Stieltjes interlacing of zeros of Laguerre polynomials from different sequences

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Received 15 September 2010; received in revised form 9 March 2011; accepted 21 April 2011

Communicated by T.H. Koornwinder

Abstract

Stieltjes’ Theorem (cf. Szegö (1959) \cite{10}) proves that if \( \{p_n\}_{n=0}^{\infty} \) is an orthogonal sequence, then between any two consecutive zeros of \( p_k \) there is at least one zero of \( p_n \) for all positive integers \( k \), \( k < n \); a property called Stieltjes interlacing. We prove that Stieltjes interlacing extends across different sequences of Laguerre polynomials \( L_\alpha^n \), \( \alpha > -1 \). In particular, we show that Stieltjes interlacing holds between the zeros of \( L_\alpha^{n+1} \) and \( L_\alpha^n \), \( \alpha > -1 \), when \( t \in \{1, \ldots, 4\} \) but not in general when \( t > 4 \) or \( t < 0 \) and provide numerical examples to illustrate the breakdown of interlacing. We conjecture that Stieltjes interlacing holds between the zeros of \( L_\alpha^{n+1} \) and those of \( L_\alpha^{n+1} \) for \( 0 < t < 4 \). More generally, we show that Stieltjes interlacing occurs between the zeros of \( L_\alpha^{n+1} \) and the zeros of the \( k \)th derivative of \( L_\alpha^n \), as well as with the zeros of \( L_{\alpha-k}^{n+1} \) for \( t \in \{1, 2\} \) and \( k \in \{1, 2, \ldots, n-1\} \). In each case, we identify associated polynomials, analogous to the de Boor–Saff polynomials (cf. de Boor and Saff (1986) \cite{3}, Ismail (2005) \cite{6}), that are completely determined by the coefficients in a mixed three-term recurrence relation, whose zeros complete the interlacing process.

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MSC: 33C50; 42C05

Keywords: Interlacing of zeros; Stieltjes’ Theorem; Laguerre polynomials; de Boor–Saff polynomials

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doi:10.1016/j.indag.2011.04.003
1. Introduction

A classical theorem of Stieltjes (cf. [10, Theorem 3.3.3]) proves that if \( \{p_n\}_{n=0}^\infty \) is any sequence of orthogonal polynomials then the zeros of \( p_k \) and \( p_n \), \( k < n \), are interlacing in the sense that each open interval of the form \((-\infty, z_1), (z_1, z_2), \ldots, (z_{k-1}, z_k), (z_k, \infty)\), where \( z_1 < z_2 < \cdots < z_k \) are the zeros of \( p_k \), contains at least one zero of \( p_n \). If we assume that \( p_k \) and \( p_n \) have no common zeros, the same argument used by Stieltjes shows that there exist \( k \) open intervals, with endpoints at successive zeros of \( p_n \), each of which contains exactly one zero of \( p_k \).

For polynomials \( p_k \) and \( p_n \) with \( k < n - 1 \), there are clearly not enough zeros of \( p_k \) to provide the required number of points to interlace fully with the zeros of \( p_n \). However, this deficit in the number of points needed to complete the interlacing process between the zeros of polynomials of non-consecutive degree in an orthogonal sequence is well understood. In [2, Theorem 4], Beardon proves that if \( \{p_n\}_{n=0}^\infty \) is any sequence of orthogonal polynomials with \( p_m \) and \( p_n \) having no common zeros for \( m \neq n \), then there exist real polynomials \( S_{n-m-1} \) of degree \( n-m-1 \) whose real simple zeros, together with the zeros of \( p_m \), interlace with the zeros of \( p_n \) for \( m < n \). The polynomials \( S_{n-m-1} \) were first observed, albeit in a rather different context, by de Boor and Saff in [3] and are also studied by Vinet and Zhedanov in [11]. An important feature of the polynomials \( S_{n-m-1} \) is that they are completely determined by the coefficients in the three-term recurrence relation satisfied by the orthogonal sequence \( \{p_n\}_{n=0}^\infty \) (cf. [2]). Independently, it also follows immediately from Segura’s result (cf [8], Theorem 1) that the zero of the linear polynomial that completes the interlacing of the zeros of \( p_{n-1} \) with those of \( p_{n+1} \) is given by one of the coefficients in the three-term recurrence relation satisfied by the orthogonal sequence \( \{p_n\}_{n=0}^\infty \).

