Mean Value Properties of Solutions to Parabolic Equations with Variable Coefficients

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1. INTRODUCTION

Let $L = \text{div}(A(x, t) \text{grad}) - D$, be a uniformly parabolic operator in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}^+$ with smooth coefficients, and let $\Gamma(x, t; y, s)$ be its fundamental solution. For a fixed point $(x_0, t_0) \in \mathbb{R}^{n+1}_+$ and $r > 0$ small enough we set

$$\psi_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{n+1}_+ | \Gamma(x_0, t_0; x, t) = (4\pi r^2)^{-n/2}\},$$

where {} indicates the closure of the set {}. $\psi_r(x_0, t_0)$ is the level set relative to the value $(4\pi r^2)^{-n/2}$ of the fundamental solution with pole at $(x_0, t_0)$, $\Gamma(x_0, t_0; \cdot, \cdot)$, of the backward parabolic operator $L^* = \text{div}(A(x, t) \text{grad}) + D$. We also define

$$\Omega_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{n+1}_+ | \Gamma(x_0, t_0; x, t) > (4\pi r^2)^{-n/2}\}.$$

We call $\psi_r(x_0, t_0)$ and $\Omega_r(x_0, t_0)$ respectively the parabolic sphere and the parabolic ball with radius $r$ and "center" at $(x_0, t_0)$. Because of well known estimates on the fundamental solution (see also (2.5)) the parabolic balls $\Omega_r(x_0, t_0)$ shrink to the center $(x_0, t_0)$ as $r \to 0$.

The main purpose of this note is to prove the following

**Theorem 1.** Let $D$ be a domain in $\mathbb{R}^{n+1}_+$, and $u$ be a solution of $Lu = 0$ in

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D. Let \((x_0, t_0) \in D\), then for every \(r > 0\) sufficiently small (depending on \((x_0, t_0)\)) we have

\[
 u(x_0, t_0) = (4\pi r^2)^{-n/2} \int_{\Omega_r(x_0, t_0)} A(x, t) \frac{(\text{grad} \Gamma(x_0, t_0; x, t)) \cdot \text{grad} \Gamma(x_0, t_0; x, t)}{\Gamma(x_0, t_0; x, t)^2} dx dt.
\]

In (1.1), \(\cdot\) indicates the inner product in \(\mathbb{R}^n\), and \(\text{grad} \Gamma\) the \((n\text{-dimensional})\) spatial gradient of \(\Gamma\).

The proof of Theorem 1 is achieved by first establishing a representation formula for solutions of \(Lu = 0\) involving integration on the level sets \(\psi_r(x_0, t_0)\) of the fundamental solution \(\Gamma\). Precisely, we show

**Theorem 2.** In the hypothesis of Theorem 1 on \(D, u, (x_0, t_0)\) we obtain for almost every (a.e.) sufficiently small \(r > 0\)

\[
 u(x_0, t_0) = \int_{\psi_r(x_0, t_0)} u(x, t) \frac{A(x, t)(\text{grad} \Gamma(x_0, t_0; x, t)) \cdot \text{grad} \Gamma(x_0, t_0; x, t)}{|(\text{grad} \Gamma(x_0, t_0; x, t), D, \Gamma(x_0, t_0; x, t)|} dH_n. \tag{1.2}
\]

In (1.2) \(|(\text{grad} \Gamma, D, \Gamma)|\) denotes the \((n + 1)\text{-dimensional}\) norm of the space-time gradient of \(\Gamma\) and \(dH_n\) denotes the \(n\text{-dimensional}\) Hausdorff measure on \(\psi_r(x_0, t_0)\).

