 1.3], hence we omit the proof.

Theorem 2.4. *An operator is in S_n^{-1} if and only if it has the upper triangular matrix representation $[t_{ij}]_{i,j=1}^n$, where $|t_{ii}| > 1$ for all i and $t_{ij} = s_{ij}(|t_{ii}|^2 - 1)^{1/2}(|t_{jj}|^2 - 1)^{1/2}$ for $i < j$ with*

$$s_{ij} = \begin{cases} \prod_{k=i+1}^{j-1} \bar{t}_{kk} & \text{if } j > i+1, \\ 1 & \text{if } j = i+1. \end{cases}$$

Notice that in the preceding matrix representation for operator B in S_n^{-1}, the entries t_{ij} are all determined, up to moduli, by the diagonal terms t_{ii}, which are the eigenvalues of B. This is not surprising since in the representation of B as the inverse of an invertible S_n-matrix, which is determined by its eigenvalues. Hence if B_1 and B_2 are in S_n^{-1}, then B_1 is unitarily equivalent to B_2 if and only if $\sigma(B_1) = \sigma(B_2)$ (counting multiplicities).

We now apply Theorem 2.4 to study the numerical ranges of S_n^{-1}-matrices. For a matrix $T \in M_n$, $\operatorname{Re} T = (T + T^*)/2$ and $\operatorname{Im} T = (T - T^*)/(2i)$ are the *real* and *imaginary* parts of T, respectively, and $\partial W(T)$ is the boundary of the numerical range of T.

Theorem 2.5. *Let B be an S_n^{-1}-matrix.*

(1) *The maximal eigenvalue of $\operatorname{Re}(e^{i\theta}B)$ is simple for all $\theta \in \mathbb{R}$.*
(2) *Let M be a proper invariant subspace for B, then $W(B|_M) \cap \partial W(B) = \emptyset$.*
(3) *If x is a unit vector in $\mathbb{C}^n$ for which $\langle Bx, x\rangle \in \partial W(B)$, then x is a cyclic vector of B.*
(4) *$\partial W(B)$ contains no line segment.*
(5) *For any point λ in $\partial W(B)$, the set $\{y \in \mathbb{C}^n : \langle By, y\rangle = \lambda \|y\|^2\}$ is a vector space of dimension one.*
(6) *B is irreducible.*
(7) *$\partial W(B)$ is a differentiable curve.*
(8) *$\partial W(B)$ lies on the real part of an irreducible algebraic curve of degree $m \geqslant 2$.*

Notice that if A is an S_n-matrix, then the preceding properties (1)–(8) hold (cf. [4,5]). For the proof of Theorem 2.5, we start with the following lemma.

Lemma 2.6. *Let B be an S_n^{-1}-matrix represented as in Theorem 2.4 and r be the maximal eigenvalue of $\operatorname{Re} B$. If $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is an eigenvector of $\operatorname{Re} B$ corresponding to the eigenvalue r, then $x_n \neq 0$.*

Proof. The proof is by induction on n.

We first check the case $n = 2$. That is,

$$\begin{bmatrix} r - \operatorname{Re} t_{11} & -\frac{t_{12}}{2} \\ -\frac{\bar{t}_{12}}{2} & r - \operatorname{Re} t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Suppose, contrary to our claim, that $x_2 = 0$. Then $x_1 \neq 0$ and $0 = x_1\bar{t}_{12}$. This contradicts the fact that $t_{12} = (|t_{11}|^2 - 1)^{1/2}(|t_{22}|^2 - 1)^{1/2} \neq 0$.

Assume the assertion of the lemma holds for $n - 1$, we will prove it for n. On the contrary, suppose that $x_n = 0$. It implies that $(rI_{n-1} - \operatorname{Re} B_{n-1})y = 0$, where $y = (x_1, \ldots, x_{n-1})^T \in \mathbb{C}^{n-1}$ and $B_{n-1} \in M_{n-1}$ is the principal submatrix of B. It follows that y is the eigenvector of $\operatorname{Re} B_{n-1}$ corresponding to the maximal eigenvalue r. Note that B_{n-1} is in S_{n-1}^{-1} from Theorem 2.4. Therefore, by the hypothesis of induction, we have $x_{n-1} \neq 0$. On the other hand, let us compute the nth and $(n-1)$th entries of $(rI_n - \operatorname{Re} B)x$, we have

$$-\frac{1}{2}\sum_{j=1}^{n-2} x_j\bar{t}_{j,n} - \frac{1}{2}x_{n-1}\bar{t}_{n-1,n} = 0 \tag{1}$$

and

$$-\frac{1}{2}\sum_{j=1}^{n-2} x_j\bar{t}_{j,n-1} + x_{n-1}(r - \operatorname{Re} t_{n-1,n-1}) = 0. \tag{2}$$

