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Radiation Flux from a Slab or Sphere

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The major objective of this note is the explicit theoretical prediction of the rate of radiative energy loss through the boundaries of an absorbing-emitting medium of finite thickness. An arbitrary distribution of internal sources is assumed known and the incident radiation on the boundaries is either zero or uniform and isotropic. A close parallelism appears in the development for planar and spherical geometries provided the latter case is limited to a single homogeneous sphere with uniform extinction coefficient. This parallelism has been noted by Davison [1] for neutron transport, and was used by Heaslet and Warming [2] to predict radiative transfer within a homogeneous sphere with a uniform distribution of internal sources.

Two methods of calculating surface flux are given. First, Green's formula for the governing integral equations is applied to the two geometries and leads in either case to predictions involving weighted quadratures of the internal source distribution. Second, an alternative evaluation in which moments of the Chandrasekhar-Ambartsumian [3] X and Y functions are used is applied to particular source distributions. Sobouti's [4] tables of the moment functions then yield accurate numerical values of flux for these special cases. The flux relations contrast in an interesting manner: the moment functions are weighted means over an angular variation at the boundaries whereas Green's formula provides an averaging over the entire optical path length through the medium.

The main contribution here is one of assembling predictions of physical interest and of demonstrating the interplay between results for the two geometries. The formulas, being exact, can of course be derived in various ways or, in retrospect, can perhaps be seen to appear implicitly in other investigations. This is true especially for the planar problem where invariant embedding techniques and photon diffusion processes have been studied in considerable detail. Particular attention should be directed to Sobolev's treatise [5] and papers by Horak [6], Horak and Lundquist [7], Bellman, Kalaba, and Ueno [8], and Ueno [9].

The presentation starts with the integral equation formulation for the source function $\mathfrak{J}$ of radiative transfer theory. The analysis is slanted toward the interests of continuum-radiation theory and the needs of the thermal engineer but, as will become apparent, the results apply also to problems involving spectral line emission and isotropic scattering. Extensions of Green's formula to more complex geometries will, at the same time, be apparent. The notation is in essential agreement with that of Chandrasekhar [3] and Kourganoff [10].

A medium with local volumetric extinction coefficient $\alpha(x)$ is confined between two planar boundaries at $x = 0$ $x = L$. At the boundaries or walls the incident radiation is uniform and isotropic. Optical length τ normal to the boundaries is introduced where $d\tau = \alpha(x)\,dx$ and $\tau = 0$ at $x = 0$ and $\tau = \tau_0$ at $x = L$. The rate of internal energy release per unit volume has the specified variation $P(\tau)$. The source function $\mathfrak{J}(\tau)$ satisfies the integral equation [3]

$$\mathfrak{J}(\tau) = \frac{P}{4\pi\alpha} + \frac{\omega q_1^+}{2\pi} E_2(\tau) + \frac{\omega q_2^-}{2\pi} E_2(\tau_0 - \tau) + \frac{\omega}{2}\int_0^{\tau_0} \mathfrak{J}(\tau_1)\, E_1(|\tau - \tau_1|)\, d\tau_1 . \tag{1}$$

In Eq. (1), ω is a parameter to be defined more specifically below. The constants q_1^+ and q_2^- are, respectively, the imposed flux (rate of energy transport per unit area) *into* the medium from the left and right boundaries; $E_n(x)$ is the exponential integral function of index n, defined as

$$E_n(x) = \int_1^\infty \frac{e^{-xx_1}}{x_1^n}\, dx_1 = \int_0^1 e^{-x/\mu} \mu^{n-2}\, d\mu.$$

Net flux normal to the boundaries and through a unit area at point τ is

$$q(\tau) = q^+(\tau) - q^-(\tau) = 2q_1^+ E_3(\tau) - 2q_2^- E_3(\tau_0 - \tau) + 2\pi \int_0^{\tau_0} \mathfrak{J}(\tau_1)\, \operatorname{sgn}(\tau - \tau_1)\, E_2(|\tau - \tau_1|)\, d\tau_1 . \tag{2}$$

Equations (1) and (2) are consistent with the differential relation

$$\omega \frac{dq}{d\tau} = \frac{P}{\alpha} - 4\pi(1 - \omega)\,\mathfrak{J}. \tag{3}$$

The sign convention for total flux is chosen such that $q(\tau) > 0$ when the net energy transport is in the positive x (or τ) direction.