Suitably rewriting (1.2) and using a generalized version of the co-area formula (see [Fe], or (2.19) below) we obtain (1.1). A basic observation is that if in (1.1) we set \(u \equiv 1\), then we have for every \(r > 0\) small enough

\[
 (4\pi r^2)^{-n/2} \int_{\Omega_r(x_0, t_0)} A(x, t)(\text{grad} \Gamma(x_0, t_0; x, t)) \cdot \text{grad} \Gamma(x_0, t_0; x, t) dx dt \equiv 1. \tag{1.3}
\]

Hence

\[
 d\mu_r^{x_0, t_0}(x, t) = (4\pi r^2)^{-n/2} \frac{A(\text{grad} \Gamma) \cdot \text{grad} \Gamma}{\Gamma^2} dx dt
\]

is a probability measure supported on \(\Omega_r(x_0, t_0)\). This fact, together with the positivity of the kernel \(A(\text{grad} \Gamma) \cdot \text{grad} \Gamma / \Gamma^2\) on \(\Omega_r(x_0, t_0)\) can be used
to show that a continuous function, \( u \), in \( D \), for which (1.1) holds, satisfies the maximum principle. Using this information, it is easy to prove that such a function must be a solution of \( Lu = 0 \) in \( D \). Therefore, solutions of \( Lu = 0 \) can be characterized by (1.1).

When \( A(x, t) \equiv \text{Id} \) and \( \Gamma(x_0, t_0; x, t) = (4\pi(t_0 - t))^{-n/2} \exp[-|x_0 - x|^2/4(t_0 - t)] \), the Gauss–Weierstrass kernel with pole at \((x_0, t_0)\), (1.1), (1.2), give back known weighted average formulas for solutions of the heat operator \( H = \Delta - D_t \); see (2.28), (2.14) below, and also \([P1, P2, F, W1]\).

Mean value theorems relative to the Laplace operator play a basic role in the theory of harmonic functions. Their striking simplicity reflects the fact that the level sets of the fundamental solution of \( Au = 0 \) in \( \mathbb{R}^n \) are spheres. The geometry relative to the operator \( H = \Delta - D_t \), in \( \mathbb{R}^{n+1} \) is not as simple, due to the special role played by the time variable. The level sets of \( \Gamma(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t) \) are smooth, football-shaped surfaces whose common “center,” the origin, is placed on the surface itself (for more details see \([W1]\), or the discussion below).

Weighted average formulas involving the level sets of \( \Gamma(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t) \) have been established and used by several authors. Pini \([P1, P2]\) has been the first to show that in \( \mathbb{R}^2 \), solutions of \( D_{xx}u - D_{tt}u = 0 \) can be represented by means of formulas like (1.1) and (1.2). (See (2.14) and (2.28) below for the corresponding heat versions of (1.1), (1.2).) He also used these formulas to characterize sub- and super-temperatures, in analogy to what is done with sub- and superharmonic functions. In \([P3]\) are discussed some criteria of regularity for boundary points which involve the sets \( \psi_\cdot(x_0, t_0) \) and \( \Omega_\cdot(x_0, t_0) \). Fulks \([F]\) has shown that the solutions of \( Hu = 0 \) in \( \mathbb{R}^{n+1} \) can be characterized by (2.14) below, thus establishing the analogue of the Gauss–Koebe theorem for harmonic functions. Watson \([W1, W2, W3]\) has developed a systematic theory of sub- and super-temperatures for the heat operator in \( \mathbb{R}^{n+1} \) based on (2.14), (2.28) below. Among other things, he has shown that subtemperatures satisfy the strong and the weak maximum principles, and used this to characterize solutions to \( Hu = 0 \). Kupcov \([K1, K2, K3]\) has stated various mean value theorems relative to the operator \( H \), or to some type of degenerate parabolic equations. He has shown in \([K1]\) that (2.28) can be used to deduce Harnack’s inequality for positive solutions of \( Hu = 0 \). Evans and Gariepy \([EG]\) have recently succeeded in proving a Wiener-type criterion for the heat equation. Their starting point is Watson’s formula (2.28), and the formula giving the derivative of (2.28) as a function of \( r > 0 \). Using these they have been able to estimate the capacity of regions nested between two level sets of the Gauss–Weierstrass kernel. Finally, we mention the note by Bauer \([B]\), in which the following result is announced: “The probability measure on \( \psi_\cdot(x_0, t_0) \) attached to the kernel (2.14) is
obtained by sweeping the Dirac measure at \((x_0, t_0)\) on the complementary of the set \(\Omega_r(x_0, t_0)\)." An application of this result to the theory of subtemperatures and a probabilistic interpretation of the measure given by (2.13) are presented.