By Theorem 2.4, we have

$$t_{j,n} = t_{j,n-1} \cdot \frac{\sqrt{|t_{n,n}|^2 - 1}}{\sqrt{|t_{n-1,n-1}|^2 - 1}} \cdot \bar{t}_{n-1,n-1}, \tag{3}$$

for $1 \leqslant j \leqslant n-2$. Substituting (3) into (1) yields

$$0 = -\frac{\sqrt{|t_{n,n}|^2 - 1}}{2\sqrt{|t_{n-1,n-1}|^2 - 1}} \cdot t_{n-1,n-1}\sum_{j=1}^{n-2} x_j\bar{t}_{j,n-1} - \frac{1}{2}x_{n-1}\bar{t}_{n-1,n} \tag{4}$$

Substituting (2) into (4) we obtain

$$0 = -\frac{\sqrt{|t_{n,n}|^2 - 1}}{\sqrt{|t_{n-1,n-1}|^2 - 1}} \cdot t_{n-1,n-1} x_{n-1} (r - \operatorname{Re} t_{n-1,n-1}) - \frac{1}{2} x_{n-1} \bar{t}_{n-1,n} \quad (5)$$

Since $x_{n-1} \neq 0$ and $\bar{t}_{n-1,n} = (|t_{n,n}|^2 - 1)^{1/2}(|t_{n-1,n-1}|^2 - 1)^{1/2} > 0$, we can rewrite (5) as

$$|t_{n-1,n-1}|^2 - 1 = -2t_{n-1,n-1}(r - \operatorname{Re} t_{n-1,n-1}). \quad (6)$$

Note that $|t_{n-1,n-1}| > 1$ and $r \geqslant \operatorname{Re} t_{n-1,n-1}$, Eq. (6) becomes

$$|t_{n-1,n-1}|^2 - 1 = 2|t_{n-1,n-1}|(r - \operatorname{Re} t_{n-1,n-1}). \quad (7)$$

Next, we will rearrange the diagonal entries of B to obtain that

$$|t_{n-2,n-2}|^2 - 1 = 2|t_{n-2,n-2}|(r - \operatorname{Re} t_{n-2,n-2}). \quad (8)$$

Indeed, since B_{n-1} is in S_{n-1}^{-1}, by Theorem 2.4, there exists a unitary matrix $V \in M_{n-1}$ such that $V^* B_{n-1} V = [t'_{ij}]_{i,j=1}^{n-1}$ which is represented as in Theorem 2.4 and $t'_{n-2,n-2} = t_{n-1,n-1}$, $t'_{n-1,n-1} = t_{n-2,n-2}$ and $t'_{i,i} = t_{i,i}$ for $i = 1, \ldots, n-3$. Let $U = V \oplus [1] \in M_n$, $B' = U^* B U$ and $x' = U^* x$. Then $(rI_n - \operatorname{Re} B')x' = 0$ and the nth entry of x' is zero. As was proved above, we can obtain that

$$|t'_{n-1,n-1}|^2 - 1 = 2|t'_{n-1,n-1}|(r - \operatorname{Re} t'_{n-1,n-1}).$$

Since $t'_{n-1,n-1} = t_{n-2,n-2}$, it follows that

$$|t_{n-2,n-2}|^2 - 1 = 2|t_{n-2,n-2}|(r - \operatorname{Re} t_{n-2,n-2}).$$

Now, note that r is the maximal eigenvalue of $\operatorname{Re} B$, thus, the submatrix

$$\begin{bmatrix} r - \operatorname{Re} t_{n-2,n-2} & -\frac{t_{n-2,n-1}}{2} \\ -\frac{\bar{t}_{n-2,n-1}}{2} & r - \operatorname{Re} t_{n-1,n-1} \end{bmatrix}$$

of $(rI_n - \operatorname{Re} B)$ is positive semidefinite. Moreover, we have

$$\begin{aligned} 0 &\leqslant \det \begin{bmatrix} r - \operatorname{Re} t_{n-2,n-2} & -\frac{t_{n-2,n-1}}{2} \\ -\frac{\bar{t}_{n-2,n-1}}{2} & r - \operatorname{Re} t_{n-1,n-1} \end{bmatrix} \\ &= (r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}) - \frac{|t_{n-2,n-1}|^2}{4}. \end{aligned}$$