The dependence on radiation frequency ν has been suppressed in the above expressions. This offers no difficulty in interpretation in at least three problems of physical interest:

(a) Transport of thermal radiation in the continuum regime and through a "gray" medium between heated walls. Here, $\omega = 1$, P/α is assumed known, $\mathfrak{J} = \sigma T^4/\pi$ where T is local temperature and σ is the Stefan-Boltzmann constant.

(b) Monochromatic, isotropic scattering where P/α is known and $\omega = \omega_0$ is the albedo of the medium.

(c) Emission from a spectral line with a rectangular profile in an isothermal environment of temperature T_0 (see, e.g., Cuperman, Engelmann, and Oxenius [11], or Sobolev [5]). Here, $0 < \omega < 1$, $P/\alpha = 4\pi(1 - \omega) B_\nu(T_0)$ and $B_\nu(T_0)$ is Planck's function for black-body radiation.

Since Eq. (1) depends linearly on the source function, the analysis can be reduced to the solution of component equations. Thus, introducing the integral operator $\Lambda_{\tau,\tau_0}[f(\tau_1)]$ where

$$\Lambda_{\tau,\tau_0}[f(\tau_1)] \equiv f(\tau) - \frac{\omega}{2} \int_0^{\tau_0} f(\tau_1) E_1(|\tau - \tau_1|)\, d\tau_1 \tag{4}$$

and the transformation

$$\pi\mathfrak{J}(\tau) = q_1^+ + (q_2^- - q_1^+)\, \Theta(\tau) + A\Theta_P(\tau) - 4q_1^+(1 - \omega)\, \Theta_U(\tau) \tag{5}$$

where A is an appropriate nondimensionalizing constant, one gets the integral equations

$$\Lambda_{\tau,\tau_0}[\Theta(\tau_1)] = \frac{\omega}{2} E_2(\tau_0 - \tau) \tag{6a}$$

$$\Lambda_{\tau,\tau_0}[\Theta_P(\tau_1)] = \frac{P}{4\alpha A} \tag{6b}$$

$$\Lambda_{\tau,\tau_0}[\Theta_U(\tau_1)] = \tfrac{1}{4}. \tag{6c}$$

The function $\Theta_U(\tau)$ is associated with a uniform distribution of internal sources, that is, $P/\alpha = A =$ const. In the study of thermal radiation or perfect scattering $\omega = 1$ and in this so-called conservative case $\Theta_U(\tau)$ and $\Theta(\tau) - \frac{1}{2}$ are, respectively, even and odd functions about $\tau = \tau_0/2$. The calculation of these functions can therefore be limited to a half portion of the thickness range. Such results have been given, for example, by Usiskin and Sparrow [12] and by Heaslet and Warming [13]. When $\omega \neq 1$ the non-conservative nature of the radiation field destroys the asymmetry of the component function and $\Theta_U(\tau)$ can, in fact, be determined from a knowledge of $\Theta(\tau)$ over the full thickness range. Thus, the identity

$$\Theta(\tau) + \Theta(\tau_0 - \tau) = 1 - 4(1 - \omega)\, \Theta_U(\tau) \tag{7}$$

shows that the source function for uniform internal energy release with no incident radiation at the walls can be extracted from the solution for $\Theta(\tau)$

alone. The fragmentation of $\mathfrak{J}(\tau)$ introduced in Eq. (5) is, however, well adapted to the conservative case and, as will be shown, yields useful results in general.

