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On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points ☆

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Abstract

In the present paper, the authors introduce two new subclasses $\mathcal{S}_s^{(k)}(\phi)$ of close-to-convex functions and $\mathcal{C}_s^{(k)}(\phi)$ of quasi-convex functions. The integral representations for functions belonging to these classes are provided, the convolution conditions, growth theorems, distortion theorems and covering theorems for these classes are also provided. The results obtained generalize some known results, and some other new results are obtained.

Keywords: Close-to-convex functions; Quasi-convex functions; Differential subordination; Hadamard product (or convolution); k-Symmetric points

1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk

$$\mathcal{U}=\big\{z\colon\ z\in C \text{ and } |z|<1\big\}.$$

Let $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathcal{U}$. Also let $\mathcal{P}$ denote the class of functions of the form

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n \quad (z\in\mathcal{U}),$$

which satisfy the condition $\Re\{p(z)\}>0$.

We denote by $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Thus, by definition, we have (see, for details, [1,2]; see also [3,4])

$$\mathcal{S}^*=\left\{f\colon\ f\in\mathcal{A} \text{ and } \Re\left\{\frac{zf'(z)}{f(z)}\right\}>0\ (z\in\mathcal{U})\right\},$$
$$\mathcal{K}=\left\{f\colon\ f\in\mathcal{A} \text{ and } \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\ (z\in\mathcal{U})\right\},$$
$$\mathcal{C}=\left\{f\colon\ f\in\mathcal{A},\ g\in\mathcal{S}^*, \text{ and } \Re\left\{\frac{zf'(z)}{g(z)}\right\}>0\ (z\in\mathcal{U})\right\},$$

and

$$\mathcal{C}^*=\left\{f\colon\ f\in\mathcal{A},\ g\in\mathcal{K}, \text{ and } \Re\left\{\frac{(zf'(z))'}{g'(z)}\right\}>0\ (z\in\mathcal{U})\right\}.$$

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)|\leqslant|z|$ and $f(z)=F(\omega(z))$, denoted by $f\prec F$ or $f(z)\prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0)=F(0)$ and $f(\mathcal{U})\subset F(\mathcal{U})$ (see [5]).

A function $f(z)\in\mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if

$$\frac{zf'(z)}{f(z)}\prec\phi(z)\quad (z\in\mathcal{U}),$$

where $\phi(z)\in\mathcal{P}$. The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [6]. And the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$.

Sakaguchi [7] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, which consists of functions $f(z)\in\mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0\quad (z\in\mathcal{U}).$$

Following him, many authors discussed this class and its subclasses (see [8–16]). Motivated by $\mathcal{S}_s^*$, we can easily obtain the following class $\mathcal{C}_s^*$ of functions convex with respect to symmetric points.

Definition 1. Let $\mathcal{C}_s^*$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{(zf'(z))'}{f'(z)+f'(-z)}\right\}>0\quad (z\in\mathcal{U}).$$

Now, we introduce the following classes of analytic functions with respect to k-symmetric points and obtain some interesting results.

Definition 2. Let $\mathcal{S}_s^{(k)}(\phi)$ denote the class of functions in $\mathcal{S}$ satisfying the condition

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$, $k \geqslant 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality:

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f\left(\varepsilon^{\nu} z\right) \quad \left(\varepsilon^k = 1\right). \tag{1.2}$$

If $k = 2$ and $\phi(z) = (1+z)/(1-z)$, then the class $\mathcal{S}_s^{(k)}(\phi)$ reduces to the class $\mathcal{S}_s^*$. If $k = 2$, then the class $\mathcal{S}_s^{(k)}(\phi)$ reduces to the class $\mathcal{S}_s^*(\phi)$, which was introduced and investigated recently by Ravichandran [14]. If $\phi(z) = (1+\beta z)/(1-\alpha\beta z)$ $(0 \leqslant \alpha \leqslant 1,\ 0 < \beta \leqslant 1)$, then the class $\mathcal{S}_s^{(k)}(\phi)$ reduces to the class $\mathcal{S}_s^{(k)}[\alpha, \beta]$, which was considered recently by Gao and Zhou [15].

Definition 3. Let $\mathcal{C}_s^{(k)}(\phi)$ denote the class of functions in $\mathcal{S}$ satisfying the condition

$$\frac{(zf'(z))'}{f_k'(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$, $k \geqslant 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2).

If $k = 2$ and $\phi(z) = (1+z)/(1-z)$, then the class $\mathcal{C}_s^{(k)}(\phi)$ reduces to the class $\mathcal{C}_s^*$. If $k = 2$, then the class $\mathcal{C}_s^{(k)}(\phi)$ reduces to the class $\mathcal{C}_s^*(\phi)$, which was also introduced and investigated recently by Ravichandran [14].