It follows that

$$\frac{|t_{n-2,n-1}|^2}{4} \leqslant (r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}) \quad (9)$$

and $(r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}) > 0$, since $|t_{n-2,n-1}|^2 = (|t_{n-2,n-2}|^2 - 1)(|t_{n-1,n-1}|^2 - 1) > 0$. On the other hand, substituting (7) and (8) into the (9), we have

$$\begin{aligned} (r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}) &\geqslant \frac{|t_{n-2,n-1}|^2}{4} \\ &= \frac{(|t_{n-2,n-2}|^2 - 1)(|t_{n-1,n-1}|^2 - 1)}{4} \\ &= |t_{n-2,n-2}| \cdot |t_{n-1,n-1}|(r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}). \end{aligned}$$

Since $(r - \operatorname{Re} t_{n-2,n-2})(r - \operatorname{Re} t_{n-1,n-1}) > 0$, the above inequality implies that $1 \geqslant |t_{n-2,n-2}| \cdot |t_{n-1,n-1}|$. This contradicts the fact that $|t_{ii}| > 1$ for all $i = 1, \ldots, n$. Therefore, we conclude that x_n is nonzero as asserted. □

We are now ready for the

Proof of Theorem 2.5. (1) Since $e^{i\theta}B$ is also in S_n^{-1} for all $\theta \in \mathbb{R}$, we may assume that $\theta = 0$ and B is represented as in Theorem 2.4. Let r be the maximal eigenvalue of $\operatorname{Re} B$ and $K = \ker(rI_n - \operatorname{Re} B)$. If $\dim K \geqslant 2$, then there exists a nonzero vector $x \in K$ such that the nth entry of x is zero. This contradicts to Lemma 2.6. Hence $\dim K = 1$ and r is a simple eigenvalue of the hermitian matrix $\operatorname{Re} B$.

(2) On the contrary, suppose that $W(B|_M) \cap \partial W(B)$ is nonempty, say, $\lambda \in W(B|_M) \cap \partial W(B)$. Through a rotation, we may assume without loss of generality that

$$\operatorname{Re} \lambda = \max \operatorname{Re} W(B) = \max \operatorname{Re} W(B|_M) = \max \sigma(\operatorname{Re}(B|_M)).$$

We can choose a suitable orthonormal basis such that B is represented as in Theorem 2.4 and $B|_M$ is unitary equivalent to the k-by-k principal submatrix B_k of B, where $k = \dim M < n$. Let $y \in \mathbb{C}^k$ be a unit eigenvector of $\operatorname{Re} B_k$ corresponding to the maximal eigenvalue $\operatorname{Re} \lambda$, and $x = \begin{bmatrix} y \\ \vec{0} \end{bmatrix} \in \mathbb{C}^n$. Then $\|x\| = 1$ and

$$\operatorname{Re} \lambda = \langle (\operatorname{Re} B_k)y, y\rangle = \langle (\operatorname{Re} B)x, x\rangle = \operatorname{Re} \langle Bx, x\rangle \leqslant \operatorname{Re} \lambda.$$

It follows that $\langle (\operatorname{Re} B)x, x\rangle = \operatorname{Re} \lambda = \max \operatorname{Re} W(B)$, consequently, x is an eigenvector of $\operatorname{Re} B$ corresponding to the maximal eigenvalue $\operatorname{Re} \lambda$. This contradicts the assertion of Lemma 2.6. Hence $W(B|_M) \cap \partial W(B) = \emptyset$.

(3) The proof is completed by showing that the space $M = \operatorname{span}\{x, Bx, \ldots, B^{n-1}x\}$ is equal to $\mathbb{C}^n$. If $M \subsetneqq \mathbb{C}^n$, then M is a proper invariant subspace for B, and $\langle (B|_M)x, x\rangle = \langle Bx, x\rangle \in W(B|_M) \cap \partial W(B)$, contrary to (2). Hence $M = \mathbb{C}^n$ as asserted.

(4) Assume that $[a, b]$ is a line segment in $\partial W(B)$. Through a rotation, we may assume without loss of generality that $\operatorname{Re} a = \operatorname{Re} b = \max \operatorname{Re} W(B) = \max \sigma(\operatorname{Re} B)$. Then there exist unit vectors x and y such that $\langle Bx, x\rangle = a$ and $\langle By, y\rangle = b$. Since $a \neq b$, x and y are linearly independent. Moreover, x and y are eigenvectors of $\operatorname{Re} B$ corresponding to the maximal eigenvalue. This leads to a contradiction since the maximal eigenvalue of $\operatorname{Re} B$ is simple. We conclude that $\partial W(B)$ contains no line segment.

(5) Through a rotation, we may assume without loss of generality that $\operatorname{Re} \lambda = \max \operatorname{Re} W(B) = \max \sigma(\operatorname{Re} B)$. Then (5) is an easy consequence of (1), because the set $\{y \in \mathbb{C}^n : \langle By, y\rangle = \lambda\|y\|^2\}$ is equal to the eigenspace of $\operatorname{Re} B$ corresponding to the maximal eigenvalue $\operatorname{Re} \lambda$.

