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Periodic Oscillations of Coefficients of Power Series That Satisfy Functional Equations

A. M. ODLYZKO

Bell Laboratories, Murray Hill, New Jersey 07974

It is shown that the coefficients a_n of the power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ which satisfies the functional equation

$$f(z) = z + f(z^2 + z^3) \tag{$*$}$$

display periodic oscillations; $a_n \sim (\phi^n/n)\, u(\log n)$ as $n \to \infty$, where $\phi = (1 + 5^{1/2})/2$ and $u(x)$ is a positive, nonconstant, continuous function which is periodic with period $\log(4 - \phi)$. Similar results are obtained for a wide class of power series that satisfy similar functional equations. Power series of these types are of interest in combinatorics and computer science since they often represent generating functions. For example, the nth coefficient of the power series satisfying ($*$) enumerates 2, 3-trees with n leaves.

1. Introduction

The basic problem motivating this paper is the enumeration of 2,3-trees, although much more general results will also be proved. A 2,3-tree (also known in the literature as a 3–2 tree and a 2–3 tree) is a rooted, oriented tree each of whose nonleaf nodes has either two or three successors, and all of whose root-to-leaf paths have the same length [13, Section 6.2.3]. Figure 1 presents some small 2,3-trees. 2,3-Trees were proposed by Hopcroft for use as data structures in situations where it is desirable to be able to insert and delete records in time that is logarithmic in the total number of records present. The purpose of this paper is to investigate a_n, the number of 2,3-trees with exactly n leaves. If we agree to regard the tree consisting of a single vertex as a 2,3-tree, then $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 2, \ldots$, as is shown in Fig. 1. In a recent paper, Miller *et al.* [16] showed that for $n \geqslant 2$,

$$a_n = \sum_{2k+3m=n} \binom{k+m}{k} a_{k+m}, \tag{1.1}$$

and deduced from this that there exist positive constants c_1 and c_2 such that

$$\frac{c_1}{n}\phi^n < a_n < \frac{c_2}{n}\phi^n, \tag{1.2}$$

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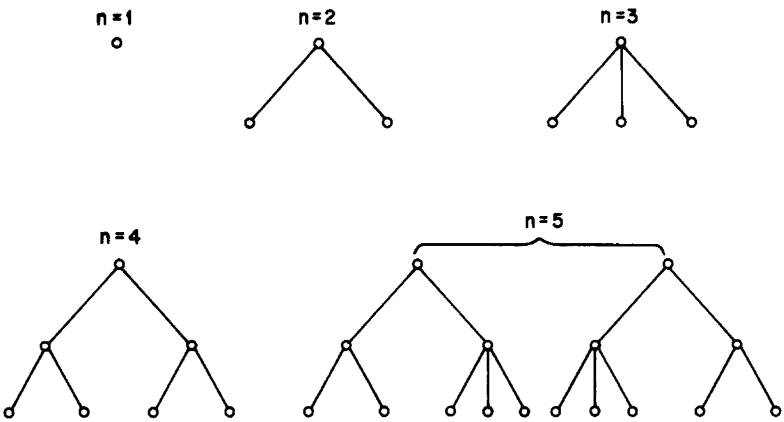

FIG. 1. All the 2,3-trees with $n \leqslant 5$ leaves.

where $\phi = (1 + 5^{1/2})/2 = 1.618...$ is the "golden ratio." The authors of that paper also asked whether there exists a constant c such that

$$a_n \sim \frac{c}{n} \phi^n \qquad \text{as} \qquad n \to \infty. \tag{1.3}$$

It was next noted by Knuth that if (1.3) holds, then the most likely value of c is $(\phi \log(4 - \phi))^{-1}$. (All logarithms in this paper are to the base e.) The basic idea is to consider the generating function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n. \tag{1.4}$$

The most important fact about $f(z)$ is that it satisfies the functional equation

$$f(z) = z + f(z^2 + z^3). \tag{1.5}$$

If we substitute $z^2 + z^3$ for z in (1.4) and use the binomial theorem, then (1.5) is readily seen to be equivalent to (1.1). (Such expansions are valid in $|z| < \phi^{-1}$, where (1.4) converges absolutely by (1.2).) Alternatively, this can be seen directly. Any single term z^n in (1.4) corresponds to some particular 2,3-tree T with n leaves. If d is the depth (distance from root to leaf) of such a tree, then $(z^2 + z^3)^n$ will correspond to the sum of z^m over all 2,3-trees with m leaves obtainable from T by attaching to each leaf either two or three descendents. Since any tree of depth $d + 1$ is obtainable in a unique way from a unique tree of depth d, we see that (1.5) holds as a relation on formal power series, and therefore by (1.2) for all z with $|z| < \phi^{-1}$.

Now that we have established (1.5), there are several ways, of varying degrees of rigor, of showing that if (1.3) holds then $c = (\phi \log(4-\phi))^{-1}$. Certainly (1.3) implies that

$$f(z) \sim -c \log(\phi^{-1} - z) \qquad \text{as} \quad z \to \phi^{-1},\ z \in (0, \phi^{-1}). \tag{1.6}$$

Let us suppose that in fact for some constant c',

$$f(z) = -c \log(\phi^{-1} - z) + c' + o(1) \qquad \text{as} \quad z \to \phi^{-1}. \tag{1.7}$$

Since for $z \in (0, \phi^{-1})$, if we let $z = \phi^{-1} - x$, then

$$z^2 + z^3 = \phi^{-1} - (4-\phi)\,x + O(x^2),$$

the functional equation (1.5) and (1.7) give us

$$\begin{aligned} &-c \log x + c' + o(1) \\ &\quad = \phi^{-1} - x - c\log(4-\phi) - c\log x + O(x^2) + c' + o(1) \end{aligned}$$

as $x \to 0$, which implies

$$c \log(4-\phi) = \phi^{-1}, \tag{1.8}$$

which is the desired conclusion.

We have shown that $c = (\phi \log(4-\phi))^{-1}$ if $f(z)$ satisfies (1.7). But (1.7) is not implied by (1.3). However, one can also prove that (1.8) gives the only possible value for c without any unproved assumptions. It can be shown by a relatively easy application of the Hardy–Littlewood–Karamata Tauberian Theorem [11] to simplified versions of the results that will be proved here that

$$\sum_{1 \leqslant n \leqslant x} a_n \phi^{-n} \sim \frac{\log x}{\phi \log(4-\phi)} \qquad \text{as} \quad x \to \infty,$$

which proves rigorously that if (1.3) holds, then c must satisfy (1.8). However, the main result of this paper is that (1.3) does not hold, and that the a_n exhibit asymptotic oscillations.

THEOREM 1. *There exists a positive nonconstant continuous function $u(x)$ which satisfies $u(x) = u(x + \log(4-\phi))$ for all real x such that*

$$a_n \sim \frac{\phi^n}{n} u(\log n) \qquad \textit{as} \quad n \to \infty.$$

The average value of $u(x)$ is $(\phi \log(4-\phi))^{-1}$.

The oscillations of $u(x)$ are demonstrted in Table I. Unfortunately we do not obtain any good expansions for $u(x)$, but numerical evidence indicates

TABLE I

Periodic Oscillations of a_n, the Number of 2,3-Trees with n Leaves

n	$a_n \cdot n \cdot \phi^{-n}$
250	0.6825
300	0.6357
350	0.7473
400	0.8064
450	0.7379
500	0.7036
550	0.7006
595	0.6827

Note. Note that $250 \cdot (4-\phi) = 595.49$.... Numbers are accurate to within one unit in the last decimal.

that $0.63 < u(x) < 0.81$ for all real x, while $(\phi \log(4-\phi))^{-1} = 0.712$.... Our results do show, however, that $u(z)$ is actually analytic in a strip $|\mathrm{Im}(z)| < c_3$, and so the Fourier coefficients of $u(x)$ go to 0 exponentially fast.