The question arises as to whether Stieltjes interlacing occurs between the zeros of two polynomials \( p_n \) and \( q_k \), \( k < n - 1 \), from different orthogonal sequences \( \{p_n\}_{n=0}^\infty \) and \( \{q_n\}_{n=0}^\infty \) and whether polynomials analogous to the de Boor–Saff polynomials exist in this more general situation. Among the classical orthogonal families of Gegenbauer, Laguerre and Jacobi polynomials, natural choices of different orthogonal sequences are those corresponding to different values of the appropriate parameter(s) and also the (orthogonal) sequences of their derivatives.

In this paper we prove that Stieltjes interlacing takes place between the zeros of Laguerre polynomials \( L_{n+1}^{\alpha+1} \) and \( L_n^{\alpha+1} \), \( n \in \mathbb{N}, \alpha > -1, t \in \{1, 2, 3, 4\} \) and we identify the polynomials that are analogous to the de Boor–Saff polynomials in these cases. We make two conjectures regarding the location of the (single) extra point that completes the interlacing process for continuous variation of the parameter \( t \) in the range \( 0 < t < 2 \) and \( 2 < t < 4 \). Our two conjectures are equivalent when \( t = 2 \). Numerical examples are provided to illustrate that Stieltjes interlacing breaks down in general when \( t = 5 \) or \( t = -1 \). Our main result proves that Stieltjes interlacing occurs between the zeros of \( L_{n+1}^{\alpha} \) and the zeros of the \( k \)th derivative of \( L_n^{\alpha} \) for all \( k \in \{1, \ldots, n-1\} \) and identifies the polynomials analogous to the de Boor–Saff polynomials. Finally, we prove that Stieltjes interlacing holds between the zeros of \( L_{n+1}^{\alpha} \) and the zeros of \( L_{n-k-1}^{\alpha+1-t} \) for \( t = 1 \) and \( t = 2 \).

We recall that, for \( \alpha > -1 \), the Laguerre polynomials \( L_n^{\alpha} \) are orthogonal with respect to the weight function \( w_{\alpha}(x) = x^\alpha e^{-x} \) on \((0, \infty)\) and satisfy the three-term recurrence relation

\[
(n+1)L_{n+1}^{\alpha}(x) = (2n+\alpha+1-x)L_n^{\alpha}(x) - (\alpha+n)L_{n-1}^{\alpha}(x). \tag{1}
\]

In Section 2, we discuss the existence of common zeros of Laguerre polynomials. Thereafter, we will assume in all our theorems, lemmas and conjectures that the polynomials under
2. Common zeros of Laguerre polynomials

Bourget’s Hypothesis for Bessel functions states that for any integers \( n \geq 0 \) and \( m \geq 1 \), the functions \( J_n \) and \( J_{n+m} \) have no common zeros except for the one at the origin. Siegel proved this famous hypothesis in [9] as a consequence of a more general result. An interesting discussion on this and related results can be found in [12, p. 484–485]. The analogue of Bourget’s Hypothesis for Laguerre polynomials, namely that \( L_n^\alpha \) and \( L_{n+m}^\alpha \) have no common zeros for any integers \( n \geq 0 \) and \( m > 1 \), does not hold for all \( n \in \mathbb{N} \) and \( \alpha > -1 \). If we compute the zeros of the resultant of \( L_n^\alpha \) and \( L_{n+m}^\alpha \) using Mathematica, we see that the two polynomials have a common zero when \( n = 2, m = 2, \) and \( \alpha = 23 \). Note that it is evident from (1) that this common zero of \( L_2^{23} \) and \( L_4^{23} \) is at the point \( x = 30 = 2n + \alpha + 1 \). It is an open question how many common zeros are possible in general for \( L_n^\alpha \) and \( L_{n+m}^\alpha, \) \( n \geq 0 \) and \( m > 1 \), although there is a sharp upper bound, namely, \( \min(k, n - k - 1) \), due to Gibson, for the maximum possible number of common zeros of \( p_k \) and \( p_n, \) \( k < n, \) in a general orthogonal sequence \( \{p_n\}_{n=0}^\infty \) (cf. [4]). Across different sequences of Laguerre polynomials, we can compute the zeros of the resultant using Mathematica to show that when \( t = 1, n = 4, m = 2 \) and \( \alpha = \sqrt{97} - 5 \), the polynomials \( L_4^{\alpha+1} \) and \( L_6^\alpha \) have a common zero at \( x = 1 + 3\sqrt{97}/3 \). Note that this common zero occurs at the point \( \alpha + n + 1 \) as can be seen from (5).