In our proposed investigation of functions in the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$, we shall also make use of the following definition.

Definition 4 *(Hadamard product or convolution).* Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

In the present paper, first we prove that the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$ are subclasses of the class of close-to-convex functions and the class of quasi-convex functions, respectively. Then we provide the integral representations for functions belonging to these classes. At last, we provide the convolution conditions, growth theorems, distortion theorems and covering theorems for these classes. The results obtained generalize some known results and some other new results are obtained.

2. Integral representations

First we give two meaningful conclusions about the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$.

Theorem 1. *Let $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, then $f_k(z) \in \mathcal{K} \subset \mathcal{S}$.*

Proof. Suppose that $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, from the definition of $\mathcal{C}_s^{(k)}(\phi)$ we can get

$$\Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}) \tag{2.1}$$

since $\Re\{\phi(z)\} > 0$. Substituting z by $\varepsilon^\mu z$ in (2.1) respectively ($\mu = 0, 1, 2, \ldots, k-1$; $\varepsilon^k = 1$), then (2.1) is also true, that is,

$$\Re\left\{\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(\varepsilon^\mu z)}\right\} > 0 \quad (z \in \mathcal{U};\ \mu = 0, 1, 2, \ldots, k-1). \tag{2.2}$$

According to the definition of $f_k(z)$ and $\varepsilon^k = 1$, we know $f_k'(\varepsilon^\mu z) = f_k'(z)$. Then inequality (2.2) becomes

$$\Re\left\{\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.3}$$

Let $\mu = 0, 1, 2, \ldots, k-1$ in (2.3) respectively, and summing them we can get

$$\Re\left\{\frac{\sum_{\mu=0}^{k-1} f'(\varepsilon^\mu z) + z\sum_{\mu=0}^{k-1} \varepsilon^\mu f''(\varepsilon^\mu z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

or equivalently,

$$\Re\left\{\frac{f_k'(z) + z f_k''(z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

that is $f_k(z) \in \mathcal{K} \subset \mathcal{S}$. □

Remark 1. From Theorem 1 and inequality (2.1), we know that if $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, then $f(z)$ is a quasi-convex function. So $\mathcal{C}_s^{(k)}(\phi)$ is a subclass of the class $\mathcal{C}^*$ of quasi-convex functions.

By applying similar method as in Theorem 1, we have

Theorem 2. *Let $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, then $f_k(z) \in \mathcal{S}^* \subset \mathcal{S}$.*

Remark 2. From Theorem 2 and the definition of $\mathcal{S}_s^{(k)}(\phi)$, we know that if $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, then $f(z)$ is a close-to-convex function. So $\mathcal{S}_s^{(k)}(\phi)$ is a subclass of the class $\mathcal{C}$ of close-to-convex functions.

In particular, if $\phi(z) = (1+\beta z)/(1-\alpha\beta z)$, then the following results of Gao and Zhou [15] are obtained as special case of Theorem 2.

Corollary 1. *Let $f(z) \in \mathcal{S}_s^{(k)}[\alpha, \beta]$, then $f_k(z) \in \mathcal{S}^* \subset \mathcal{S}$.*

Now, we give the integral representations of functions belonging to the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$.

Theorem 3. *Let $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, then we have*

$$f_k(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{\phi(\omega(t)) - 1}{t}\,dt\right\} d\zeta, \tag{2.4}$$

where $f_k(z)$ is defined by equality (1.2)*, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.*

Proof. Suppose that $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, from the definition of $\mathcal{C}_s^{(k)}(\phi)$ we have

$$\frac{(zf'(z))'}{f_k'(z)} = \phi\big(\omega(z)\big), \tag{2.5}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting z by $\varepsilon^\mu z$ in (2.5) respectively ($\mu = 0, 1, 2, \ldots, k-1$; $\varepsilon^k = 1$), we have

$$\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(\varepsilon^\mu z)} = \phi\big(\omega\big(\varepsilon^\mu z\big)\big) \quad (\mu = 0, 1, 2, \ldots, k-1). \tag{2.6}$$

It is easy to know that $f_k'(\varepsilon^\mu z) = f_k'(z)$, summing (2.6) we can obtain

$$\frac{(zf_k'(z))'}{f_k'(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1}\phi\big(\omega\big(\varepsilon^\mu z\big)\big), \tag{2.7}$$

from equality (2.7) we get

$$\frac{(zf_k'(z))'}{zf_k'(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\mu=0}^{k-1}\frac{\phi(\omega(\varepsilon^\mu z)) - 1}{z}. \tag{2.8}$$

Integrating equality (2.8) we have

$$\log\big\{f_k'(z)\big\} = \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z \frac{\phi(\omega(\varepsilon^\mu \zeta)) - 1}{\zeta}\,d\zeta = \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{\phi(\omega(t)) - 1}{t}\,dt,$$

that is,

$$f_k'(z) = \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{\phi(\omega(t)) - 1}{t}\,dt\right\}. \tag{2.9}$$