There are other combinatorial enumeration problems which involve functional equations similar to (1.5). For examples, B-trees [13] are another type of data structures that have been considered in computer science. A B-tree of order m is defined similarly to a 2,3-tree, except that every nonleaf node has between $[(m+1)/2]$ and m successors. (Thus B-trees of order 3 are just 2,3-trees.) The enumerator $f(z)$ of the number of B-trees with a given number of leaves satisfies the functional equation

$$f(z) = z + f(z^{[(m+1)/2]} + \cdots + z^m).$$

Our method easily generalizes to cover many equations od this sort. We will prove the following theorem.

THEOREM 2. *Let $P(z)$ and $Q(z)$ be nonzero polynomials with real, nonnegative coefficients, which satisfy $P(0) = Q(0) = Q'(0) = 0$. Write*

$$Q(z) = \sum_{k=0}^{K} q_k z^{e_k}, \qquad 2 \leqslant e_0 < e_1 < \cdots < e_K, \tag{1.9}$$

where $q_k > 0$ for $0 \leqslant k \leqslant K$, and assume that the greatest common divisor (g.c.d.) *of $0 = e_0 - e_0$, $e_1 - e_0$, $e_2 - e_0, \ldots, e_K - e_0$ is 1. (We adopt the convention that the* g.c.d. *of the single number* 0 *is infinite, so as to rule out* *the possibility that* $K=0$.) *Let* α *be the (unique) positive root of* $Q(z)=z$, *and set* $\beta=Q'(\alpha)$. *If*

$$f(z)=\sum_{n=1}^{\infty} a_n z^n \tag{1.10}$$

is a formal power series in z which satisfies

$$f(z)=P(z)+f(Q(z)), \tag{1.11}$$

then

$$a_n \sim \frac{\alpha^{-n}}{n} u(\log n) \qquad as \quad n\to\infty, \tag{1.12}$$

where $u(x)$ *is a nonconstant positive continuous function which is periodic with period* $\log\beta$. *Further, the average value of* $u(x)$ *is* $P(\alpha)(\log\beta)^{-1}$.

Theorem 1 follows immediately from Theorem 2, since one easily computes that for $P(z)=z$, $Q(z)=z^2+z^3$, we have $\alpha=\phi^{-1}$, $\beta=4-\phi$.

Theorem 2 can be generalized further. It is not essential, for example, that $P(z)$ and $Q(z)$ be polynomials nor that all their coefficients be nonnegative, nor that $Q'(0)=0$, provided that some additional assumptions are satisfied. However, the requirements that $P(0)=Q(0)=0$ are much harder to dispense with. Also, the greatest common divisor condition is crucial. For example, if $f(z)$, defined as in (1.10), satisfies

$$f(z)=z+f(z^2+z^5),$$

then $a_n=0$ for all $n\equiv 0 \pmod 3$, while for $n\not\equiv 0 \pmod 3$, the asymptotic behavior of a_n depends on whether $n\equiv 1 \pmod 3$ or $n\equiv 2 \pmod 3$, as can be shown by an extension of the methods of this paper.

Our proof of Theorem 2 will be based on an extensive study of the behavior of $f(z)$ as an analytic function, and this behavior will be determined with the help of the functional equation (1.11). It will be shown that $f(z)$ can be continued analytically to a region C_0 which is much larger than the disk $|z|<\alpha$, which is the region of convergence of the Taylor series (1.10) of $f(z)$. (Figures 2 and 3 show the boundary of C_0 for the case of 2,3-trees, $f(z)=z+f(z^2+z^3)$.) The region C_0 contains all of the disk $|z|\leqslant\alpha$ in its interior with the exception of $z=\alpha$ itself. (When the g.c.d. condition of Theorem 2 is not satisfied, there are other points on the circle $|z|=\alpha$ which are not in C_0, which accounts for the behavior of a_n when $f(z)=z+f(z^2+z^5)$.) The behavior of the a_n will be deduced from the properties

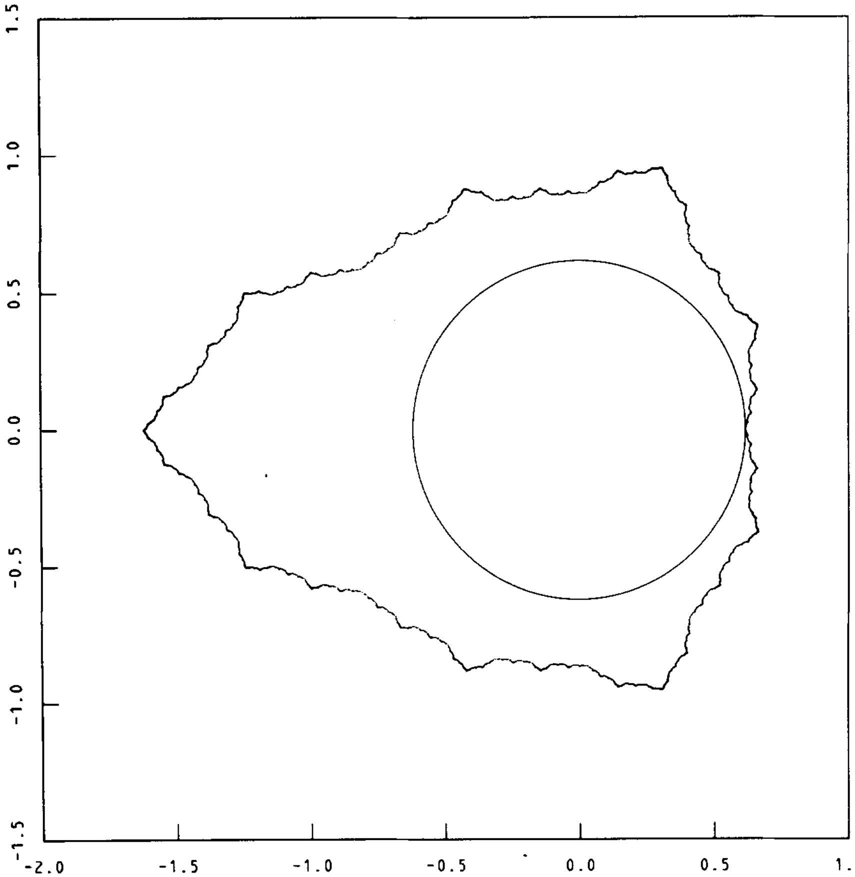

FIG. 2. Boundary of $C_0 = \{z\colon Q_n(z) \to 0 \text{ as } n \to \infty\}$ for $Q(z) = z^2 + z^3$, and of the circle $|z| = \alpha = \phi^{-1} = \phi - 1$.

of $f(z)$ in a neighborhood of $z = \alpha$. It will be shown that instead of something like (1.7), we actually have

$$f(z) = -c \log(\alpha - z) + w(\log(\alpha - z)) + O(|\alpha - z|) \tag{1.13}$$

as $z \to \alpha$, where $c = P(\alpha)(\log \beta)^{-1}$, and $w(z)$ is a nonconstant analytic function which is periodic with period $\log \beta$. Moreover, (1.13) holds as $z \to \alpha$ in a sector of the form $|\operatorname{Arg}(\alpha - z)| < \pi/2 + \varepsilon$ for some $\varepsilon > 0$. Theorem 2 will then be derived from (1.13).