Once Eqs. (6) are solved, $\mathfrak{J}(\tau)$ is given by Eq. (5) and the flux $q(\tau)$ can then be predicted by a simple quadrature of equation (3) provided total flux at one boundary is known. But from Eqs. (2) and (5), flux at the right boundary is

$$q(\tau_0) = (q_2^- - q_1^+) Q + AQ_P - 4q_1^+(1 - \omega) Q_U \tag{8a}$$

where

$$Q = -1 + 2 \int_0^{\tau_0} \Theta(\tau) E_2(\tau_0 - \tau) \, d\tau \tag{8b}$$

$$Q_P = 2 \int_0^{\tau_0} \Theta_P(\tau) E_2(\tau_0 - \tau) \, d\tau \tag{8c}$$

$$Q_U = 2 \int_0^{\tau_0} \Theta_U(\tau) E_2(\tau_0 - \tau) \, d\tau. \tag{8d}$$

Total wall flux can actually be calculated from a knowledge of $\Theta(\tau)$ alone for arbitrary variations of P/α. To show this, consider Green's formula for the operator Λ (see, e.g., Tricomi [14], p. 53)

$$\int_0^{\tau_0} \Theta_P(\tau) \Lambda_{\tau,\tau_0}[\Theta(\tau_1)] \, d\tau = \int_0^{\tau_0} \Theta(\tau) \Lambda_{\tau,\tau_0}[\Theta_P(\tau_1)] \, d\tau. \tag{9}$$

From Eqs. (6), (8), and (9)

$$Q_P = \frac{1}{\omega} \int_0^{\tau_0} \frac{P}{\alpha A} \Theta(\tau) \, d\tau \tag{10}$$

so that all flux components are now given by proper averages of $\Theta(\tau)$. When $\omega = 1$ and, in addition P/α is symmetric about the mid-plane, the oddness of $\Theta(\tau)$ about $\tau = \tau_0/2$, $\Theta = \frac{1}{2}$ reduces equation (10) to the intuitively obvious result that the flux component AQ_P at the boundary is one-half the rate of energy production of the internal sources.

Alternative evaluations will next be given for a linear variation of internal sources $P/\alpha A = a + b\tau$. These results follow from the application of the invariance principles of radiation theory and are expressed in terms of the moment functions α_n and β_n defined as follows

$$\alpha_n(\tau_0, \omega) = \int_0^1 X(\tau_0, \omega, \mu) \mu^n \, d\mu, \qquad \beta_n(\tau_0, \omega) = \int_0^1 Y(\tau_0, \omega, \mu) \mu^n \, d\mu \tag{11}$$

Sobouti [4] has tabulated these functions for $n = 0, 1, 2$ and $0 < \tau_0 \leqslant 3$. Extensions of these tables to greater optical thicknesses are not difficult if

Sobolev's [15] asymptotic forms of X and Y are used. The predictions, in closed form, are a direct consequence of the methods discussed in [3] and [5]. The somewhat lengthy algebra leads to the final expressions

$$Q = \omega(\alpha_0\alpha_1 - \beta_0\beta_1) - 2\alpha_1, \qquad \omega \leqslant 1$$

$$Q = -\beta_0(\alpha_1 + \beta_1), \qquad \omega = 1$$

$$Q_P = \frac{a}{2}\frac{\alpha_1 - \beta_1}{1 - \frac{\omega}{2}(\alpha_0 - \beta_0)} + \frac{b}{2}\left\{\frac{\beta_2 - \alpha_2 - \tau_0\beta_1}{1 - \frac{\omega}{2}(\alpha_0 - \beta_0)} + \frac{\alpha_1 + \beta_1}{1 - \omega}\left[\tau_0\left(1 - \frac{\omega}{2}\alpha_0\right) - \frac{\omega}{2}(\alpha_1 - \beta_1)\right]\right\}, \qquad \omega \neq 1$$

$$Q_P = \frac{a}{2}\tau_0 + \frac{b}{2}\left\{\frac{\tau_0^2}{2} - \frac{\alpha_2 - \beta_2}{\beta_0} + \frac{3}{4}(\alpha_1 + \beta_1)\left[2(\alpha_3 - \beta_3) + \tau_0(\alpha_2 - \beta_2) + \frac{\tau_0^2}{6}(\alpha_1 - \beta_1)\right]\right\}, \qquad \omega = 1. \qquad (12)$$