(6) Assume that B is reducible, that is, B is unitarily equivalent to a direct sum $B_1 \oplus B_2$. From (2) we have $W(B_1) \cap \partial W(B) = \emptyset$ and $W(B_2) \cap \partial W(B) = \emptyset$. By (4), the boundary of $W(B)$ consists of extreme points of $W(B)$, hence that $\operatorname{conv}(W(B_1) \cup W(B_2)) \subsetneqq W(B)$, a contradiction. We conclude that B is irreducible.

(7) The differentiability of $\partial W(B)$ follows easily from (6) since any nondifferentiable point λ of $\partial W(B)$ is a reducing eigenvalue of A (i.e., $By = \lambda y$ and $B^*y = \bar{\lambda} y$ for some nonzero vector y) (cf. [10, Theorems 1.6.3 and 1.6.6]).

(8) After translation, we may assume that B is an n-by-n non-Hermitian matrix such that the numerical range $W(B)$ contains 0 as an interior point. Under this assumption, the connected component $\triangle$ of

$$\{(x, y) \in \mathbb{R}^2 : p_B(x, y, z) \neq 0\}$$

containing $(0, 0)$ is a bounded convex set. From (1) and (7), we infer that the boundary of the compact convex set $\triangle$ has no sharp point and lies on the real part of an irreducible component $q(x, y, z) = 0$ of the curve $p_B(x, y, z) = 0$ with $\deg(q) \geqslant 2$. That is, $\partial W(B)$ lies on the real part of the dual curve of $q(x, y, z) = 0$. This completes the proof. □

It is known that an S_n-matrix is determined by its numerical range, namely, matrices A_1 and A_2 in S_n are unitarily equivalent if and only if $W(A_1) = W(A_2)$ (cf. [4, Theorem 3.2]). We have an analogous result for S_n^{-1}-matrices.

Theorem 2.7. *The following statements are equivalent for matrices $B_1 \in S_m^{-1}$ and $B_2 \in S_n^{-1}$:*

(1) $n = m$ *and* B_1 *is unitarily equivalent to* B_2;
(2) $W(B_1) = W(B_2)$;
(3) $p_{B_1}(x, y, z) = p_{B_2}(x, y, z)$, *where* $p_{B_1}(x, y, z) = \det(x\mathrm{Re}\,B_1 + y\mathrm{Im}\,B_1 + zI_m)$ *and* $p_{B_2}(x, y, z) = \det(x\mathrm{Re}\,B_2 + y\mathrm{Im}\,B_2 + zI_n)$;
(4) $\sigma(B_1) = \sigma(B_2)$ (*counting multiplicities*).

Note that p_{B_j} and $W(B_j)$ are related by a result of Kippenhahn [11]: the numerical range of an n-by-n matrix T equals the convex hull of the real points $(u/w, v/w)$ of the dual $\{[u, v, w] \in \mathbb{C}P^2 : ux + vy + wz = 0$ is a tangent line of $p_T(x, y, z) = 0\}$ of the curve $p_T(x, y, z) = 0$ in the complex projective plane $\mathbb{C}P^2$, where p_T is the degree-n homogeneous polynomial in x, y and z given by

$$p_T(x, y, z) = \det(x\mathrm{Re}\,T + y\mathrm{Im}\,T + zI_n).$$

Let us recall some other known properties of curves in the complex projective plane $\mathbb{C}P^2$. Let $p(x, y, z)$ be a degree-n homogeneous polynomial and Γ be the dual curve of $p(x, y, z) = 0$. If $ax + by + z$ is not a factor of p, Bézout's theorem [12, Theorem 3.1] implies that the intersection of the curve $p = 0$ and the line $ax + by + z = 0$ consists of exactly n points (counting multiplicities). By duality, there are exactly n tangent lines of Γ passing through the point (a, b) for any point $(a, b) \notin \Gamma$ (counting multiple lines). Among other things, a point $\lambda = a + ib$, a, b real, is called a *real focus* of Γ if $p(1, \pm i, -(a \pm ib)) = 0$ is satisfied. Consequently, the eigenvalues of an n-by-n matrix T are exactly the real foci of the dual curve of $p_T = 0$ (cf. [11, Theorem 11]).

To prove the preceding theorem, we need the following lemmas.