The fact that $f(z)$ is analytic in a sector $|\operatorname{Arg}(\alpha - z)| < \pi/2 + \varepsilon$, $0 < |\alpha - z| < \delta$ will be quite important in the proof of Theorem 2 that is presented here. One could obtain the conclusions of Theorem 2 without

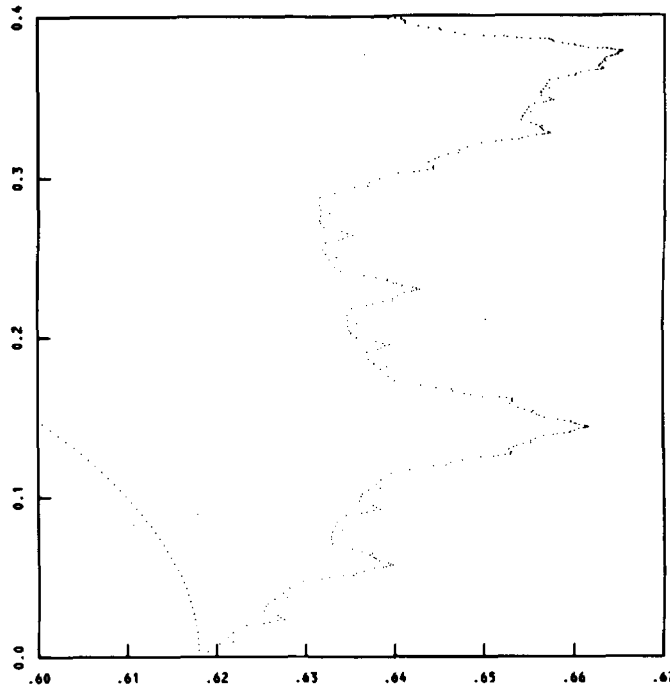

FIG. 3. Boundary of C_0 and of the circle $|z| = \alpha$ near $\alpha = \phi - 1$. ($Q(z) = z^2 + z^3$.) Note that the scale on the x-axis is different from the one on the y-axis.

proving that fact, which would allow one to shorten Section 3 drastically. However, if that course were followed, one would have to proceed much more carefully in later sections. We have therefore preferred the present approach, especially since it gives more information about the function $u(x)$.

The proof that is presented here yields a result somewhat stronger than the assertion of the theorem, namely, that

$$a_n = n^{-1}\alpha^{-n}u(\log n) + O(n^{-2}\alpha^{-n}) \qquad \text{as} \quad n \to \infty.$$

With additional work one can obtain an even more complete asymptotic development of the a_n.

A problem which is left open by the present proof of Theorem 2 is that of obtaining an explicit representation of the periodic function $u(x)$. It will follow from the proof that the kth coefficient in the Fourier expansion of $u(x)$ is $O(\exp(-\varepsilon\,|k|))$ for some $\varepsilon > 0$, but that coefficient will be given only in terms of a cofficient of a Fourier expansion of another periodic function. The coefficients in the expansion of this other function are in principle computable numerically, but it would be desirable to have a closed-form expression for them.

The proof of Theorem 2 is completely analytic. The combinatorial reason for the oscillations in a_n, the number of 2,3-trees with n leaves, should probably be ascribed to the requirement in the definition of 2,3-trees that all leaves be at the same distance from the root. If, for a fixed d, we consider the number $a(d, n)$ of 2,3-trees with n leaves which are at distance d from the root, then this number is nonzero precisely form $2^d \leqslant n \leqslant 3^d$. If we consider the normalized counting function $a^*(d, n) = n\phi^{-n}a(d, n)$, then numerical evidence, as well as heuristic reasoning, suggests that $a^*(d, n)$ achieves a maximum for $n \sim c_4(4-\phi)^d$, and that for such n, $a^*(d', n)$ is quite small for $d' \neq d$. An n increases from the value that maximizes $a^*(d, n)$, $a^*(d, n)$ decreases, while $a^*(d+1, n)$ increases. However,, this decrease and increase seem not to compensate for each other exactly, giving rise to oscillations. (A somewhat similar phenomenon has been proved to occur in another setting, for prefix-synchronized codes [8].)

Before proceeding to the proof of Theorem 2, let us mention various related results by other authors. De Bruijn [3] has also studied functional equations similar to (1.11). He even allowed $P(z)$ and $Q(z)$ to be rather general continuous functions. Because of the great generality of the allowed form of P and Q, he considered such functional equations only for real values of z, but he succeeded in deducing the existence of a periodic term in $f(z)$.

The literature on tree enumeration is very extensive [2, 9, 10, 17]. However, most of the generating functions encountered there have only a single algebraic singularity on the circle of convergence, which does not give rise to any untoward periodic phenomena. On the other hand, periodicities seem to be quite common in the case of partition functions. This seems to have been shown first by De Bruijn [4] in the case of Mahler's partition problem [15], but it is quite common in other settings as well, as shown, for example, in [6].

The literature on functional equations is by now voluminous, as is shown by [1, 14] and the references given there. Our results, however, seem to be new. A large part of our work is related to Fatou's treatise [7] on iteration of rational functions, but the thrust of our arguments is different in that we need to study very intensively some very special properties of the functions $f(z)$ satisfying the hypotheses of Theorem 2.

In the sections that follow we will prove Theorem 2. Section 2 contains a proof (considerably simpler than the one in [16]) of a generalization of (1.2), namely, that for large n, a_n is bounded above and below by positive constants times $n^{-1}\alpha^{-n}$. The upper bound for a_n will enable us to conclude that the formal power series (1.10) for $f(z)$ is actually analytic in $|z| < \alpha$. Section 3 proves some basic results about the region to which $f(z)$ can be continued. Section 4 then shows that in that region, $f(z)$ has an expansion of the form (1.13). That expansion is then used in Section 5 to prove the asymptotically oscillating behavior of the a_k.

In this paper, $c_1, c_2, \ldots$, will denote positive constants which depend only on the $P(z)$ and $Q(z)$ being considered.

2. Rough Bounds for the a_n

From now on it will be assumed that $f(z)$, $P(z)$ and $Q(z)$ satisfy the hypotheses of Theorem 2. The main result of this section can be stated as follows.

Proposition 1. *There exist positive constants c_5, c_6 and c_7 such that for $n \geqslant c_5$,*

$$c_6 n^{-1} \alpha^{-n} < a_n < c_7 n^{-1} \alpha^{-n}. \tag{2.1}$$

The proofs of both of the upper and lower bounds will use induction. As a first step towards the proofs, let

$$Q_0(z) = z, \qquad Q_1(z) = Q(z), \qquad Q_n(z) = Q_{n-1}(Q(z))$$

for $n \geqslant 2$. Note that each $Q_n(z)$ is a polynomial in z. When we iterate the functional equation (1.11), we find that for any $n \in \mathbf{Z}^+$,

$$\begin{aligned} f(z) &= P(z) + f(Q(z)) \\ &= P(Q_0(z)) + f(Q_1(z)) \\ &= P(Q_0(z)) + P(Q_1(z)) + f(Q_2(z)) \\ &= \sum_{k=0}^{n-1} P(Q_k(z)) + f(Q_n(z)). \end{aligned} \tag{2.2}$$

Since $Q_n(z)$ has only terms of degrees $\geqslant e_0^n \geqslant 2^n$ and $P(z)$ has no constant term, we can let $n \to \infty$ in (2.2) to obtain the expansion

$$f(z) = \sum_{k=0}^{\infty} P(Q_k(z)), \tag{2.3}$$

valid in the ring of formal power series in z. (A large part of Section 3 will be devoted to investigating the region where (2.3) holds as an identity for analytic functions.) The expansion (2.3) shows that $a_n \geqslant 0$ for all n.

In our proof we will use the following auxiliary result, which we state as a separate lemma, since it will be used in the following sections.

LEMMA 1. *If* $z \in \mathbf{C}$, $|z| \leqslant \alpha$, $z \neq 0$, $z \neq \alpha$, *then* $|Q(z)| < |z|$.