Therefore, integrating equality (2.9) we can obtain equality (2.4). □

Theorem 4. *Let $f(z) \in \mathcal{C}_s^{(k)}(\phi)$, then we have*

$$f(z) = \int_0^z \frac{1}{\xi}\int_0^\xi \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{\phi(\omega(t)) - 1}{t}\,dt\right\}\cdot\phi\big(\omega(\zeta)\big)\,d\zeta\,d\xi, \tag{2.10}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{C}_s^{(k)}(\phi)$; from equalities (2.5) and (2.9) we have

$$\big(zf'(z)\big)' = f_k'(z) \cdot \phi\big(\omega(z)\big) = \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{\phi(\omega(t))-1}{t}\,dt\right\} \cdot \phi\big(\omega(z)\big). \tag{2.11}$$

Integrating equality (2.11), we can obtain

$$f'(z) = \frac{1}{z}\int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{\phi(\omega(t))-1}{t}\,dt\right\} \cdot \phi\big(\omega(\zeta)\big)\,d\zeta. \tag{2.12}$$

Therefore, integrating equality (2.12) we can obtain equality (2.10). □

By applying similar method as in Theorem 3, we have

Theorem 5. *Let $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, then we have*

$$f_k(z) = z \cdot \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{\phi(\omega(t))-1}{t}\,dt\right\},$$

where $f_k(z)$ is defined by equality (1.2)*, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.*

Corollary 2. [15] *Let $f(z) \in \mathcal{S}_s^{(k)}[\alpha, \beta]$, then we have*

$$f_k(z) = z \cdot \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))}\,dt\right\},$$

where $f_k(z)$ is defined by equality (1.2)*, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.*

By applying similar method as in Theorem 4, we have

Theorem 6. *Let $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, then we have*

$$f(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{\phi(\omega(t))-1}{t}\,dt\right\} \cdot \phi\big(\omega(\zeta)\big)\,d\zeta,$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Corollary 3. [15] *Let $f(z) \in \mathcal{S}_s^{(k)}[\alpha, \beta]$, then we have*

$$f(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))}\,dt\right\} \cdot \frac{1+\beta\omega(\zeta)}{1-\alpha\beta\omega(\zeta)}\,d\zeta,$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

3. Convolution conditions

In this section, we provide the convolution conditions for the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$.

Theorem 7. *Let $f(z) \in \mathcal{A}$ and $\phi(z) \in \mathcal{P}$, then $f(z) \in \mathcal{S}_s^{(k)}(\phi)$ if and only if*

$$\frac{1}{z}\left[f * \left(\frac{z}{(1-z)^2} - \phi\big(e^{i\theta}\big)h(z)\right)\right] \neq 0 \tag{3.1}$$

for all $z \in \mathcal{U}$ and $0 \leqslant \theta < 2\pi$, where $h(z)$ is given by (3.6).

Proof. Suppose that $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, since

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z)$$

if and only if

$$\frac{zf'(z)}{f_k(z)} \neq \phi\big(e^{i\theta}\big) \tag{3.2}$$

for all $z \in \mathcal{U}$ and $0 \leqslant \theta < 2\pi$. It is easy to know that the condition (3.2) can be written as

$$\frac{1}{z}\big(zf'(z) - f_k(z)\phi\big(e^{i\theta}\big)\big) \neq 0. \tag{3.3}$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{3.4}$$

And from the definition of $f_k(z)$, we know

$$f_k(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (f * h)(z), \tag{3.5}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{3.6}$$

for

$$c_n = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases}$$

Substituting (3.4) and (3.5) into (3.3), we can get (3.1). This completes the proof of Theorem 7. □

By applying similar method as in Theorem 7, we have

Theorem 8. *Let $f(z) \in \mathcal{A}$ and $\phi(z) \in \mathcal{P}$, then $f(z) \in \mathcal{C}_s^{(k)}(\phi)$ if and only if*

$$\frac{1}{z}\left\{f * \left[z\left(\frac{z}{(1-z)^2} - \phi\big(e^{i\theta}\big)h(z)\right)'\right]\right\} \neq 0$$

for all $z \in \mathcal{U}$ and $0 \leqslant \theta < 2\pi$, where $h(z)$ is given by (3.6).