The value of the source function at $\tau = \tau_0$ can also be expressed in terms of the moment functions. In the analysis of thermal radiation between heated walls, the so-called "temperature slip" at the walls follows directly from such formulas. The desired relations are

$$\Theta(\tau_0) = \frac{\omega}{2}\alpha_0; \qquad \Theta(0) = \frac{\omega}{2}\beta_0, \qquad \omega \leqslant 1$$

$$\Theta(\tau_0) + \Theta(0) = 1, \qquad \omega = 1$$

$$\Theta_P(\tau_0) = \left(a + \frac{b}{2}\tau_0\right)\frac{1}{[2 - \omega(\alpha_0 - \beta_0)]} - \Theta_P(0), \qquad \omega \leqslant 1$$

$$= \frac{a}{2}\frac{1}{[2 - \omega(\alpha_0 - \beta_0)]} + \frac{b}{4(1 - \omega)}\left[\tau_0\left(1 - \frac{\omega}{2}\alpha_0\right) - \frac{\omega}{2}(\alpha_1 - \beta_1)\right], \qquad \omega \neq 1$$

$$\Theta_P(\tau_0) = \frac{a}{4\beta_0} + \frac{b}{4}\left\{\frac{\tau_0}{2\beta_0} + \frac{3}{4}\left[2(\alpha_3 - \beta_3) + \tau_0(\alpha_2 - \beta_2) + \frac{\tau_0^2}{6}(\alpha_1 - \beta_1)\right]\right\}, \qquad \omega = 1. \qquad (13)$$

HOMOGENEOUS SPHERE. Attention is limited here to a single sphere with constant α but with an arbitrary radial distribution of internal sources. Let ρ be optical length where $\rho = \alpha r$ and r is radial distance, $0 \leqslant r \leqslant R$. If the incident radiation at the single boundary is isotropic, $P = P(\rho)$ is the volumetric rate of internal energy release, and $\rho_0 = \alpha R$, the integral equation for the source function $\mathfrak{J}(\rho)$ may be written in operational form

$$\chi_{\rho,\rho_0}[\rho_1 \mathfrak{J}(\rho_1)] = \frac{\rho P}{4\pi\alpha} + \frac{\omega q_2^-}{2\pi} H(\rho, \rho_0) \tag{14}$$

where the χ operator is defined as

$$\chi_{\rho,\rho_0}[f(\rho_1)] \equiv f(\rho) - \frac{\omega}{2} \int_0^{\rho_0} f(\rho_1)\,[E_1(|\,\rho - \rho_1\,|) - E_1(|\,\rho + \rho_1\,|)]\, d\rho_1 \tag{15}$$

and

$$H(\rho, \rho_0) = \rho_0 E_2(\rho_0 - \rho) + E_3(\rho_0 - \rho) - \rho_0 E_2(\rho_0 + \rho) - E_3(\rho_0 + \rho). \tag{16}$$

The imposed isotropic flux into the medium at the boundary is q_2^-.

Net flux at the optical radius ρ satisfies the differential relation

$$\omega \frac{d(\rho^2 q)}{d\rho} = \frac{\rho^2 P}{\alpha} - 4\pi(1 - \omega)\,\rho^2 \mathfrak{J}. \tag{17}$$

Once $\mathfrak{J}$ is determined, flux can be calculated through integration since it vanishes at $\rho = 0$ and a boundary condition is therefore known. Flux at the boundary can also be expressed by means of the integral form

$$\rho_0^2 q(\rho_0) = \frac{q_2^-}{2}\,[1 - 2\rho_0^2 - e^{-2\rho_0}(1 + 2\rho_0)] + 2\pi \int_0^{\rho_0} \rho \mathfrak{J}(\rho)\, H(\rho, \rho_0)\, d\rho \tag{18}$$