Proof of Lemma 1. Since $Q(z) = z$ for $z \neq 0$ is equivalent to

$$\sum_{k=0}^{K} q_k z^{e_k - 1} = 1,$$

and $q_k > 0$, it is clear that $Q(z) = z$ has exactly one positive solution α. Suppose that $|z| < \alpha$, $z \neq 0$. Then, since $Q(z)$ has nonnegative coefficients,

$$|Q(z)| \leqslant Q(|z|) < |z|.$$

Next, suppose that $|z| = \alpha$, $z \neq \alpha$. Then

$$\begin{aligned}|Q(z)| &= \alpha^{e_0} |q_0 + q_1 z^{e_1 - e_0} + \cdots| < \alpha \\ &= Q(\alpha) = \alpha^{e_0}(q_0 + q_1 \alpha^{e_1 - e_0} + \cdots),\end{aligned}$$

since the only way for equality to hold would be to have

$$z^{e_k - e_0} = \alpha^{e_k - e_0}, \qquad 1 \leqslant k \leqslant K,$$

which would force $z = \alpha\zeta$, where ζ is an $(e_k - e_0)$th root of unity for $1 \leqslant k \leqslant K$. Since g.c.d. $(e_1 - e_0, \ldots, e_K - e_0) = 1$, this is impossible for $z \neq \alpha$, which completes the proof of the lemma.

We now return to the proof of the proposition. We first consider the easier upper bound. Suppose that for some large $N \in \mathbf{Z}^+$ and some $b \in \mathbf{R}^+$ we have

$$a_n < b n^{-1} \alpha^{-n} \qquad \text{for } N/e_K \leqslant n \leqslant N/2. \tag{2.4}$$

We wish to show that a_N is not too large. Now a_N is by (1.11) and (2.4) no bigger than the coefficient of z^N in

$$b \sum_{N/e_K \leqslant n \leqslant N/2} n^{-1} \alpha^{-n} Q(z)^n. \tag{2.5}$$

Since the coefficient of z^N in (2.5) is equal to the coefficient y_N of z^N in

$$b \sum_{n=1}^{\infty} n^{-1} \alpha^{-n} Q(z)^n = -b \log(1 - \alpha^{-1} Q(z)), \tag{2.6}$$

it will suffice to obtain an estimate for y_N.

By Lemma 1, the only root of $Q(z) = \alpha$ in $|z| \leqslant \alpha$ is $z = \alpha$. Hence $(Q(z) - \alpha)/(z - \alpha) \neq 0$ for $|z| \leqslant \alpha + c_8$. If we write

$$\log(1 - \alpha^{-1} Q(z)) = \log(1 - \alpha^{-1} z) + R(z),$$

then $R(z)$ is analytic in $|z| < \alpha + c_8$, and therefore its Nth Taylor series coefficient is $\leqslant c_9(\alpha + c_{10})^{-N}$ in absolute value. Hence if $N \geqslant c_{11}$ and (2.4) holds, then

$$y_N \leqslant bn^{-1}\alpha^{-N}(1 + e^{-c_{12}N}).$$

We now easily complete the upper bound proof. Given any $H \geqslant c_{11}e_K$, say, there is certainly a constant b such that

$$a_n < bn^{-1}\alpha^{-n} \qquad \text{for } 1 \leqslant n \leqslant H.$$

Then the argument above shows that

$$a_n < bn^{-1}\alpha^{-n}(1 + e^{-c_{12}H}) \qquad \text{for } 1 \leqslant n \leqslant 2H.$$

Repeating the argument, we find that

$$a_n < bn^{-1}\alpha^{-n} \prod_{j=0}^{J-1} (1 + e^{-c_{12}H2^j}) \qquad \text{for } 1 \leqslant n \leqslant H2^J,$$

and so

$$a_n < bn^{-1}\alpha^{-n} \prod_{j=0}^{\infty} (1 + e^{-c_{12}H2^j}) \qquad \text{for } n \geqslant 1,$$

which is the desired upper bound.

The lower bound of Proposition 1 follows in a similar way, but there is a technical complication. If for some large N and some $b \in \mathbf{R}^+$ we have

$$a_n > bn^{-1}\alpha^{-n} \qquad \text{for } N/e_K \leqslant n \leqslant N/2, \tag{2.7}$$

then again we find that a_N is not less than the coefficient of z^N in the expansion of the function in (2.5), and so equals the coefficient y_N of z^N in the function in (2.6). But the arguments following Eq. (2.6) show that

$$y_N \geqslant bn^{-1}\alpha^{-N} - bc_{13}n^{-1}\alpha^{-N}e^{-c_{14}N}.$$

Therefore, if $N \geqslant c_{15}$, and (2.7) holds, then

$$y_N \geqslant bn^{-1}\alpha^{-N}(1 - e^{-c_{16}N}). \tag{2.8}$$

If we knew a value of $H \geqslant c_{15}$ such that

$$a_n > bn^{-1}\alpha^{-n} \qquad \text{for } H \leqslant n \leqslant He_K \tag{2.9}$$

for some $b = b(H) > 0$, then we could use (2.8) together with an inductive argument analogous to the one used for the upper bound in order to obtain the lower bound of Proposition 1.

The technical complication in the lower bound proof arises from the need to prove (2.9) for some very large values of H. (The corresponding problem in the upper bound proof was trivial.) Note that it suffices to prove that for some $H \geqslant c_{15}$ we have $a_n > 0$ for $H \leqslant n \leqslant He_K$, since (2.9) will then be satisfied for some sufficiently small b.

By (2.3), $a_n > 0$ for infinitely many n. Choose some n with $a_n > 0$. Then by (1.11) the power series for $f(z)$ contains all the terms of $Q(z)^n$, and so $a_m > 0$ for every m that can be represented as

$$m = \sum_{i=0}^{K} k_i e_i \qquad \text{with} \qquad \sum_{i=0}^{K} k_i = n,\ k_i \geqslant 0. \tag{2.10}$$

Writing $d_i = e_i - e_0$, $1 \leqslant i \leqslant K$, we find that (2.10) is equivalent to

$$m = \sum_{i=1}^{K} k_i d_i + ne_0 \qquad \text{with} \qquad \sum_{i=1}^{K} k_i \leqslant n,\ k_i \geqslant 0.$$

We will use the following auxiliary result, which is related to a linear diophantine problem of Frobenius, for which full references can be found in [5, 18].

LEMMA 2. *Let* $1 \leqslant d_1 < d_2 \cdots < d_K$ *be positive integers such that* g.c.d. $(d_1, \ldots, d_K) = 1$. *Then there exists an integer* $M = M(d_1, \ldots, d_K)$ *such that for any* $n \in \mathbf{Z}^+$, *any integer* m *in the interval* $[M, nd_K - M]$ *can be expressed as*

$$m = \sum_{i=1}^{K} k_i d_i \qquad \textit{with} \qquad \sum_{i=1}^{K} k_i \leqslant n,\ k_i \geqslant 0.$$

(If $a > b$, *we regard* $[a, b]$ *as the empty interval.)*

Lemma 2 will be proved at the end of this section, while we will now use it to complete the proof of Proposition 1. Let $M = M(d_1, \ldots, d_K)$. Then if $a_n > 0$, Lemma 2 shows that $a_m > 0$ for all

$$m \in [ne_0 + M, ne_K - M]. \tag{2.11}$$

Suppose that n was chosen so large that $n > 100M$. Then applying the same argument to the interval in (2.11) shows that $a_m > 0$ for all

$$m \in [ne_0^2 + Me_0 + M, ne_K^2 - Me_K - M].$$

Repeating the argument r times shows that $a_m > 0$ for all

$$m \in \left[ne_0^r + M\frac{e_0^r - 1}{e_0 - 1}, ne_K^r - M\frac{e_K^r - 1}{e_K - 1}\right].$$

This proves the existence of arbitrarily large H which satisfy (2.9) and completes the proof of Proposition 1.

Proof of Lemma 2. The proof we present here is very simple-minded. No attempt has been made to obtain the best possible value of M, which might be a problem of independent interest (cf. [18]).