We close this section with some concluding remarks. Formulas (1.1) and (1.2) can be used to investigate several open questions in the parabolic potential theory of operators with variable coefficients. For instance, starting from (1.1) one can develop a theory of subparabolic functions, i.e., solutions of the inequality \(Lu \leq 0\), along the lines of \([W1, W2]\). Another stimulating open problem concerns the regularity of the boundary points for a parabolic operator with variable coefficients. The results of Evans and Gariepy strongly suggest that (1.1) should represent a key tool for such investigation. Finally, we mention that it would be interesting to show that (1.1) can be used to deduce Harnack's inequality for positive solutions of \(Lu = 0\). This requires a more detailed analysis of the kernel appearing in (1.1) which we defer to a forthcoming study.

2. Proofs of Theorems 1 and 2

In what follows \(\cdot\) and \(\langle \cdot, \cdot \rangle\) will denote respectively the inner product in \(\mathbb{R}^n\) and \(\mathbb{R}^{n+1}\). \(A(x, t) = (a_{ij}(x, t))\) will be a \(C^\infty\), symmetric, \(n \times n\) matrix-valued function on \(\mathbb{R}^{n+1}\), for which there exists a \(\lambda \in (0, 1)\) such that for every \((x, t) \in \mathbb{R}^{n+1}\) and \(\xi \in \mathbb{R}^n\)

\[
\lambda |\xi|^2 \leq A(x, t) \cdot \xi \leq \lambda^{-1} |\xi|^2.
\]

We let

\[
L = \text{div}(A(x, t) \text{grad}) - D,
\]

be the parabolic operator corresponding to \(A\), and

\[
L^* = \text{div}(A(x, t) \text{grad}) + D,
\]

be the formal adjoint of \(L\). Our starting point will be the identity

\[
uL^*v - vLu = \text{div}(uA(\text{grad } v) - vA(\text{grad } u)) + D(\langle uv \rangle), \tag{2.1}
\]

which holds for any couple of smooth functions \(u, v\) on \(\mathbb{R}^{n+1}\). Let \(D\) be a bounded piecewise smooth domain in \(\mathbb{R}^{n+1}\). We will denote with \(n\) the outward normal to \(\partial D\), whereas \(dH_n\) will indicate the \(n\)-dimensional Hausdorff measure on \(\partial D\). We will need to distinguish the spatial from the time component of \(n\), so we will write \(n = (N_x, N_t)\). If \(F\) indicates the \((n+1)\)-dimen-
sional vector field on $\mathbb{R}^{n+1}_+$ defined by $F = (uA(\text{grad } v) - vA(\text{grad } u), uv)$, from (2.1) and the divergence theorem we obtain

$$\int_{D} (uL^*v - vLu) \, dx \, dt$$

$$= \int_{\partial D} \langle F, n \rangle \, dH$$

$$= \int_{\partial D} \{ [uA(\text{grad } v) - vA(\text{grad } u)] \cdot N_x + uvN_t \} \, dH. \quad (2.2)$$

Assume now that $Lu = L^*v = 0$ in $D$. Equation (2.2) gives

$$\int_{\partial D} \{ [uA(\text{grad } v) - vA(\text{grad } u)] \cdot N_x + uvN_t \} \, dH = 0. \quad (2.3)$$

If in (2.3) we take $v = 1$ we obtain

$$\int_{\partial D} [A(\text{grad } u) \cdot N_x - uN_t] \, dH = 0, \quad (2.4)$$

which is the analogue of Gauss' theorem for solutions to parabolic equations.