Lemma 2.8. *Let T be an n-by-n matrix and $q(x, y, z)$ be a factor of $p_T(x, y, z)$. If Γ is the dual curve of $q(x, y, z) = 0$, then the real foci of Γ are in the convex hull of the real points of Γ.*

Proof. Let d be the degree of the homogeneous polynomial $q(x, y, z)$. If λ is a real focus of Γ, then $q(1, i, -\lambda) = 0$, that is, the curve Γ has the complex tangent line $x + iy - \lambda = 0$ through the focus λ. Therefore, it is impossible that there are d real tangent lines of Γ passing through the focus λ. We complete the proof by showing that if $z_0 \in \mathbb{C}$ is not in the convex hull of the real points of Γ, then there are d real tangent lines of Γ passing through the point z_0. Indeed, for each $\theta \in \mathbb{R}$, the equation $p_T(\cos\theta, \sin\theta, -z) = \det(\mathrm{Re}(e^{-i\theta}T) - zI_n) = 0$ has n real roots. Thus $q(\cos\theta, \sin\theta, -z) = 0$ has d real roots. Let $\lambda_1(\theta) \geqslant \lambda_2(\theta) \geqslant \cdots \geqslant \lambda_d(\theta)$ be the roots of the equation $q(\cos\theta, \sin\theta, -z) = 0$. Then $\lambda_j(\theta)$ is continuous in θ for each $j = 1, 2, \ldots, d$, because q is a polynomial. Moreover, the convex hull of the real points of Γ is equal to the intersection

$$\bigcap_{\theta \in [0, 2\pi)} \{z \in \mathbb{C} : \mathrm{Re}(e^{-i\theta}z) \leqslant \lambda_1(\theta)\}.$$

Since z_0 is not in the convex hull of the real points of Γ, then $\mathrm{Re}(e^{-i\theta_0}z_0) > \lambda_1(\theta_0)$ for some θ_0. By symmetry, we also have

$$\mathrm{Re}(e^{-i(\pi+\theta_0)}z_0) = -\mathrm{Re}(e^{-i\theta_0}z_0) < -\lambda_1(\theta) = \lambda_d(\pi + \theta_0).$$

For each $j = 1, 2, \ldots, d$, let $f_j(\theta) = \mathrm{Re}(e^{-i\theta}z_0) - \lambda_j(\theta)$ for $\theta \in \mathbb{R}$, then

$$f_j(\theta_0) \geqslant \mathrm{Re}(e^{-i\theta_0}z_0) - \lambda_1(\theta_0) > 0$$

and

$$f_j(\pi + \theta_0) \leqslant \mathrm{Re}(e^{-i(\pi+\theta_0)}z_0) - \lambda_d(\pi + \theta_0) < 0.$$

Since f_j is continuous on $\mathbb{R}$, by the Intermediate Value Theorem, there exists a $\theta_j \in (\theta_0, \pi + \theta_0)$ such that $f_j(\theta_j) = 0$ or $\mathrm{Re}(e^{-i\theta_j}z_0) = \lambda_j(\theta_j)$. This implies that $q(\cos\theta_j, \sin\theta_j, -\mathrm{Re}(e^{-i\theta_j}z_0)) = 0$, that is, Γ has the real tangent lines $(\cos\theta_j)x + (\sin\theta_j)y - \mathrm{Re}(e^{-i\theta_j}z_0) = 0$ for $j = 1, 2, \ldots, d$. On the other hand, it is easily seen that the point z_0 lies on these d real tangent lines, because $(\cos\theta_j)\mathrm{Re}\,z_0 + (\sin\theta_j)\mathrm{Im}\,z_0 - \mathrm{Re}(e^{-i\theta_j}z_0) = 0$ for all $j = 1, 2, \ldots, d$. This completes the proof. $\square$

Remark. Let $p(x, y, z)$ be a real homogeneous polynomial of degree n. Recall that $p(x, y, z)$ is *hyperbolic with respect to* $(0, 0, 1)$ if any line containing $(0, 0, 1)$ has n real points of intersection with the curve $p = 0$ (counting multiplicities). That is, the equation $p(\cos\theta, \sin\theta, z) = 0$ has n real roots for all $\theta \in \mathbb{R}$. In [1], it is proved that a factor $q(x, y, z)$ of a form $p(x, y, z)$ hyperbolic with respect to $(0, 0, 1)$ is also hyperbolic with respect to $(0, 0, 1)$. On the other hand, $p_T(x, y, z)$ is hyperbolic with respect to $(0, 0, 1)$ for any n-by-n matrix T. Notice that the essential part of the proof of Lemma 2.8 is independent of the matrix T. The proof essentially only depend on the hyperbolicity of q. Therefore, the core of Lemma 2.8 is formulated in the following.