We use induction on K. If $K=1$, then $d_1 = 1$ and the assertion of the lemma is true with $M=0$. Suppose that the lemma is true for $K-1$. Let

$$d=(d_1,..., d_{K-1}) \qquad \text{and} \qquad M' = M(d_1/d,..., d_{K-1}/d).$$

Then every multiple m of d in $[M'd, rd_K - M'd]$ can be represented as

$$m = \sum_{i=1}^{K-1} k_i d_i, \qquad \sum_{i=1}^{K-1} k_i \leqslant r, \quad k_i \geqslant 0. \tag{2.12}$$

Choose r such that $rd_{K-1} > 2M'd + d_1 d$. Then every $m \in \{M'd, (M'+1)\,d,..., (M'+d_1-1)\,d\}$ can be represented as in (2.12).

Since $(d_K, d) = (d_1, d_2,..., d_K) = 1$, every integer s can be represented as

$$s = xd_K + yd, \qquad x, y \in \mathbf{Z}, \tag{2.13}$$

and this representation is unique if we require that $M' \leqslant y \leqslant M' + d_K - 1$. Hence every integer $s \in [(M' + d_K - 1)\,d,\ nd_K - rd_K - M'd]$ can be represented in the form (2.13) with $M' \leqslant y \leqslant M' + d_K - 1$, $0 \leqslant x \leqslant n - r$, and therefore in the form

$$s = \sum_{i=1}^{K} k_i d_i, \qquad \sum_{i=1}^{K} k_i \leqslant n, \quad k_i \geqslant 0.$$

Therefore the assertion of the lemma holds with $M(d_1,..., d_K) = \max(M'd + d_K d, M'd + rd)$, and this completes the proof.

3. Overconvergent Expansion of $f(z)$

The bounds of Proposition 1 of the previous section show that the series (1.10) for the generating function $f(z)$, which was regarded as a formal power series, is convergent in $|z| < \alpha$ but not beyond. (Since $a_n \geqslant 0$, this also shows that $f(z)$ has a singularity at $z = \alpha$.) We will now show that $f(z)$ can be continued analytically beyond its circle of convergence. (Actually, for the proof of this section we only need to know that $f(z)$ is analytic in some open neighborhood of 0, which can be proved very simply. However, we will need the full strength of the lower bound of Proposition 1 later.)

When we iterate the functional equation (1.11), as in Section 2, we obtain for any $n \in \mathbf{Z}^+$

$$f(z) = \sum_{k=0}^{n-1} P(Q_k(z)) + f(Q_n(z)), \tag{3.1}$$

valid for z sufficiently small. Suppose now that for some z_0, $Q_n(z_0) \to 0$ as $n \to \infty$. Since $|Q(z)| < |z|/2$ for $|z|$ sufficiently small, we must have $Q_n(z) \to 0$ as $n \to \infty$ for z in some open neighborhood S of z_0. Furthermore, for $z \in S$, $f(Q_n(z)) \to 0$ as $n \to \infty$, and since $|P(z)| \leqslant c_{17} |z|$ for $|z| \leqslant 1$, say, we have

$$\sum_{k=0}^{\infty} |P(Q_k(z))| = O\left(\sum_{k=0}^{\infty} 2^{-k}\right) = O(1).$$

Hence

$$\sum_{k=0}^{\infty} P(Q_k(z))$$

is analytic in the open set

$$C = \{z : Q_n(z) \to 0 \text{ as } n \to \infty\}.$$

The above reasoning also shows that in some neighborhood $S' \subseteq C$ of 0, we can let $n \to \infty$ in the expansion (3.1), since $f(Q_n(z)) \to 0$ for $z \in S'$. The result is that

$$f(z) = \sum_{k=0}^{\infty} P(Q_k(z)) \tag{3.2}$$

holds for $z \in S'$. Since the right side of (3.2) is analytic in C, it provides an analytic continuation of $f(z)$ to all of the connected component C_0 of C that includes $z = 0$. The remainder of this section is devoted to showing that C_0 is considerably larger than the circle of convergence of the Taylor series (1.10) of $f(z)$, and thus that (3.2) presents an example of an overconvergent expansion (i.e., a regrouping of the Taylor series which converges in a larger region than the Taylor series itself).

From Lemma 1 of Section 2 one immediately deduces that

$$\{z : |z| \leqslant \alpha, z \neq \alpha\} \subseteq C_0. \tag{3.3}$$

This already proves that $f(z)$ can be continued beyond its circle of convergence, since, for example, it follows from the fact that C_0 is open that it contains a set of the type

$$\{z : |z| \leqslant \alpha + c_{18}, |z - \alpha| > 2c_{18}\}.$$

On the other hand, C_0 contains no points of the segment (α, ∞), since $Q_n(z) \to \infty$ as $n \to \infty$ for $z \in \mathbf{R}$, $z > \alpha$. Thus α is on the boundary of C_0. Our next goal will be to obtain more information about this boundary in the vicinity of α.

In order to show how wild C_0 can be, Figs. 2 and 3 exhibit the boundary of C_0 in the case of 2,3-trees, where $Q(z) = z^2 + z^3$. The points which are shown there are some of the roots of equations $Q_n(z) = \alpha$ for various n, $1 \leqslant n \leqslant 8$. Note that if $Q_n(z_0) = \alpha$, and $z \to z_0$, then by (3.1) $f(z)$ equals a polynomial in z plus $f(Q_n(z))$, which has a singularity at $z = z_0$. Hence $f(z)$ cannot be analytic at z_0, and $z_0 \notin C$. It follows from Fatou's [7] general theory of iteration of rational functions that in this particular case $C = C_0$, and the boundary of C is a nowhere differentiable Jordan curve.

Although we do not need to know the precise shape of C_0, it will be convenient to obtain a better estimate of that part of it which borders on α. Since α is the only singularity of $f(z)$ on its circle of convergence, it is natural to expect that the size of the coefficients a_n in the Taylor series of $f(z)$ will be determined largely by the behavior of $f(z)$ near α. Although it is not essential for the proof of Theorem 2, the following result does simplify that proof considerably. (Figure 3 shows the boundary of C_0 near α for $Q(z) = z^2 + z^3$.)

LEMMA 3. *There exist positive constants c_{19} and c_{20} such that*

$$D = \{z : |\operatorname{Arg}(\alpha - z)| \leqslant \pi/2 + c_{19}, 0 < |\alpha - z| < c_{20}\} \subseteq C_0.$$

Proof. We first establish a very important auxiliary result. Since $Q(\alpha) = \alpha$ is equivalent to

$$\sum_{k=0}^{K} q_k \alpha^{e_k - 1} = 1,$$

we find that

$$\beta = Q'(\alpha) = \sum_{k=0}^{K} q_k e_k \alpha^{e_k - 1} > \sum_{k=0}^{K} q_k \alpha^{e_k - 1} = 1.$$

This means that α is a repulsive fixed point of the transformation $z \to Q(z)$ (see [7, 14]); i.e., if $|z - \alpha| < \varepsilon$ for some small ε, then $|Q(z) - \alpha| > (1 + c_{21}) |z - \alpha|$.