Now let $\Gamma$ be the fundamental solution of $L$ in $\mathbb{R}^{n+1}_+$. It is well known that for $(x_0, t_0) \in \mathbb{R}^{n+1}_+$ fixed, if we set $v(x, t) = \Gamma(x_0, t_0; x, t)$, then $v \in C^\infty(\mathbb{R}^{n+1}_+ \setminus \{(x_0, t_0)\})$, $v = 0$ in the half-space $t > t_0$, and $L^*v = 0$ in $\mathbb{R}^{n+1}_+ \setminus \{(x_0, t_0)\}$ (see [Fr]). Moreover, by the results in [A] there exist constants $C_1, C_2, \alpha_1, \alpha_2$, depending only on $\lambda$ and $n$, such that

$$C_1 \Gamma_{x_1}(x - \xi, t - \tau) \leq \Gamma(x, t; \xi, \tau) \leq C_2 \Gamma_{x_2}(x - \xi, t - \tau) \quad (2.5)$$

for all $(x, t), (\xi, \tau) \in \mathbb{R}^{n+1}_+$ with $(x, t) \neq (\xi, \tau)$, where $\Gamma_{x_i}$ is the fundamental solution of the heat operator $\alpha_i A - D_i$, $i = 1, 2$. A known estimate on the gradient of $\Gamma$ is the following (see, e.g., [Fr]):

$$|\text{grad } \Gamma(x, t; \xi, \tau)| \leq C(4\pi(t - \tau))^{-n/2 - 1/2} \exp\left[ -\mu \frac{|x - \xi|^2}{A(t - \tau)} \right] \quad (2.6)$$

for all $(x, t), (\xi, \tau) \in \mathbb{R}^{n+1}_+$ with $(x, t) \neq (\xi, \tau)$. Here $C$ and $\mu$ are two constants depending on $\lambda, u$, and the $L^\infty$ norms of the gradients of the $a_i$'s. An immediate corollary of (2.5) is that for a fixed $(x, t) \in \mathbb{R}^{n+1}_+$

$$\text{grad } \Gamma(x, t; \cdot, \cdot) \in L^p_{\text{loc}}(\mathbb{R}^n \times (0, t)), \quad \text{for } p < (n + 2)/(n + 1). \quad (2.7)$$
A more general result that does not depend on the smoothness of the coefficients $a_{ij}$ is given in \[A\]. Theorem 5 in that paper asserts that (2.7) holds with a bound on the $L^p$ norm of $\text{grad} \, I$ which depends only on $\lambda$ and $n$.

We now fix a point $(x_0, t_0) \in \mathbb{R}^{n+1}_+$ and for $r > 0$ define

$$
\psi_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{n+1}_+ \mid I(x_0, t_0; x, t) = (4\pi r^2)^{-\frac{n}{2}}\}.
$$

Since $I(x_0, t_0; \cdot, \cdot) \in C^\infty(\mathbb{R}^n \times (0, t_0))$ Sard's theorem (see, e.g., \[S\]) implies that for a.e. sufficiently small $r > 0$, $\psi_r(x_0, t_0)$ is a smooth regular $n$-dimensional manifold in $\mathbb{R}^{n+1}_+$. Because of (2.5), $\psi_r(x_0, t_0)$ is a bounded set nested between the two smooth surfaces

$$
\psi_i(x_0, t_0) = \{(x, t) \in \mathbb{R}^{n+1}_+ \mid I_{x}(x_0 - x, t_0 - t) = C_i^{-1}(4\pi r^2)^{-\frac{n}{2}}\},
$$

$i = 1, 2$. It is clear that for a.e. small enough $r > 0$, $\psi_r(x_0, t_0)$ is the (smooth) boundary of the bounded domain

$$
\Omega_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{n+1}_+ \mid I(x_0, t_0; x, t) > (4\pi r^2)^{-\frac{n}{2}}\}.
$$