Lemma 2.8′. *Let $q(x, y, z)$ be a real form hyperbolic with respect to $(0, 0, 1)$. If Γ is the dual curve of $q(x, y, z) = 0$, then the real foci of Γ are in the convex hull of the real points of Γ.*

In [3], Fiedler gave a conjecture: if $p(x, y, z)$ is a real homogeneous polynomial of degree n which is hyperbolic with respect to $(0, 0, 1)$, then there exist Hermitian matrices H_1 and H_2 such that

$$p(x, y, z) = \det(xH_1 + yH_2 + zI_n).$$

We remark that if Fiedler's conjecture is true, then Lemma 2.8' follows easily from Kippenhahn's result. Indeed, by Fiedler's conjecture, we may assume that

$$q(x, y, z) = \det(xK_1 + yK_2 + zI_d)$$

for some Hermitian matrices K_1, K_2. Then Kippenhahn's theorems (cf. [11, Theorems 10 and 11]) imply that

$$\lambda \in \sigma(K_1 + iK_2) \subseteq W(K_1 + iK_2) = \mathrm{conv}(\Gamma_{re})$$

for all real foci λ of Γ. However, the proof of Lemma 2.8 is independent of the validity of Fiedler's conjecture for q.

Lemma 2.9. *Let B be an S_n^{-1}-matrix and $\lambda_1(\theta) \geqslant \lambda_2(\theta) \geqslant \cdots \geqslant \lambda_n(\theta)$ be the eigenvalues of* $\mathrm{Re}(e^{-i\theta}B)$ *for $\theta \in \mathbb{R}$.*

(1) $|\lambda_2(\theta)| \leqslant 1$ *for all* $\theta \in \mathbb{R}$.
(2) *The homogeneous polynomial $p_B(x, y, z)$ is irreducible.*

Proof. (1) Let $K = \ker(I_n - B^*B)$, then $\dim K = n - d_B = n - 1$ since B is an S_n^{-1}-matrix. Let $B' = P_K B|_K$ the compression of B on K, where P_K is the (orthogonal) projection from $\mathbb{C}^n$ onto K. For any vector $x \in K$,

$$\|(P_K B|_K)x\|^2 \leqslant \|Bx\|^2 = \langle Bx, Bx\rangle = \langle B^*Bx, x\rangle = \langle x, x\rangle = \|x\|^2,$$

showing that B' is a contraction. Consequently, $W(B') \subseteq \overline{\mathbb{D}}$. Let $\rho(\theta)$ be the maximal eigenvalue of $\mathrm{Re}(e^{-i\theta}B')$ for $\theta \in \mathbb{R}$, it follows that $|\rho(\theta)| \leqslant 1$ for all θ. Since B' is an $(n-1)$-dimensional compression of B, by the interlacing inequality [2, Corollary III.1.5], we have

$$1 \geqslant \rho(\theta) \geqslant \lambda_2(\theta) \geqslant \lambda_{n-1}(\theta) = -\lambda_2(\theta + \pi) \geqslant -\rho(\theta + \pi) \geqslant -1.$$

Thus $|\lambda_2(\theta)| \leqslant 1$ for all $\theta \in \mathbb{R}$ as asserted.

(2) Assume that $p_B(x, y, z) = \prod_{j=1}^m p_j(x, y, z)$, where the p_j's are irreducible homogeneous polynomials in x, y and z with real coefficients. By Theorem 2.5 (8), we may assume, for convenience, that the boundary of $W(B)$ is generated by $p_1(x, y, z) = 0$, that is, $p_1(\cos\theta, \sin\theta, -\lambda_1(\theta)) = 0$ for all θ, and let $q = \prod_{j=2}^m p_j$. For each $\theta \in \mathbb{R}$, $\gamma(\theta)$ denotes the maximal root of the equation $q(\cos\theta, \sin\theta, -z) = 0$, note that $\gamma(\theta)$ is also a root of the equation $p_B(\cos\theta, \sin\theta, -z) = 0$, thus $\gamma(\theta) \leqslant \lambda_2(\theta) \leqslant 1$ by (1). On the other hand, if Γ is the dual curve of $q(x, y, z) = 0$, then the convex hull of the real points of Γ is equal to the intersection

$$\bigcap_{\theta\in[0,2\pi)}\{z\in\mathbb{C}:\mathrm{Re}(e^{-i\theta}z)\leqslant\gamma(\theta)\}$$

and contained in the closed unit disc. Moreover, Lemma 2.8 follows that all real foci of Γ are in the closed unit disc, which means that B has some eigenvalue with modulus less than or equal to one, since the real foci of Γ are also the eigenvalues of B. But B is an S_n^{-1}-matrix, that is, every eigenvalue of B has modulus greater than one, we reach a contradiction. This proves the irreducibility of p_B. □