The fragmentation of Eqs. (14) and (18) can be carried out in an analogous manner to that used for the slab in Eqs. (5) and (8a). The absence of an inner boundary introduces simplifications, moreover, since only incident radiation flux q_2^- at the outer boundary is specified and q_1^+ is removed from the problem. The following relations are therefore introduced

$$\pi \mathfrak{J}(\rho) = q_2^- \Omega(\rho) + A \Omega_P(\rho) \tag{19}$$

$$\rho_0^2 q(\rho_0) = q_2^- Q + A Q_P \tag{20}$$

From Eqs. (14) and (19) one has

$$\chi_{\rho,\rho_0}[\rho_1 \Omega(\rho_1)] = \frac{\omega}{2} H(\rho, \rho_0) \tag{21a}$$

$$\chi_{\rho,\rho_0}[\rho_1 \Omega_P(\rho_1)] = \frac{\rho P}{4\alpha A} \tag{21b}$$

In a recent paper Kuznetsov [16] has derived equivalent equations for $\omega = 1$ and nonconstant α. His paper also gives the conservative forms of

Eqs. (21) but does not note the important simplification that can be used to eliminate $H(\rho, \rho_0)$ as the forcing term in the calculation of $\rho\Omega(\rho)$. Direct calculation does, in fact, establish the equality

$$\Omega(\rho) = 1 - 4(1 - \omega)\,\Omega_U(\rho) \tag{22}$$

where Ω_U is Ω_P when $P/\alpha = A = \text{const}$. For the conservative case Eq. (22) reduces to $\Omega(\rho) = 1$. When $\omega = 1$ and no internal sources are present one knows that uniform incident radiation on the single boundary of a closed region can produce only a uniform effect within the region if equilibrium conditions are maintained. One of Kuznetsov's two integral equations thus has a trivial solution, that is, the unknown function is a constant.

An analytic advantage does, however, accrue from the dual equations (21). Thus, invoking Green's formula for the operator χ and applying the formula to the functions $\rho\Omega$ and $\rho\Omega_P$, Eqs. (18), (20), and (22) permit one to establish the following formula

$$\begin{aligned} AQ_P &= \frac{1}{\omega}\int_0^{\rho_0} \frac{P\rho^2}{\alpha}\,\Omega(\rho)\,d\rho \\ &= \frac{1}{\omega}\int_0^{\rho_0} \frac{P\rho^2}{\alpha}\,d\rho - \frac{4(1-\omega)}{\omega}\int_0^{\rho_0} \frac{P\rho^2}{\alpha}\,\Omega_U(\rho)\,d\rho. \end{aligned} \tag{23}$$

The energy loss from the system resulting from the internal energy release is predicted by a weighted average involving only the function $\Omega_U(\rho)$. As a check on the formula one notes that for the conservative case the second integral is removed from the expression and the total loss is then given by a volumetric integration of the energy released inside the sphere, a result that is intuitively clear. The application to energy loss associated with spectral line emission is less obvious intuitively.

If Eq. (23) is used together with Eqs. (20) and (22) the complete expression of flux at the outer boundary is determined finally as

$$\begin{aligned} \omega\rho_0^2 q(\rho_0) &= -\,4q_2^-(1-\omega)\left[\frac{\rho_0^2}{3} - 4(1-\omega)\int_0^{\rho_0} \rho^2\Omega_U(\rho)\,d\rho\right] \\ &\quad + \int_0^{\rho_0} \frac{P\rho^2}{\alpha}\,d\rho - 4(1-\omega)\int_0^{\rho_0} \frac{P\rho^2}{\alpha}\,\Omega_U(\rho)\,d\rho. \end{aligned} \tag{24}$$

We consider next a reduction of the integral equation for the homogeneous sphere to a simpler form. It is sufficient to consider the operator introduced in Eqs. (14) and (15). If the definition of the source function is extended such that $\mathfrak{J}(\rho) = \mathfrak{J}(-\rho)$, the transformations

$$2\rho_0 = \tau_0\,, \qquad \rho = \tau - \frac{\tau_0}{2}\,, \qquad \pi\rho\mathfrak{J}(\rho) = F(\tau)$$

yield the operational equivalence

$$\chi_{\rho,\rho_0}[\pi\rho_1\mathfrak{J}(\rho_1)] = \Lambda_{\tau,\tau_0}[F(\tau_1)]. \tag{25}$$