We will call the set $\Omega_r(x_0, t_0)$ the parabolic ball with "center" at $(x_0, t_0)$ and radius $r$. Analogously, we will call $\psi_r(x_0, t_0)$ the parabolic sphere with "center" at $(x_0, t_0)$ and radius $r$. The section of the surface $\psi_i(x_0, t_0)$, $i = 1, 2$, with a hyperplane perpendicular to the $t$-axis is an $n$-dimensional sphere. The square of the radius of this sphere is given by the function of $t$

$$
\rho_i(t) = 2n\xi_i(t_0 - t) \log \left[ -\frac{r^2 C_i^{2/n}}{\xi_i(t_0 - t)} \right], \quad t_0 - C_i^{2/n} \xi_i^{-1} r^2 \leq t < t_0.
$$

It is then clear that $\rho_i(t)$ is zero at $t = t_0$ and at $t = t_0 - C_i^{2/n} \xi_i^{-1} r^2$. Moreover, $\rho_i(t)$ has a maximum at $t = t_0 - (c\xi_i)^{-1} C_i^{2/n} r^2$ given by $\rho_i^{\text{max}} = 2nC_i^{2/n} r^2$.

Assume now that $r$ is fixed so that $\psi_r(x_0, t_0)$ is a smooth surface. For each $s \in (t_0 - r^2, t_0)$ set

$$
\Omega_i^s(x_0, t_0) = \{(x, t) \in \Omega_i(x_0, t_0) \mid t < s\},
$$

$$
I_i^s(x_0, t_0) = \{(x, t) \in \Omega_i(x_0, t_0) \mid t = s\},
$$

$$
\psi_i^s(x_0, t_0) = \{(x, t) \in \psi_i(x_0, t_0) \mid t < s\}.
$$

Let $u$ be a solution of $Lu = 0$. We can thus apply (2.3) to $u$ and $v = I(x_0, t_0; \cdot, \cdot)$ on $D = \Omega_i^s(x_0, t_0)$ to obtain

$$
\int_{\psi_i(x_0, t_0) \cup \Omega_i^s(x_0, t_0)} \left[ uA(\text{grad} \, I(x_0, t_0; \cdot, \cdot))
- I(x_0, t_0; \cdot, \cdot) A(\text{grad} \, u) \right] \cdot N_x + uI(x_0, t_0; \cdot, \cdot) N_r \cdot dH_n = 0.
$$

(2.8)
Observe that on $I(x_0, t_0)N_\chi = 0$ and $N_\xi = 1$, and that on $\psi_t(x_0, t_0)$ we have $I(x_0, y_0; \cdot, \cdot) \equiv (4\pi r^2)^{-n/2}$, we obtain from (2.8)

$$
\int_{\psi_t(x_0, t_0)} uA(\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)) \cdot N_\chi \, dH_n
$$

$$
+ (4\pi r^2)^{-n/2} \int_{\psi_t(x_0, t_0)} \left[ -A(\text{grad } u) \cdot N_\chi + uN_\xi \right] \, dH_n
$$

$$
+ \int_{I(x_0, t_0)} u\Gamma(x_0, t_0; \cdot, \cdot) \, dH_n = 0. \quad (2.9)
$$

Now we let $s \to t_0^-$ in (2.9). Since $\Gamma(x_0, t_0; \cdot, \cdot)$ is the fundamental solution with pole at $(x_0, t_0)$ we have

$$
\lim_{s \to t_0^-} \int_{I_s(x_0, t_0)} u\Gamma(x_0, t_0; \cdot, \cdot) \, dH_n = u(x_0, t_0). \quad (2.10)
$$

Therefore (2.9) becomes

$$
u(x_0, t_0) = \int_{\psi_t(x_0, t_0)} \left[ -uA \text{grad } \Gamma(x_0, t_0; \cdot, \cdot) \right] \cdot N_\chi \, dH_n
$$

$$
+ (4\pi r^2)^{-n/2} \int_{\psi_t(x_0, t_0)} \left[ A(\text{grad } u) \cdot N_\chi - uN_\xi \right] \, dH_n. \quad (2.11)
$$