The assumption throughout this paper that the pair of polynomials under consideration in each of the lemmas and theorems have no common zeros is necessary in order to ensure that “full” Stieltjes interlacing takes place. However, each of our proofs can be modified to accommodate the situation where there are common zeros and we provide the details of the required modification in the remark after the proof of Theorem 3. Essentially, the common zeros are factored out and Stieltjes interlacing occurs between the remaining (non-common) zeros of the two polynomials under consideration.

3. Main results

Our first theorem proves that the zeros of \( L_{n+1}^{\alpha+t} \), together with a point on the real line whose location depends on \( t, \alpha \) and \( n \), interlace with the zeros of \( L_{n+1}^\alpha \) for integer values of \( t, \) \( t \in \{0, 1, 2, 3, 4\} \). The case \( t = 0 \) was proved by Segura in [8, Section 3.1]. For the convenience of the reader, we provide an alternative proof of the case \( t = 0 \).

**Theorem 1.** (i) The zeros of \( L_{n-1}^\alpha \), together with the point \( \alpha + 1 + 2n \), interlace with the zeros of \( L_{n+1}^\alpha \).

(ii) The zeros of \( L_{n-1}^{\alpha+1} \), together with the point \( \alpha + 1 + n \), interlace with the zeros of \( L_{n+1}^\alpha \);

(iii) The zeros of \( L_{n-1}^{\alpha+2} \), together with the point \( \alpha + 1 + 2n \), interlace with the zeros of \( L_{n+1}^\alpha \);

(iv) The zeros of \( L_{n-1}^{\alpha+3} \), together with the point \( \frac{(\alpha+1)(\alpha+2)}{(\alpha+2n)(\alpha+3)}, \) interlace with the zeros of \( L_{n+1}^\alpha \);

(v) The zeros of \( L_{n-1}^{\alpha+4} \), together with the point \( \frac{(\alpha+1)(\alpha+3)}{(\alpha+3+2n)}, \) interlace with the zeros of \( L_{n+1}^\alpha \).

**Remarks.** (a) The extra interlacing points in (iv) and (v) respectively are the upper bounds for the smallest zero \( w_1 \) of the Laguerre polynomial \( L_{n+1}^\alpha \) obtained by Gupta and Muldoon in [5, eqns. 2.9 and 2.10], namely \( w_1 < \frac{(\alpha+1)(\alpha+2)}{\alpha+2+n} \) and \( w_1 < \frac{(\alpha+1)(\alpha+3)}{\alpha+3+2n} \).
(b) We see that Stieltjes interlacing can break down when the restrictions stated in Theorem 1 are not satisfied. For example, Mathematica shows that the zeros of $L_{n+1}^\alpha(x)$ and $L_{n-1}^{\alpha+t}(x)$ do not interlace when $n = 5$, $\alpha = -0.9$ and $t = 5$ or $t = -1$. Similarly, the result in Theorem 3 that the zeros of $L_{n+1}^{\alpha}$ and $L_{n-k}^{\alpha+k+t}$ are Stieltjes interlacing for $t \in \{0, 1, 2\}$ does not hold for $t > 2$ or $t < 0$ and this can be illustrated by computing the zeros of $L_{n+1}^{\alpha}$ and $L_{n-k}^{\alpha+k+t}(x)$ when $n = 7$, $\alpha = 1.34$, $k = 3$ with $t = 3$ or $t = -1$ respectively.

**Conjectures.** (a) The $(n-1)$ zeros of $L_{n-1}^{\alpha+t}$, together with the point $\alpha + 1 + (2-t)n$, interlace with the zeros of $L_{n+1}^{\alpha}$ for all $0 < t < 2$.

(b) The $(n-1)$ zeros of $L_{n-1}^{\alpha+t}$, together with the point $(\alpha+1)(\alpha+(t-1))$, interlace with the zeros of $L_{n+1}^{\alpha}$ for all $2 < t < 4$.