If ω is a fixed point of a function $R(z)$, then $R_2'(\omega) = (R'(\omega))^2$, $R_2''(\omega) = R''(\omega) R'(\omega)(R'(\omega) + 1)$. Hence if we let $\gamma = Q''(\alpha)$, then

$$Q_{2^k}'(\alpha) = \beta^{2^k}, \qquad Q_{2^k}''(\alpha) = \gamma \beta^{2^k - 1} \prod_{j=0}^{k-1} (1 + \beta^{2^j}).$$

We claim that if $n = 2^k$ is large enough, then

$$(Q_n'(\alpha))^2 < \alpha Q_n''(\alpha). \tag{3.4}$$

To establish this claim, note that as $k \to \infty$,

$$\frac{(Q_n'(\alpha))^2}{Q_n''(\alpha)} \to \beta^2 \gamma^{-1} \prod_{r=0}^{\infty} (1 + \beta^{-2^r})^{-1} = \beta(\beta - 1)\, \gamma^{-1}.$$

Hence (3.4) will hold for large $n = 2^k$ if we show that

$$\beta^2 < \alpha\gamma + \beta. \tag{3.5}$$

Now

$$1 = \sum_k q_k \alpha^{e_k - 1},$$

$$\beta = \sum_k q_k e_k \alpha^{e_k - 1},$$

$$\alpha\gamma = \sum_k q_k e_k (e_k - 1)\, \alpha^{e_k - 1} = \sum_k q_k e_k^2 \alpha^{e_k - 1} - \beta,$$

so by the Cauchy inequality,

$$\beta^2 = \left(\sum_k q_k e_k \alpha^{e_k - 1} \right)^2 \leqslant \left(\sum_k q_k e_k^2 \alpha^{e_k - 1} \right) \left(\sum_k q_k \alpha^{e_k - 1} \right) = \alpha\gamma + \beta.$$

For equality to hold, we would have to have

$$q_k e_k^2 \alpha^{e_k - 1} = c_{22} q_k \alpha^{e_k - 1} \qquad \text{for } 0 \leqslant k \leqslant K,$$

which would allow $q_k > 0$ for only one value of k, which is ruled out by the hypotheses of the theorem.

We have shown that if $n = 2^k$ is large enough, then (3.4) holds. Fix an $n = 2^k$ for which (3.4) holds, let

$$\beta_n = Q_n'(\alpha), \qquad \gamma_n = Q_n''(\alpha),$$

and consider $z = \alpha + x + iy$, where $x, y \in \mathbf{R}$, $|x|, |y| \leqslant 1$, say. Then

$$Q_n(z) = \alpha + \beta_n(x + iy) + \tfrac{1}{2}\gamma_n (x + iy)^2 + O(|x|^3 + |y|^3), \tag{3.6}$$

and so

$$|Q_n(z)|^2 \leqslant \alpha^2 + 2\alpha\beta_n x + (\beta_n^2 + \alpha\gamma_n)\, x^2 + (\beta_n^2 - \alpha\gamma_n)\, y^2 + O(|x|^3 + |y|^3). \tag{3.7}$$

It now follows from (3.6) and (3.4) that for some $c_{22}, c_{23}, c_{24} \in \mathbf{R}^+$,

$$T = \{z = \alpha + x + iy : |y| < c_{22}, x < c_{23} |y|^2, |x| < c_{24}\}$$
$$\subseteq \{z : |Q_n(z)| < \alpha\}.$$

Since

$$\{z : |Q_n(z)| < \alpha\} \subseteq C,$$

C_0 contains the disk $|z| < \alpha$, nd

$$T \cup \{z : |z| < \alpha\}$$

is connected, we can conclude that $T \subseteq C_0$.

Since $\beta_n = \beta^n > 1$, we can find a $b > 1$ such that if $|\alpha - z| < c_{25}$, then (by (3.6)),

$$b|\alpha - z| \leqslant |\alpha - Q_n(z)| \leqslant 2\beta_n |\alpha - z|,$$

and

$$|\mathrm{Arg}(Q_n(z) - \alpha)| \geqslant |\mathrm{Arg}(z - \alpha)|.$$

We can also find positive constants c_{26} and c_{27} ($c_{27} < c_{25}$) such that

$$T_1 = \{z : ||\mathrm{Arg}(\alpha - z)| - \pi/2| < c_{26}, (10\beta_n)^{-1} \leqslant |\alpha - z| < c_{27}\} \subseteq T.$$

But then $T_2 = \alpha + (T_1 - \alpha)/b$ satisfies $Q_n(T_2) \subseteq T$, and since $T_2 \cup T$ is connected, we must have $T_2 \subseteq C_0$. Next, $T_3 = \alpha + (T_2 - \alpha)/b$ satisfies $Q_n(T_3) \subseteq T \cup T_2$, and $T_3 \cup T_2$ is connected, so $T_3 \subseteq C_0$. Continuing in this way, we discover that $T_m = \alpha + (T_1 - \alpha) b^{1-m} \subseteq C_0$ for all $m \in \mathbf{Z}^+$, which then completes the proof of the lemma.

4. Periodicity of the Generating Function $f(z)$

In this section we will study the behavior of $f(z)$ in the vicinity of α. Heuristic arguments similar to those used in the Introduction for 2,3-trees suggest that

$$f(z) \sim -\frac{P(\alpha)}{\log \beta} \log(\alpha - z) \qquad \text{as} \quad z \to \alpha. \tag{4.1}$$

However, for our purposes it will be important to prove not only (4.1), but also an estimate for the difference of the two sides of (4.1). To this end, let us define

$$g(z) = f(z) + c \log(1 - \alpha^{-1} z), \tag{4.2}$$

where

$$c = P(\alpha)(\log \beta)^{-1}. \tag{4.3}$$

Then the functional equation (1.11) yields

$$g(z) = h(z) + g(Q(z)), \tag{4.4}$$

where

$$h(z) = P(z) - c \log \frac{\alpha - Q(z)}{\alpha - z}. \tag{4.5}$$

Note that $h(z)$ is analytic in C_0 and at α, and

$$|h(z)| = O(|z|) \qquad \text{for } |z| \leqslant \alpha,$$

$$|h(z)| = O(|\alpha - z|) \qquad \text{for } |\alpha - z| \leqslant c_{28}.$$

Also, $|g(z)| = O(|z|)$ for $|z| < \alpha/2$, so we can iterate (4.4) to obtain the absolutely convergent expansion

$$g(z) = \sum_{n=0}^{\infty} h(Q_n(z)), \tag{4.6}$$

valid for all $z \in C_0$.

Since $\beta = Q'(\alpha) > 1$, there is a disk

$$D_1 = \{z : |\alpha - z| < c_{29}\}$$

such that for $z \in D_1$, the equation $Q(u) = z$ has a unique solution $u \in D_1$, which is moreover an analytic function of z. Let us denote $u = Q_{-1}(z)$. We can then define analytic functions $Q_{-n}(z) = Q_{-1}(Q_{-(n-1)}(z))$ for $n \geqslant 2$, $z \in D_1$. If D_1 is chosen sufficiently small, we will have

$$\sum_{n=1}^{\infty} |h(Q_{-n}(z))| = O(|\alpha - z|), \qquad z \in D_1, \tag{4.7}$$

and

$$\sum_{n=1}^{\infty} h(Q_{-n}(z))$$

will be analytic for $z \in D_1$.

Let us now define

$$g^*(z) = \sum_{n=-\infty}^{\infty} h(Q_n(z)), \qquad z \in D_2 = C \cap D_1. \tag{4.8}$$

By (4.7),

$$g(z) = g^*(z) + O(|\alpha - z|), \qquad z \in D_2. \tag{4.9}$$

For us, the most important property of $g^*(z)$ is that

$$g^*(z) = g^*(Q(z)). \tag{4.10}$$

A priori, (4.10) holds only for those $z \in D_2$ such that $Q(z) \in D_2$, since $g^*(z)$ was only defined there. However, we can now use (4.10) to continue $g^*(z)$ analytically into C_0. (In fact, by Fatou's results [7], $g^*(z)$ can be continued this way into all of C_0 but again this is not needed for our proof.)

Our next step will be to approximate $g^*(z)$ by a periodic function. Consider

$$g_n^*(z) = g^*(\alpha + (z - \alpha)\beta^{-n}), \qquad n \in \mathbf{Z}^+,\ z \in D_3 = D_2 \cap D.$$

(D is defined as in Lemma 3.) Since for $z \in D_3$ we have

$$Q_{-1}(z) = \alpha + \beta^{-1}(z - \alpha) + O(|z - \alpha|^2), \tag{4.11}$$

we also have

$$g_n^*(z) - g_m^*(z) = O(|z - \alpha|^2 \beta^{-2m})$$

if $z \in D_3$, $n, m \in \mathbf{Z}^+$, $m < n$. Hence

$$v(z) = \lim_{n \to \infty} g_n^*(z) = \lim_{n \to \infty} g^*(\alpha + (z - \alpha)\beta^{-n}) \tag{4.12}$$

exists for $z \in D_3$, and is analytic there. We also have (by (4.11))

$$g^*(z) - v(z) = O(|z - \alpha|^2), \qquad z \in D_3. \tag{4.13}$$

Furthermore, it is clear from (4.12) that

$$v(z) = v(z\beta^{-1}), \qquad z \in D_3.$$

Hence if we define

$$w(s) = v(\alpha - e^s), \tag{4.14}$$

then $w(s)$ is analytic for $|\text{Im}(s)| < \pi/2 + c_{19}$, $\text{Re}(s) \leqslant -c_{30}$. What is most important,

$$w(s - \log \beta) = w(s) \tag{4.15}$$

for s in that region.