Now since $u$ is a solution of $Lu = 0$, (2.4) implies that the last addend in the right-hand side of (2.11) is zero. Hence we obtain

$$
u(x_0, t_0) = \int_{\psi_t(x_0, t_0)} \left[ -uA \text{grad } \Gamma(x_0, t_0; \cdot, \cdot) \right] \cdot N_\chi \, dH_n
$$

$$
= \int_{\psi_t(x_0, t_0)} u(x, t) Q(x_0, t_0; x, t) \, dH_n, \quad (2.12)
$$

where we have set $Q(x_0, t_0; x, t) = -A(\text{grad } \Gamma(x_0, t_0; x, t)) \cdot N_\chi$. Here

$$
N_\chi = -\frac{\text{grad } \Gamma(x_0, t_0; x, t)}{|\text{grad } \Gamma(x_0, t_0; x, t), D_1(\Gamma(x_0, t_0; x, t))|}.
$$

This completes the proof of Theorem 2.

It is interesting to explicate the expression for the surface kernel $Q$ in (2.12) in the case in which $L = A - D_1$, the heat operator. In this case $A \equiv $ Identity and

$$
\Gamma(x_0, t_0; x, t) = (4\pi(t_0 - t))^{-n/2} \exp \left[ -\frac{|x_0 - x|^2}{4(t_0 - t)} \right]
$$
for \( t < t_0 \), \( \Gamma(x_0, t_0; x, t) \equiv 0 \) for \( t \geq t_0 \). A computation gives

\[
Q(x_0, t_0; x, t) = \frac{|x_0 - x|^2 \Gamma(x_0 - x; t_0 - t)}{\sqrt{4(t_0 - t)^2 |x_0 - x|^2 + (|x_0 - x|^2 - 2n(t_0 - t))^2}}, \quad t < t_0,
\]

\[
Q(x_0, t_0) = 1 \quad \text{(to be defined as a limit).} \tag{2.13}
\]

Recalling that on \( \psi^r(x_0, t_0) \), \( \Gamma(x_0 - x, t_0 - t) = (4\pi r^2)^{-n/2} \), (2.12) gives

\[
u(x_0, t_0) = (4\pi r^2)^{-n/2} \int_{\psi^r(x_0, t_0)} u(x, t) \tag{2.14}
\]

\[
\times \frac{|x_0 - x|^2}{[4(t_0 - t)^2 |x_0 - x|^2 + (|x_0 - x|^2 - 2n(t_0 - t))^2]^{1/2}} \, dH_n.
\]

Formula (2.14) is well known; see [P1, P2] for the case \( n = 1 \) and [F, W1] for the case \( n \geq 2 \). For applications of (2.14) to parabolic potential theory see [P1, P2, P3, W1, W2, W3, K1, K2, K3, B] and, especially, the important paper [EG].

If we take \( u = 1 \) in (2.12) we obtain that for a.e. \( r > 0 \) the measure

\[
dv^{(x_0, t_0)}_r = Q(x_0, t_0; r, \cdot) \, dH_n
\]

\[
\frac{A(\nabla \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \nabla \Gamma(x_0, t_0; \cdot, \cdot)}{|(\nabla \Gamma(x_0, t_0; \cdot, \cdot), D, \Gamma(x_0, t_0; \cdot, \cdot))|} \, dH_n
\]

is a finite measure on \( \psi^r(x_0, t_0) \) whose total mass is one.