We are now ready for the

Proof of Theorem 2.7. The implication (4) ⇒ (1) is an easy consequence of Theorem 2.4. The implication (2) ⇒ (3) follows from Lemma 2.9 (2) and [9, Corollary 2.4]. Since (1) ⇒ (2) is trivial, to complete the proof we need only show (3) ⇒ (4). If (3) holds, then m = degree of p_{B_1} = degreeof $p_{B_2} = n$ and $\det(B_1+zI_n)=\det(\mathrm{Re}\,B_1+i\mathrm{Im}\,B_1+zI_n)=p_{B_1}(1,i,z)=p_{B_2}(1,i,z)=\det(B_2+zI_n)$ for all z, which implies that B_1 and B_2 have the same eigenvalues (counting multiplicities). This completes the proof. □

We now restrict our attention to the numerical ranges of reducible companion matrices. Let C be an n-by-n reducible companion matrix with $r(C)>1$, then C is unitarily equivalent to $A\oplus B$, where $A\in S_k$ and $B\in S_{n-k}^{-1}$ with $r(B)>1$, $1\leqslant k\leqslant n-1$. It is clear that $W(C)=\mathrm{conv}(W(A)\cup W(B))$. But $r(B)>1$ and $W(A)$ is contained in the closed unit disc, it is impossible for $W(A)$ to contain $W(B)$. Therefore, the boundary of the numerical range of C must contain part of $\partial W(B)$ which lies outside the closed unit disc. Now, we first show that a reducible companion matrix is completely determined by its numerical range.

Theorem 2.10. *Let C_1 and C_2 be n-by-n reducible companion matrices. Then $W(C_1)=W(C_2)$ if and only if $C_1=C_2$.*

Proof. Assume that $W(C_1)=W(C_2)$, then C_1 and C_2 have some common eigenvalues from [9, Proposition 2.3]. If C_1 or C_2 is unitary, from Proposition 1.1, $W(C_1)=W(C_2)$ is a regular n-sided polygon, which implies $\sigma(C_1)=\sigma(C_2)$ or $C_1=C_2$ as asserted. Therefore, we may assume that C_1 and C_2 are nonunitary. By Corollary 2.3, we may assume that C_j is unitarily equivalent to $A_j\oplus B_j$, where $A_j\in S_{k_j}$ and $B_j\in S_{n-k_j}^{-1}$, $1\leqslant k_j\leqslant n-1$, for $j=1,2$. Since $\mathrm{conv}(W(A_1)\cup W(B_1))=W(C_1)=W(C_2)=\mathrm{conv}(W(A_2)\cup W(B_2))$ and $W(A_j)\not\supseteq W(B_j)$ for $j=1,2$, we deduce that $\partial W(B_1)\cap\partial W(B_2)$ contains some common arc. By Theorem 2.5 (8) and Bézout's theorem (cf. [12, Theorem 3.1]), the homogeneous polynomials p_{B_1} and p_{B_2} have a common factor. The irreducibility of p_{B_j} (cf. Lemma 2.9 (2)) implies that $p_{B_1}=p_{B_2}$. From Theorem 2.7, we have $n-k_1=n-k_2$ and $\sigma(B_1)=\sigma(B_2)$, hence $\sigma(A_1)=\sigma(A_2)$ by Proposition 1.1, which shows that $\sigma(C_1)=\sigma(C_2)$ or $C_1=C_2$, completing the proof. □

Next, we give some equivalent conditions for $W(C)=W(B)$. Recall that the *numerical radius* of an n-by-n matrix T is $w(T)=\max\{|z|:z\in W(T)\}$.

Theorem 2.11. *Let C be an n-by-n reducible companion matrix which is unitarily equivalent to $A\oplus B$, where $A\in M_k$ and $B\in M_{n-k}$ with $r(B)>1$, $1\leqslant k\leqslant n-1$. Then the following are equivalent:*

(1) $W(C)=W(B)$;
(2) $W(A)\subseteq W(B)$;
(3) $W(J_{n-1})\subseteq W(B)$;
(4) $W(A)\subseteq W(J_{n-1})$;
(5) $w(A)\leqslant\cos(\pi/n)$.

Proof. The implications (1) $\Leftrightarrow$ (2), (4) $\Leftrightarrow$ (5) and (1) $\Rightarrow$ (3) are trivial. We need only prove (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1).

We now check (4) $\Rightarrow$ (1). Assume that $W(A) \subseteq W(J_{n-1})$, then

$$W(C) = \text{conv}(W(A) \cup W(B)) \subseteq \text{conv}(W(J_{n-1}) \cup W(B)) \subseteq W(C).$$

It follows that $W(C) = \text{conv}(W(J_{n-1}) \cup W(B))$. If $W(J_{n-1}) \nsubseteq W(B)$, then $\partial W(J_{n-1}) \cap \partial W(C)$ contains a common arc. Since $W(J_{n-1})$ is a circular disc, [8, Theorem] implies that $0 \in \sigma(C)$, contradicting the invertibility of C (cf. Proposition 1.1). Therefore, we conclude that $W(J_{n-1}) \subseteq W(B)$ and $W(C) = \text{conv}(W(J_{n-1}) \cup W(B)) = W(B)$ as asserted.