**Remark.** It is evident from Theorem 1(i)–(iii) that the interval $(w_k, w_{k+1})$ with endpoints at successive zeros of $L_{n+1}^{\alpha}$ that does not contain a zero of $L_{n-1}^{\alpha}$, $t \in \{0, 1, 2\}$ changes from the interval containing the point $\alpha + 1 + 2n$ when $t = 0$ to the interval containing the point $\alpha + 1 + n$ when $t = 1$ and the interval containing the point $\alpha + 1$ when $t = 2$. It is clear that since the variation of the extra interlacing point is continuous in $t$, the point $\alpha + 1 + (2-t)n$ will coincide with a zero of $L_{n+1}^{\alpha}$ as $t$ varies between 0 and 2. It would be reasonable to expect that when $L_{n+1}^{\alpha}(\alpha + 1 + (2-t)n) = 0$, the point $\alpha + 1 + (2-t)n$ will also be a zero of $L_{n-1}^{\alpha}$ since this is the case when $t = 0, 1, 2$ as is evident from (1), (5) and (8) respectively. Numerical data generated by Mathematica indicates that, modulo the extra interlacing point coinciding with a zero of $L_{n+1}^{\alpha}$, the above conjectures are true.

We will need the following lemma in the proof of Theorem 3.

**Lemma 2.** For each $k \in 1, \ldots, n - 1$, $n \in \mathbb{N}$,

$$x^k L_{n-k}^{\alpha+k}(x) = G_k(x)L_n^\alpha(x) - H_{k-1}(x)L_{n+1}^\alpha(x)$$

(2)

where $G_k$ and $H_k$ are polynomials of degree $k$.

**Theorem 3.** Let $n \in \mathbb{N}$ and $k \in \{1, \ldots, n - 1\}$ and let $G_k$ and $H_k$ be the polynomials defined by (2). Then

(i) the zeros of the $k$th derivative of $L_n^\alpha$, together with the $k$ zeros of $G_k$, interlace with the zeros of $L_{n+1}^\alpha$;

(ii) the zeros of $L_{n-k}^{\alpha+k+1}$, together with the $k$ zeros of $G_k - H_{k-1}$, interlace with the zeros of $L_{n+1}^\alpha$;

(iii) the zeros of $L_{n-k}^{\alpha+k+2}$, together with the $k$ zeros of $\alpha(G_k - H_{k-1})(x) - xH_{k-1}(x)$, interlace with the zeros of $L_{n+1}^\alpha$.

Our final result shows the symmetric structure of the zeros of the de Boor–Saff polynomial associated with $L_{n+1}^\alpha$ and $L_{n-2}^\alpha$.

**Theorem 4.** The zeros of $L_{n-2}^\alpha$, together with the points $2n + \alpha + \sqrt{1+n(n+\alpha)}$ and $2n + \alpha - \sqrt{1+n(n+\alpha)}$, interlace with the zeros of $L_{n+1}^\alpha$. 
4. Proofs

Proof of Theorem 1. (i) Since \( L_{n+1}^\alpha(\alpha + 1 + 2n) \neq 0 \) by (1) and our assumption that \( L_n^\alpha \) and \( L_n^\alpha \) are co-prime, evaluating (1) at successive zeros \( w_k \) and \( w_{k+1} \) of \( L_{n+1}^\alpha \), we obtain

\[
\frac{L_n^\alpha(w_k) L_n^\alpha(w_{k+1})}{L_{n-1}^\alpha(w_k) L_{n-1}^\alpha(w_{k+1})} = \frac{(\alpha + n)^2}{(2n + \alpha + 1 - w_k)(2n + \alpha + 1 - w_{k+1})}. \quad (3)
\]

The right-hand side of (3) is positive if and only if 2\( n + \alpha + 1 \notin (w_k, w_{k+1}) \), while \( L_n^\alpha(w_k) L_n^\alpha(w_{k+1}) < 0 \) for each \( k \in \{1, \ldots, n\} \) because the zeros of \( L_{n+1}^\alpha \) and \( L_n^\alpha \) are interlacing. We deduce that, provided \( 2n + \alpha + 1 \notin (w_k, w_{k+1}) \), \( L_n^\alpha \) has a different sign at successive zeros of \( L_{n+1}^\alpha \) and therefore has an odd number of zeros in each interval \((w_j, w_{j+1})\), \( j \in \{1, \ldots, n\} \), that does not contain the point \( \alpha + 1 + 2n \). It follows from the Intermediate Value Theorem that the \( n - 1 \) zeros of \( L_{n-1}^\alpha \), together with the point \( \alpha + 1 + 2n \), interlace with the \( n + 1 \) zeros of \( L_{n+1}^\alpha \).