We now turn to the proof of Theorem 1. For a.e. \( l > 0 \) sufficiently small we write

\[
\nu(x_0, t_0) = \int_{\psi^r(x_0, t_0)} \left[ -uA(\nabla \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \mathbf{N}_x \right] \, dH_n \tag{2.15}
\]

where \( \psi^r(x_0, t_0) = \{(x, t) \in \mathbb{R}_+^{n+1} \mid \Gamma(x_0, t_0; x, t) = (4\pi l)^{-n/2}\} \). Multiplying both sides of (2.15) by \( l^{n/2 - 1} \) and integrating between 0 and \( r^2 \) we obtain

\[
u(x_0, t_0) = \frac{n}{2} r^{-n} \int_0^{r^2} l^{n/2 - 1}
\]

\[
\times dl \int_{\Gamma(x_0, t_0; \cdot, \cdot) = (4\pi l)^{-n/2}} \left[ -uA(\nabla \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \mathbf{N}_x \right] \, dH_n. \tag{2.16}
\]
We now make the change of variable \((4\pi r^2)^{-\eta/2} = c\). This gives

\[
u(x_0, t_0) = (4\pi r^2)^{-\eta/2} \int_{(4\pi r^2)^{-\eta/2}} c^{-2} \times dc \int_{\Gamma(x_0, t_0; \cdot, \cdot)} \left[ -uA(\nabla \Gamma(x_0, t_0; \cdot, \cdot) \cdot \mathbf{N}_x \right] dH_n
\]

\[
= (4\pi r^2)^{-\eta/2} \int_{(4\pi r^2)^{-\eta/2}} c^{-2} \times dc \int_{\Gamma(x_0, t_0; \cdot, \cdot)} \left[ -\frac{uA(\nabla \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \mathbf{N}_x}{\Gamma(x_0, t_0; \cdot, \cdot)^2} \right] dH_n, \tag{2.17}
\]

where in the last equality in (2.17) we have used the fact that the inner integration is performed on the set where \(\Gamma = c\). By Sard's theorem we know that for a.e. \(c > 0\) the spatial component of the outward normal to \(\{ (x, t) \in \mathbb{R}^{n+1}_+ \mid \Gamma(x_0, t_0, x, t) = c \} \) is

\[
\mathbf{N}_x = -\frac{\nabla \Gamma(x_0, t_0; \cdot, \cdot)}{|(\nabla \Gamma(x_0, t_0; \cdot, \cdot), D, \Gamma(x_0, t_0; \cdot, \cdot))|},
\]

Therefore (2.17) gives

\[
u(x_0, t_0) = (4\pi r^2)^{-\eta/2} \int_{(4\pi r^2)^{-\eta/2}} c^{-2} \times dc \int_{\Gamma(x_0, t_0; \cdot, \cdot)} A(\nabla \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \nabla \Gamma(x_0, t_0; \cdot, \cdot) \frac{\nabla \Gamma(x_0, t_0; \cdot, \cdot)}{\Gamma(x_0, t_0; \cdot, \cdot)^2 |(\nabla \Gamma(x_0, t_0; \cdot, \cdot), D, \Gamma(x_0, t_0; \cdot, \cdot))|} dH_n. \tag{2.18}
\]

Now we use the generalized co-area formula (see [Fe], Theorem 3.2.12, p. 249).

Let \(F \in L^1(\mathbb{R}^{n+1}), g \in \text{Lip}(\mathbb{R}^{n+1})\), then

\[
\int_{\mathbb{R}^{n+1}} F(x, t) |(\nabla g, D, g)| dx dt = \int_{-\infty}^{\infty} dx \left( \int_{g=x}^{\infty} F(y, s) dH_n \right), \tag{2.19}
\]

with obvious meaning of the symbols. Provided that on \(\{(x, t) \in \mathbb{R}^{n+1} \mid g(x, t) = a\}\) the vector \((\nabla g, D, g)\) a.e. does not vanish and that \(F/(|\nabla \Gamma, D, \Gamma)|) \in L^1(\mathbb{R}^{n+1})\) we can rewrite (2.19) in a way that is more suitable for our purposes:

\[
\int_{\mathbb{R}^{n+1}} F(x, t) dx dt = \int_{-\infty}^{\infty} da \left( \int_{g=a}^{\infty} \frac{F(y, s)}{|(\nabla g, D, g)|} dH_n \right). \tag{2.20}
\]
We now wish to apply (2.20) with \( g = \Gamma(x_0, t_0; \cdot, \cdot) \) and
\[
F = u \frac{A(\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \text{grad } \Gamma(x_0, t_0; \cdot, \cdot)}{\Gamma(x_0, t_0; \cdot, \cdot)^2} \chi_{\Omega,(x_0, t_0)}
\]
where \( \chi_{\Omega,(x_0, t_0)} \) is the indicator function of the parabolic ball \( \Omega,(x_0, t_0) \). To do this we must check that \( F/|\text{grad } \Gamma, D, \Gamma| \in L^1(\mathbb{R}^{n+1}), \) i.e., that
\[
u A(\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \text{grad } \Gamma(x_0, t_0; \cdot, \cdot) \quad \in L^1(\Omega,(x_0, t_0)).
\]
(2.21)

Since \( u \) is a solution of \( Lu = 0 \), it is bounded on \( \Omega,(x_0, t_0) \). On the other hand, using the ellipticity, the obvious observation that
\[
|\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)| \leq 1,
\]
and the fact that, on \( \Omega,(x_0, t_0) \), \( \Gamma(x_0, t_0; \cdot, \cdot) > (4\pi r^2)^{-n/2} \), we conclude that on \( \Omega,(x_0, t_0) \)
\[
A(\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)) \cdot \text{grad } \Gamma(x_0, t_0; \cdot, \cdot) \quad \leq \lambda^{-1} (4\pi r^2)^n |\text{grad } \Gamma(x_0, t_0; \cdot, \cdot)|.
\]
(2.22)

Equation (2.7) allows us to conclude that \( \text{grad } \Gamma(x_0, t_0; \cdot, \cdot) \) is in \( L^1(\Omega,(x_0, t_0)) \), hence (2.21) holds. We can thus use formula (2.20) in (2.18) obtaining
\[
u(x_0, t_0) = (4\pi r^2)^{-n/2} \int_{\Omega,(x_0, t_0)} u(x, t) K(x_0, t_0; x, t) \, dx \, dt.
\]
(2.23)

We rewrite (2.23) as
\[
u(x_0, t_0) = (4\pi r^2)^{-n/2} \int_{\Omega,(x_0, t_0)} u(x, t) K(x_0, t_0; x, t) \, dx \, dt,
\]
(2.24)
where
\[
K(x_0, t_0; x, t) = \frac{A(x, t)(\text{grad } \Gamma(x_0, t_0; x, t)) \cdot \text{grad } \Gamma(x_0, t_0; x, t)}{\Gamma(x_0, t_0; x, t)^2}.
\]
(2.25)
The proof of Theorem 1 is completed.

We conclude with some remarks. Setting \( u \equiv 1 \) in (2.24) gives for every \( r > 0 \) sufficiently small

\[
(4\pi r^2)^{-n/2} \int_{\Omega_r(x_0, t_0)} K(x_0, t_0; x, t) \, dx \, dt = 1. \tag{2.26}
\]

This incidentally implies, using the ellipticity, that

\[
\text{grad} \log I(x_0, t_0; \cdot, \cdot) \in L^2(\Omega_r(x_0, t_0)), \tag{2.27}
\]

with a bound on the \( L^2 \) norm depending only on the bounds for the matrix \( A(x, t) \).

In the case in which \( L = A - D_t \), one can compute (2.25) explicitly. Since

\[
\text{grad} I(x_0 - x, t_0 - t) = -((x_0 - x)/2(t_0 - t)) I(x_0 - x, t_0 - t),
\]

we have

\[
K(x_0, t_0; x, t) = \frac{1}{4}(|x_0 - x|^2/(t_0 - t)^2). \tag{2.25}
\]

Equation (2.25) takes the form

\[
u(x_0, t_0) = 4^{-1} (4\pi r^2)^{-n/2} \int_{\Omega_r(x_0, t_0)} u(x, t) \frac{|x_0 - x|^2}{(t_0 - t)^2} \, dx \, dt, \tag{2.28}
\]

which is precisely the formula found by Watson [W1].

**References**