4. Growth, distortion and covering theorems

Finally, we provide the growth, distortion and covering theorems for the classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$. For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\mathcal{U}$, $\phi(\mathcal{U})$ is convex and symmetric with respect to the real axis, $\phi(0)=1$ and $\phi'(0)>0$. The functions $k_{\phi n}(z)$ $(n=2,3,\ldots)$ defined by $k_{\phi n}(0)=k'_{\phi n}(0)-1=0$ and

$$1+\frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)}=\phi\left(z^{n-1}\right)$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}(z)$ satisfying $h_{\phi n}(z)=zk'_{\phi n}(z)$ are examples of functions in $\mathcal{S}^*(\phi)$. Write $k_{\phi 2}(z)$ simply as $k_\phi(z)$ and $h_{\phi 2}(z)$ simply as $h_\phi(z)$.

In order to prove our next theorem, we shall require the following lemma.

Lemma 1. [17] *Let $f(z)=z+a_{k+1}z^{k+1}+\cdots\in\mathcal{K}(\phi)$, then we have*

$$\left[k'_\phi\left(-r^k\right)\right]^{1/k}\leqslant\left|f'(z)\right|\leqslant\left[k'_\phi\left(r^k\right)\right]^{1/k}.$$

Now we give the following theorem.

Theorem 9. *Let $\min_{|z|=r}|\phi(z)|=\phi(-r)$, $\max_{|z|=r}|\phi(z)|=\phi(r)$, $|z|=r<1$. If $f(z)\in\mathcal{C}_s^{(k)}(\phi)$, then we have*

$$\frac{1}{r}\int_0^r\phi(-t)\left[k'_\phi\left(-t^k\right)\right]^{1/k}dt\leqslant\left|f'(z)\right|\leqslant\frac{1}{r}\int_0^r\phi(t)\left[k'_\phi\left(t^k\right)\right]^{1/k}dt, \tag{4.1}$$

$$\int_0^r\frac{1}{s}\int_0^s\phi(-t)\left[k'_\phi\left(-t^k\right)\right]^{1/k}dt\,ds\leqslant\left|f(z)\right|\leqslant\int_0^r\frac{1}{s}\int_0^s\phi(t)\left[k'_\phi\left(t^k\right)\right]^{1/k}dt\,ds, \tag{4.2}$$

and

$$f(\mathcal{U})\supset\left\{\omega\colon\ |\omega|\leqslant\int_0^1\frac{1}{s}\int_0^s\phi(-t)\left[k'_\phi\left(-t^k\right)\right]^{1/k}dt\,ds\right\}. \tag{4.3}$$

These results are sharp.

Proof. Suppose that $f(z)\in\mathcal{C}_s^{(k)}(\phi)$, and $\phi(z)$ is convex and symmetric with respect to the real axis; it follows that

$$\begin{aligned}f_k(z)&=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu}f\left(\varepsilon^\nu z\right)=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu}\left[\varepsilon^\nu z+\sum_{n=2}^{\infty}a_n\left(\varepsilon^\nu z\right)^n\right]\\&=z+\sum_{l=1}^{\infty}a_{lk+1}z^{lk+1}\in\mathcal{K}(\phi).\end{aligned}$$

Thus, by Lemma 1, we have

$$\left[k'_\phi\left(-r^k\right)\right]^{1/k}\leqslant\left|f'_k(z)\right|\leqslant\left[k'_\phi\left(r^k\right)\right]^{1/k}.$$

Now, for $|z| = r < 1$, we have

$$\phi(-r)\big[k'_\phi\big(-r^k\big)\big]^{1/k} \leqslant \big|\big(zf'(z)\big)'\big| = \left|\frac{(zf'(z))'}{f'_k(z)} \cdot f'_k(z)\right| \leqslant \phi(r)\big[k'_\phi\big(r^k\big)\big]^{1/k}. \tag{4.4}$$

By integrating (4.4) from 0 to r, we can get (4.1). (4.2) follows form (4.1). And (4.3) follows form (4.2), since

$$\int_0^r \frac{1}{s} \int_0^s \phi(-t)\big[k'_\phi\big(-t^k\big)\big]^{1/k}\, dt\, ds$$

is increasing in (0, 1) and bounded by 1.

The results are sharp for the function

$$f(z) = \int_0^z \frac{1}{s} \int_0^s \phi(-t)\big[k'_\phi\big(-t^k\big)\big]^{1/k}\, dt\, ds \in \mathcal{C}_s^{(k)}(\phi),$$

since it has real coefficients and is in $\mathcal{K}(\phi)$. □

The proof of Theorem 10 below is much akin to that of Theorem 7 in [14], here we omit the details.

Theorem 10. *Let* $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r < 1$. *If* $f(z) \in \mathcal{S}_s^{(k)}(\phi)$, *then we have*

$$h'_\phi(-r) \leqslant \big|f'(z)\big| \leqslant h'_\phi(r), \qquad -h_\phi(-r) \leqslant \big|f(z)\big| \leqslant h_\phi(r),$$

and

$$f(\mathcal{U}) \supset \big\{\omega\colon\ |\omega| \leqslant -h(-1)\big\}.$$

These results are sharp.

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