(ii) From [7, p.203]

\[(n + 1)L_{n+1}^\alpha(x) = (\alpha + n + 1)L_n^\alpha(x) - xL_{n-1}^\alpha(x). \quad (4)\]

From [1, eqn(22.7.30)] and (4)

\[
(n + 1)L_n^\alpha = (\alpha + n + 1)L_n^\alpha(x) - x[L_n^{\alpha+1}(x) + L_n^\alpha(x)]
= (\alpha + n + 1 - x)L_n^\alpha(x) - xL_{n-1}^\alpha(x). \quad (5)
\]

Since \( L_{n+1}^\alpha \) and \( L_{n-1}^{\alpha+1} \) are co-prime by assumption, evaluating (5) at successive zeros \( w_k \) and \( w_{k+1} \) of \( L_{n+1}^\alpha \) we obtain

\[
\frac{L_n^\alpha(w_k) L_n^\alpha(w_{k+1})}{L_{n-1}^\alpha(w_k) L_{n-1}^{\alpha+1}(w_{k+1})} = \frac{w_k w_{k+1}}{(\alpha + n + 1 - w_k)(\alpha + n + 1 - w_{k+1})}. \quad (6)
\]

Since \( w_k > 0 \) for all \( k \in \{1, \ldots, n+1\} \), the stated result follows from (6) in the same way as the result in (i) followed from (3).

(iii) Replacing \( \alpha \) by \( \alpha + 1 \) in (5) and \( n \) by \( n + 1 \) in [1, eqn(22.7.29)], we obtain

\[(n + 1)L_{n+1}^{\alpha+1}(x) = (\alpha + n + 2 - x)L_n^{\alpha+1}(x) - xL_{n-1}^{\alpha+1}(x), \quad (7)\]

and

\[xL_{n+1}^{\alpha+1}(x) = (x - n - 1)L_{n+1}^\alpha(x) + (\alpha + n + 1)L_n^\alpha(x). \quad (8)\]

Substituting from (4) and (7) into (8) yields

\[(n + 1)(\alpha + 1)L_{n+1}^\alpha(x) = (\alpha + n + 1)(\alpha + 1 - x)L_n^\alpha(x) - x^2L_{n-1}^{\alpha+2}(x). \quad (9)\]

Since \( L_{n+1}^\alpha \) and \( L_{n-1}^{\alpha+2} \) are co-prime by assumption, evaluating (9) at successive zeros \( w_k \) and \( w_{k+1}, k \in \{1, \ldots, n\} \) of \( L_{n+1}^\alpha \), we obtain

\[
\frac{L_n^\alpha(w_k) L_n^\alpha(w_{k+1})}{L_{n-1}^\alpha(w_k) L_{n-1}^{\alpha+2}(w_{k+1})} = \frac{w_k^2 w_{k+1}^2}{(\alpha + n + 1)^2(\alpha + 1 - w_k)(\alpha + 1 - w_{k+1})}. \quad (10)
\]

Again, the result follows from (10) using the same argument as in the proof of (i).

(iv) Replacing \( \alpha \) by \( \alpha + 1 \) in (9), we have

\[x^2L_{n-1}^{\alpha+3}(x) = (\alpha + n + 2)(\alpha + 2 - x)L_n^{\alpha+1}(x) - (n + 1)(\alpha + 2)L_{n+1}^{\alpha+1}(x) \quad (11)\]
and substituting from (4) and (8) into (11) yields

\[ x^3 L_{n-1}^{\alpha+3}(x) = (\alpha + n + 1)[(\alpha + 2)(\alpha + 1) - (\alpha + 2 + n)x]L_n^{\alpha}(x) \]
\[ - (n + 1)[(\alpha + 2)(\alpha + 1) - nx]L_{n+1}^{\alpha}(x). \]  

(12)

The result follows as in (i).