Because of the periodicity of $w(s)$, we can continue it analytically throughout the region $|\text{Im}(s)| < \pi/2 + c_{19}$. Furthermore, we can expand $w(s)$ in a Fourier series:

$$w(s) = \sum_{k=-\infty}^{\infty} b_k \exp\left\{\frac{2\pi iks}{\log \beta}\right\}, \tag{4.16}$$

$$b_k = (\log \beta)^{-1} \int_0^{\log \beta} w(x) \exp\left\{-\frac{2\pi ikx}{\log \beta}\right\} dx.$$

Since $w(x) \in \mathbf{R}$ for $x \in \mathbf{R}$, $b_{-k} = \bar{b}_k$. More important, since $w(s)$ is analytic for $|\text{Im}(s)| < \pi/2 + c_{19}$, we have

$$b_k = O\left(\exp\left\{-\frac{(\pi^2 + c_{31})|k|}{\log \beta}\right\}\right) \tag{4.17}$$

for some $c_{31} > 0$.

If we now collect together (4.2), (4.9), (4.14), and (4.17), we find that for $z \in D_3$,

$$f(z) = c \log(1 - \alpha^{-1}z) + w(\log(\alpha - z)) + O(|\alpha - z|), \tag{4.18}$$

which is what we will use in the next section to study the coefficients a_k of $f(z)$. First, however, we will consider another question about $w(s)$. We have shown that it is periodic with period $\log \beta$, and that the coefficients in its Fourier expansion decrease very rapidly. The final result of this section will be a proof that $w(s)$ is not a constant.

Suppose that $w(s) = t$ holds identically in s for some constant t (which has to be real). Then also $v(z) = t$, and hence $g^*(z) = t$, since $g^*(z)$ and $v(z)$ take on exactly the same values in D_3. Since

$$\sum_{n=1}^{\infty} h(Q_{-n}(z))$$

is analytic in a disk D_1 around α, this means that

$$g(z) = g^*(z) - \sum_{n=1}^{\infty} h(Q_{-n}(z))$$

can be continued analytically in D_1. But then $f(z)$ can be continued in

$$D_1^* = D_1 - \{z : z \in \mathbf{R}, z \geqslant \alpha\}.$$

To produce a contradiction, it will suffice to find a $z_0 \in D_1^*$ such that $Q_n(z_0) = \alpha$ for some $n \in \mathbf{Z}^+$, since we have remarked already that $f(z)$ has singularities at all such points. The existence of such a z_0 is guaranteed by the following lemma, which thus completes our proof of the fact that $w(s)$ is not a constant.

LEMMA 4. *Given any $\varepsilon > 0$, there is a positive integer n and a number $z \in \mathbf{C}$, $z \notin \mathbf{R}$, $|z - \alpha| < \varepsilon$, such that $Q_n(z) = \alpha$.*

Proof. The equation $Q(z) = \alpha$ has $z = \alpha$ as a root. However, since α is a simple zero of $Q(z) = \alpha$, and $\deg Q(z) \geqslant 3$, there must be at least one other root $\alpha_1 \neq \alpha$ of $Q(z) = \alpha$.

We wish to show that one can find $n \in \mathbf{Z}^+$ and $z \in \mathbf{C}$, $|z - \alpha| < \varepsilon$, $z \neq \alpha$ such that $Q_n(z) = \alpha$ or $Q_n(z) = \alpha_1$. If this is impossible then $\{Q_n(z)\}$ forms a sequence of functions meromorphic in $\varepsilon/2 < |z - \alpha| < \varepsilon$, each one of which omits there the three values α, α_1, and ∞. Hence by Montel's theorem [12, Vol. 2, p. 248], the $Q_n(z)$ form a normal family in $\varepsilon/2 < |z - \alpha| < \varepsilon$, and so one can find a subsequence of them which converges to some meromorphic function. But if $z \in \mathbf{R}$, $z \in (\alpha - \varepsilon, \alpha)$, then $Q_n(z) \to 0$ as $n \to \infty$, so the limit function must be identically zero. On the other hand, $Q_n(z) \to \infty$ for $z \in \mathbf{R}$, $z \in (\alpha, \alpha + \varepsilon)$, so the limit function must equal infinity, which gives a contradiction.

The contradiction we have obtained above shows that there are $m \in \mathbf{Z}^+$ and $z \in \mathbf{C}$, $0 < |z - \alpha| < \varepsilon$, such that $Q_m(z) = \alpha_1$ or $Q_m(z) = \alpha$. But then $n = m + 1$ satisfies $Q_n(z) = \alpha$. Finally $z \notin \mathbf{R}$, snce $0 < Q(x) < \alpha$ for $0 < x < \alpha$ and $Q(x) > \alpha$ for $x > \alpha$. This completes the proof of the lemma.

5. PERIODICITY OF THE COEFFICIENTS a_k

In the previous section we showed that if c is defined by (4.3), then

$$f(z) = -c \log(1 - \alpha^{-1} z) + w(\log(\alpha - z)) + r(\alpha - z) \tag{5.1}$$

holds for

$$z \in D_3 = \{z : |\operatorname{Arg}(\alpha - z)| \leqslant \pi/2 + c_{19}, 0 < |\alpha - z| < c_{32}\}, \tag{5.2}$$

where $w(s)$ is periodic with period $\log \beta$ and has a Fourier expansion (4.16) with coefficients satisfying (4.17), and where $|r(\alpha - z)| = O(|\alpha - z|)$ for

$z \in D_3$. Because of the periodicity of $w(s)$, we can extend $w(\log(\alpha - z))$ analytically to all of the domain

$$D_4 = \{z: |\mathrm{Arg}(\alpha - z)| \leqslant (\pi + c_{19})/2, z \neq \alpha\},$$

where, because of the periodicity, it will be bounded. Furthermore, $f(z)$ is analytic in C_0, so we can use (5.1) to define $r(\alpha - z)$ for all $z \in C_0 \cap D_4$. Since $f(z)$ is bounded in

$$D_5 = \{z: |z| < \alpha + c_{33}, |z - \alpha| > c_{33}\},$$

(where we may take $c_{33} < c_{32}/10$), it follows that (5.1) holds in $D_3 \cup D_3$, and that $|r(\alpha - z)| = O(|\alpha - z|)$ there.