Finally, we prove (3) $\Rightarrow$ (4). For every $\theta \in \mathbb{R}$, let

$$\lambda_A(\theta) = \max \sigma(\text{Re}(e^{i\theta}A))$$

and

$$\lambda_B(\theta) = \max \sigma(\text{Re}(e^{i\theta}B)).$$

If (3) holds, we have

$$\lambda_B(\theta) \geqslant \cos\frac{\pi}{n} \quad \text{for all } \theta \in \mathbb{R}.$$

On the contrary, suppose that $W(A) \nsubseteq W(J_{n-1})$, then there exists a real θ_0 such that $\lambda_A(\theta_0) > \cos(\pi/n)$. By the continuity of $\lambda_A(\theta)$ (cf. [2, Corollary III.2.6]), there is an $\varepsilon > 0$ such that

$$\lambda_A(\theta) > \cos\frac{\pi}{n} \quad \text{for all } \theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon).$$

Note that $\lambda_A(\theta)$ and $\lambda_B(\theta)$ are eigenvalues of $\text{Re}(e^{i\theta}C)$ for each $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ and $\text{Re}(e^{i\theta}J_{n-1})$ is an $(n-1)$-dimensional compression of $\text{Re}(e^{i\theta}C)$, by the interlacing inequality [2, Corollary III.1.5], we deduce that $\lambda_B(\theta) = \cos(\pi/n)$ for all $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, that is, $\partial W(B) \cap \partial W(J_{n-1})$ contains a common arc. It forces that $0 \in \sigma(B)$ by [8, Theorem]. This contradicts the fact that B is invertible (cf. Proposition 1.1). Hence we conclude that $W(A) \subseteq W(J_{n-1})$ as asserted. □

We end this paper by giving a test for a reducible companion matrix C to determine when $W(C)$ is an elliptic disc.

Theorem 2.12. *Let C be an n-by-n reducible companion matrix. Then $W(C)$ is an elliptical disc if and only if C is unitarily equivalent to $A \oplus B$, where $A \in M_{n-2}, B \in M_2$ with $\sigma(B) = \{a\omega_1, a\omega_2\}$, $\omega_1^n = \omega_2^n = 1$, $\omega_1 \neq \omega_2$, and*

$$|a| \geqslant \frac{|\omega_1 + \omega_2| + \sqrt{|\omega_1 + \omega_2|^2 + 4(1 + 2\cos(\pi/n))}}{2}. \tag{10}$$

Proof. Assume that $W(C)$ is an elliptic disc, obviously, C is not unitary. Thus C is unitarily equivalent to $A \oplus B$, where $A \in M_{n-k}, B \in M_k$ and $1 \leqslant k \leqslant n-1$. Since $\partial W(C) \cap \partial W(B)$ contains a common arc and $\partial W(B)$ is an algebraic curve (cf. Theorem 2.5 (8)), Bézout's theorem implies that p_B has a quadratic factor. By the irreducibility of p_B (cf. Lemma 2.9 (2)), B must be a 2-by-2 matrix and $W(B) = W(C)$. From Proposition 1.1, we may assume that $\sigma(B) = \{a\omega_1, a\omega_2\}$, where $\omega_1^n = \omega_2^n = 1$ and $\omega_1 \neq \omega_2$, then $W(B)$ is the elliptic disc having foci at $a\omega_1, a\omega_2$ and minor axis of length $|a|^2 - 1$ from Theorem 2.4. By Theorem 2.11, we have $W(J_{n-1}) \subseteq W(B)$. Since $|a\omega_1| = |a\omega_2|$, then $W(J_{n-1}) \subseteq W(B)$ if and only if

$$\cos\frac{\pi}{n} + \left|\frac{a\omega_1 + a\omega_2}{2}\right| \leqslant \frac{|a|^2 - 1}{2}.$$

Moreover, a direct computation shows that the above inequality is equivalent to (10). This proves our assertion.

Conversely, by our assumption, we have $|a| > 1$, consequently, $B \in S_2^{-1}$ by Corollary 2.3, and $W(B)$ is the elliptic disc having foci at $a\omega_1, a\omega_2$ and minor axis of length $|a|^2 - 1$ from Theorem 2.4. As was proved above, the inequality (10) implies that $W(J_{n-1}) \subseteq W(B)$. Our assertion follows from Theorem 2.11. We complete the proof. □

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