(v) Replacing \( \alpha \) by \( \alpha + 1 \) in (12) yields

\[ x^3 L_{n-1}^{\alpha+4}(x) = (\alpha + n + 2)[(\alpha + 3)(\alpha + 2) - (\alpha + 3 + n)x]L_n^{\alpha+1}(x) \]
\[ - (n + 1)[(\alpha + 3)(\alpha + 2) - nx]L_{n+1}^{\alpha+1}(x). \]  

(13)

Using (4) and (8) together with (13) we obtain

\[ x^4 L_{n-1}^{\alpha+4}(x) = (\alpha + 2)(\alpha + n + 1)[(\alpha + 1)(\alpha + 3) - (\alpha + 3 + 2n)x]L_n^{\alpha}(x) \]
\[ + (n + 1)[nx^2 + 2nx(\alpha + 2) - (\alpha + 1)(\alpha + 2)(\alpha + 3)]L_{n+1}^{\alpha}(x) \]

and the stated result again follows using the same argument as in (i).  \[ \square \]

**Proof of Lemma 2.** We prove the result by induction on \( k \). For \( k = 1 \), (5) yields (2) with \( G_1(x) = \alpha + n + 1 - x \) and \( H_0(x) = - (n + 1) \) so the result is true for \( k = 1 \). Now assume that, for \( m = 1, 2, \ldots, k \),

\[ x^m L_{n-m}^{\alpha+m}(x) = G_m(x)L_n^{\alpha}(x) - H_{m-1}(x)L_{n+1}^{\alpha}(x) \]  

(14)

with \( \deg(G_m) = m \) and \( \deg(H_m) = m \). Then

\[ x^{k+1} L_{n-k-1}^{\alpha+k+1}(x) = x \left[ x^k L_{(n-1)-k}^{(\alpha+1)+k}(x) \right] \]
\[ = G_k(x)xL_{n-1}^{\alpha+1}(x) - H_{k-1}(x)xL_n^{\alpha+1}(x) \quad \text{from (14)} \]
\[ = G_k(x) \left[ (\alpha + n + 1 - x)L_n^{\alpha}(x) - (n + 1)L_{n+1}^{\alpha}(x) \right] \]
\[ - H_{k-1} \left[ (\alpha + n + 1)L_n^{\alpha}(x) - (n + 1)L_{n+1}^{\alpha}(x) \right] \quad \text{from (5), (4)} \]
\[ = G_{k+1}(x)L_n^{\alpha}(x) - H_k(x)L_{n+1}^{\alpha}(x) \]

where

\[ G_{k+1}(x) = (\alpha + n + 1 - x)G_k(x) - (\alpha + n + 1)H_{k-1}(x) \]

and

\[ H_k(x) = (n + 1)[G_k(x) - H_{k-1}(x)]. \]

Therefore (14) holds for \( m = k + 1 \) and the result follows by induction on \( k \).  \[ \square \]

**Proof of Theorem 3.** (i) We note (cf. [1]) that \( D^k L_n^{\alpha} = (-1)^k L_{n-k}^{\alpha+k}, k \in \{0, 1, \ldots, n - 1\} \),

where \( D^k \) denotes the \( k \)-th derivative. From (2), if \( L_{n+1}^{\alpha}(x) \neq 0 \), we have

\[ \frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^{\alpha}(x)} = \frac{G_k(x) L_n^{\alpha}(x)}{L_{n+1}^{\alpha}(x)} - H_{k-1}(x). \]  

(15)

Now (cf. [2])

\[ \frac{L_n^{\alpha}(x)}{L_{n+1}^{\alpha}(x)} = \sum_{j=1}^{n+1} \frac{A_j}{x - w_j} \]
where \( \{w_j\} \) are the zeros of \( L_{n+1}^\alpha \) and \( A_j > 0 \) for \( j = 1, \ldots, n + 1 \), so that (15) can be written as

\[
\frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^\alpha(x)} = \sum_{j=1}^{n+1} \frac{G_k(x)A_j}{x - w_j} - H_{k-1}(x). \tag{16}
\]