After these preliminaries we can now proceed to the evaluation of the Taylor series coefficients a_n of $f(z)$. Since each function on the right of (5.1) is analytic at $z = 0$, they all have convergent power series expansions around $z = 0$, and a_n is the sum of the coefficients of z^n for the three terms, which we write as

$$a_n = a_n' + a_n'' + a_n'''.$$

The coefficient a_n' of z^n in the expansion of $-c \log(1 - \alpha^{-1} z)$ is simply

$$a_n' = (c/n)\, \alpha^{-n}. \tag{5.3}$$

The coefficient a_n'' of $w(\log(\alpha - z))$ is the interesting one, as it gives rise to the oscillations of a_n. First, however, we dispose of the coefficient a_n''' of $r(\alpha - z)$, which will turn out to be small. By Cauchy's theorem,

$$a_n''' = \frac{1}{2\pi i} \int_E \frac{r(\alpha - z)}{z^{n+1}}\, dz,$$

where E is any simple closed Jordan curve in $D_3 \cup D_5$ that encloses $z = 0$. We choose $E = E_1 \cup E_2$, where

$$E_1 = \left\{z: |z| = \alpha + \frac{1}{2} c_{33}, |\mathrm{Arg}(z - \alpha)| \geqslant \frac{\pi}{2} + \frac{1}{2} c_{19}\right\},$$

$$E_2 = \left\{z: |\mathrm{Arg}(z - \alpha)| = \frac{1}{2} (\pi + c_{19}), |z| \leqslant \alpha + \frac{1}{2} c_{33}\right\}.$$

(To be completely rigorous, we should replace a section of E_2 near $z = \alpha$ by a semicircle lying in $D_3 \cup D_5$, and then let its radius shrink to zero.) Then

$$\left| \frac{1}{2\pi i} \int_{E_1} \frac{r(\alpha - z)}{z^{n+1}}\, dz \right| = O\left(\left(\alpha + \frac{1}{2} c_{33} \right)^{-n} \right).$$

On E_2, if we let $z=\alpha+x+iy$, then $y=\pm c_{34}x$, $|z|\geqslant \alpha+x$, and $|r(\alpha-x)|=O(x)$. Hence

$$\left|\frac{1}{2\pi i}\int_{E_2}\frac{r(\alpha-z)}{z^{n+1}}\,dz\right|=O\left(\int_0^\infty \frac{x}{(\alpha+x)^{n+1}}\,dx\right)=O(\alpha^{-n}n^{-2}).$$

Therefore

$$a_n'''=O(\alpha^{-n}n^{-2}). \tag{5.4}$$

We now consider a_n'', the coefficient of z^n in the Taylor series of $w(\log(\alpha-z))$ around $z=0$. We use the following auxiliary result.

LEMMA 5. *If we let, for* $t\in\mathbf{R}$, $t\neq 0$, $z\in\mathbf{C}$, $|z|<\alpha$,

$$\exp\{it\log(1-\alpha^{-1}z)\}=\sum_{n=1}^{\infty} v_n z^n, \tag{5.5}$$

then for $|t|\geqslant n$,

$$v_n=(n^{1+it}\alpha^n\Gamma(-it))^{-1}\exp\{O(t^2/n+|t|/n)\} \tag{5.6}$$

and

$$|v_n|\leqslant|n\alpha^n\Gamma(-it)|^{-1} \tag{5.7}$$

in all cases.

Proof. Since

$$\exp\{it\log(1-\alpha^{-1}z)\}=(1-\alpha^{-1}z)^{it},$$

the expansion (5.5) is valid for $|z|<\alpha$ with

$$\begin{aligned}
v_n&=(-1)^n\alpha^{-n}\binom{it}{n}=(-1)^n\,\alpha^{-n}\,\frac{it(it-1)\cdots(it-n+1)}{n!}\\
&=\alpha^{-n}\frac{it}{it-n}\prod_{k=1}^{n}\left(1-\frac{it}{k}\right)\\
&=\alpha^{-n}\frac{it}{it-n}\exp\left\{-it\sum_{k=1}^{n}\frac{1}{k}\right\}\frac{\prod_{k=1}^{n}(1-it/k)\,e^{it/k}}{\prod_{k=n+1}^{\infty}(1-it/k)\,e^{it/k}}\\
&=\frac{\alpha^{-n}}{n-it}\Gamma(-it)^{-1}\frac{\exp\{it(\gamma-\sum_{k=1}^{n}1/k)\}}{\prod_{k=n+1}^{\infty}(1-it/k)\,e^{it/k}},
\end{aligned}$$

where γ now denotes Euler's constant, rather than $Q''(\alpha)$. In particular,

$$|v_n| \leqslant |n\alpha^n \Gamma(-it)|^{-1},$$

which gives (5.7). If $|t| \geqslant n$, then

$$it\left(\gamma - \sum_{k=1}^{n} \frac{1}{k}\right) = -it \log n + O(|t|/n),$$

$$\log \left| \prod_{k=n+1}^{\infty} \left(1 - \frac{it}{k}\right) e^{it/k} \right| = O\left(\sum_{k=n+1}^{\infty} t^2/k^2\right) = O(t^2/n),$$

and so we obtain (5.6), which finishes the proof of the lemma.

We now return to the problem of estimating the a_n''. The expansion (4.16) of $w(s)$ and Lemma 5 show that for $n \geqslant 1$,

$$\begin{aligned} a_n'' = &\sum_{\substack{|k| \leqslant \log^2 n \\ k \neq 0}} \frac{\alpha^{-n}}{n} b_k \Gamma\left(-\frac{2\pi ik}{\log \beta}\right)^{-1} \\ &\times \exp\left\{\frac{2\pi ik}{\log \beta}(\log \alpha - \log n) + O(k^2/n)\right\} \\ &+ O\left(\sum_{|k| > \log^2 n} \frac{\alpha^{-n}}{n} |b_k| \left|\Gamma\left(-\frac{2\pi ik}{\log \beta}\right)\right|^{-1}\right). \end{aligned}$$

Now for $y \in \mathbf{R}$,

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh(\pi y)},$$

so by (4.17),

$$\sum_{|k| > \log^2 n} |b_k| \left|\Gamma\left(-\frac{2\pi ik}{\log \beta}\right)\right|^{-1} \leqslant c_{35} \sum_{|k| > \log^2 n} e^{-c_{36}|k|} = O(n^{-1}),$$

and

$$\begin{aligned} &\sum_{\substack{|k| < \log^2 n \\ k \neq 0}} b_k \Gamma\left(-\frac{2\pi ik}{\log \beta}\right)^{-1} \exp\left\{\frac{2\pi ik}{\log \beta}(\log \alpha - \log n) + O(k^2/n)\right\} \\ &= \sum_{\substack{|k| < \log^2 n \\ k \neq 0}} b_k \Gamma\left(-\frac{2\pi ik}{\log \beta}\right)^{-1} \exp\left\{\frac{2\pi ik}{\log \beta}(\log \alpha - \log n)\right\} + O(n^{-1}). \end{aligned}$$

Therefore if we define

$$u^*(z) = \sum_{k \neq 0} b_k \Gamma\left(-\frac{2\pi i k}{\log \beta}\right)^{-1} \exp\left\{\frac{2\pi i k}{\log \beta}(\log \alpha - z)\right\}, \tag{5.8}$$

then $u^*(z)$ is analytic in a strip $|\mathrm{Im}(z)| < c_{37}$, is periodic with period $\log \beta$, has mean zero, and

$$a_n'' = n^{-1}\alpha^{-n}u^*(\log n) + O(n^{-2}\alpha^{-n}). \tag{5.9}$$

Combining (5.9), (5.4), and (4.2), we find that

$$a_n = n^{-1}\alpha^{-n}u(\log n) + O(n^{-2}\alpha^{-n}), \tag{5.10}$$

where

$$u(z) = P(\alpha)(\log \beta)^{-1} + u^*(z) \tag{5.11}$$

is analytic in $|\mathrm{Im}(z)| < c_{37}$, is periodic with period $\log \beta$, and has mean $P(\alpha)(\log \beta)^{-1}$.

The Fourier coefficients of $u(z)$ are given by (5.8). Unfortunately, without a good representation for the b_k we cannot conclude that $u(x) > 0$ for all real x. Clearly $u(x) \geqslant 0$, since if $u(x)$ were ever negative, then we would have $a_n < 0$ for infinitely many n, while (2.3) show that $a_n \geqslant 0$ for all n. However, in order to conclude that

$$a_n \sim n^{-1}\alpha^{-n}u(\log n) \qquad \text{as} \quad n \to \infty,$$

we have to show that $u(x) > 0$ for all x. This, however, follows from Proposition 1 of Section 2, thus completing our proof.

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