Since \( L_{n+1}^\alpha \) and \( L_{n+1}^\alpha \) are always co-prime while \( L_{n-k}^\alpha \) and \( L_{n-k}^{\alpha+k} \) are co-prime by assumption, it follows from (2) that \( G_k(w_j) \neq 0 \) for any \( j \in \{1, 2, \ldots, n + 1\} \). Now, if \( G_k(x) \) has no zeros in the interval \((w_j, w_{j+1})\) for some \( j = 1, 2, \ldots, n \), then \( G_k \) does not change sign in this interval while \( H_{k-1} \) is bounded for all \( x \in (w_j, w_{j+1}) \) and \( A_j > 0 \). However, the expression on the right hand side of (16) will take (arbitrarily) large positive and negative values for \( x \in (w_j, w_{j+1}) \) and we deduce that \( L_{n-k}^{\alpha+k} \) must have an odd number of zeros in this interval. Since there are \( n \) intervals \((w_j, w_{j+1})\) while \( G_k \) has degree \( k \) and \( L_{n-k}^{\alpha+k} \) has \( n - k \) real simple zeros, it follows that \( G_k \) must have \( k \) real simple zeros and we deduce that the zeros of the product \( G_k L_{n-k}^{\alpha+k} \) interlace with the zeros of \( L_{n+1}^\alpha \).

(ii) Replacing \( \alpha \) by \( \alpha + 1 \) in (2) and using (4) and (8) we obtain

\[
x^{k+1} L_{n-k}^{\alpha+k+1}(x) = A_k(x)L_{n}^\alpha(x) - B_k(x)L_{n+1}^\alpha(x) \tag{17}
\]

where

\[A_k(x) = (\alpha + n + 1)[G_k(x) - H_{k-1}(x)]\]

and

\[B_k(x) = (n + 1)G_k(x) + (x - n - 1)H_{k-1}(x).\]

The stated result follows from the same argument used in (i).

(iii) Replacing \( \alpha \) by \( \alpha + 1 \) in (17) and using (4) and (8) yields

\[
x^{k+2} L_{n-k}^{\alpha+k+2}(x) = C_k(x)L_{n}^\alpha(x) - D_{k+1}(x)L_{n+1}^\alpha(x)
\]

where

\[C_k(x) = (\alpha + n + 1)[A_k(x) - B_k(x)]\]

and

\[D_{k+1}(x) = (x - n - 1)B_k(x) + (n + 1)A_k(x).\]

The result now follows as in (i). \( \Box \)

**Remark.** The proof of Theorem 3(i) in the case when \( L_{n-k}^{\alpha+k} \) and \( L_{n+1}^\alpha \) have common zeros can be modified as follows. We observe from (2) that if \( L_{n+1}^\alpha \) and \( L_{n-k}^{\alpha+k} \) have any common zeros, these must also be zeros of \( G_k \) since \( L_n^\alpha \) and \( L_{n+1}^\alpha \) are always co-prime. Let \( r(x) \) be the polynomial of degree \( r \leq \max(n - k, (n + 1) - (n - k) - 1) = \max(k, n - k) \), having zeros at the \( r \) common (simple) zeros of \( L_{n-k}^{\alpha+k} \) and \( L_{n+1}^\alpha \). Then in (16), canceling out all common zeros of \( L_{n-k}^{\alpha+k} \) and \( L_{n+1}^\alpha \), we obtain

\[
\frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^\alpha(x)} = \sum_{j=1}^{n+1-r} \frac{G_{k-r}(x)A_j}{x - w_j} - H_{k-1}(x)
\]

where \( G_{k-r}(x) = \frac{G_k(x)}{r(x)} \) is a polynomial of degree \( k - r \). The argument proceeds exactly as before and we obtain Stieltjes interlacing of the (non-common) zeros of \( L_{n-k}^{\alpha+k} \) and \( L_{n+1}^\alpha \).
Proof of Theorem 4. Replacing \( n \) by \( n - 1 \) in (1) and eliminating \( L_{n-1}^{\alpha} \) from the resulting equation and (1), we obtain

\[
(n + 1)(2n + \alpha - 1 - x)L_{n+1}^{\alpha}(x) = ((2n + \alpha + 1 - x)(2n + \alpha - 1 - x) - n(\alpha + n))L_n^{\alpha}(x) - (\alpha + n)(\alpha + n - 1)L_{n-2}^{\alpha}(x).
\]

A straightforward calculation yields the result. \( \square \)

Acknowledgments

The authors wish to thank the referees for their pertinent observations, remarks and suggestions. The work of K. Driver and K. Jordaan was supported by the National Research Foundation of South Africa under grant numbers 2053730 and 2054423 respectively.

References