The integral operator for the sphere thus reduces formally to the integral operator for the slab. In particular, setting

$$\begin{aligned}\pi\rho\mathfrak{J}(\rho) &= q_2^- \rho\Omega(\rho) + A\rho\Omega_P(\rho) \\ &= q_2^- \Sigma(\tau) + A\Sigma_P(\tau)\end{aligned} \tag{26}$$

one gets the integral equation

$$\Lambda_{\tau,\tau_0}[\Sigma_P(\tau_1)] = \frac{P\,[\,|\,\tau - (\tau_0/2)\,|\,]}{4\alpha A}. \tag{27}$$

Also, when $A = P/\alpha = \text{const}$ one has $\Sigma_U(\tau) = \Sigma_P(\tau)$ and

$$\Sigma(\tau) = \left(\tau - \frac{\tau_0}{2}\right) - 4(1-\omega)\,\Sigma_U(\tau). \tag{28}$$

The transformation thus establishes the relationship between spherical and planar problems: the solution of the former reduces to the latter provided the optical thickness of the slab is equal to the optical diameter of the sphere and the internal energy release within the slab is $[\tau - (\tau_0/2)]$ times the specified function for the sphere.

Numerical predictions of $\Omega_U(\rho)$ have been given by Cuperman, Engelmann, and Oxenius [11], and Heaslet and Warming [2] have also given graphical results for $\omega = 1$. These solutions can be used directly in Eq. (24) with arbitrary variations of P/α. When P/α is expressible as a polynomials in τ, it is possible to predict flux and the source function at the boundary in terms of moments of the Chandrasekhar-Ambartsumian functions. The actual calculations have been carried out in [2] for the special case $P/\alpha = \text{const}$ and the results are repeated here. If $\alpha_n = \alpha_n(2\rho_0\,, \omega)$, $\beta_n = \beta_n(2\rho_0\,, \omega)$, surface flux associated with uniform energy release is

$$\begin{aligned}\left[\frac{\rho_0^2 q(\rho_0)}{P/\alpha}\right]_{q_2^-=0} &= Q_U \\ &= \tfrac{1}{3}\rho_0^3, \qquad \omega = 1 \\ &= \frac{[(\alpha_2+\beta_2)+\rho_0(\alpha_1+\beta_1)]}{4(1-\omega)}\{\rho_0[2-\omega(\alpha_0-\beta_0)] - \omega(\alpha_1-\beta_1)\} \\ &\qquad - \frac{[(\alpha_3-\beta_3)+\rho_0(\alpha_2-\beta_2)]}{2-\omega(\alpha_0-\beta_0)}, \qquad \omega \neq 1.\end{aligned} \tag{29}$$

Surface values for the source function satisfy the relations

$$\left[\frac{\pi\rho_0\mathfrak{J}(\rho_0)}{P/\alpha}\right]_{q_2^-=0} = \rho_0\Omega_U(\rho_0)$$

$$= \frac{\rho_0[1-(\omega/2)(\alpha_0-\beta_0)]-(\omega/2)(\alpha_1-\beta_1)}{4(1-\omega)}, \qquad \omega \neq 1$$

$$= \tfrac{1}{8}\rho_0^2(\alpha_1-\beta_1)+\tfrac{3}{8}\rho_0(\alpha_2-\beta_2)+\tfrac{3}{8}(\alpha_3-\beta_3), \qquad \omega = 1. \tag{30}$$

The analytic and numerical properties of the moment functions are now well established (see [4], [15]), and as a consequence Eqs. (12) (13), (29), and (30) provide a standard of excellence for approximate methods that may be of interest in more complex problems. For arbitrary variations of energy release Eqs. (10) and (24) should be of general utility and highlight the need for accurate evaluations of the functions $\Theta(\tau)$ and $\Omega_U(\rho)$ for the full parametric

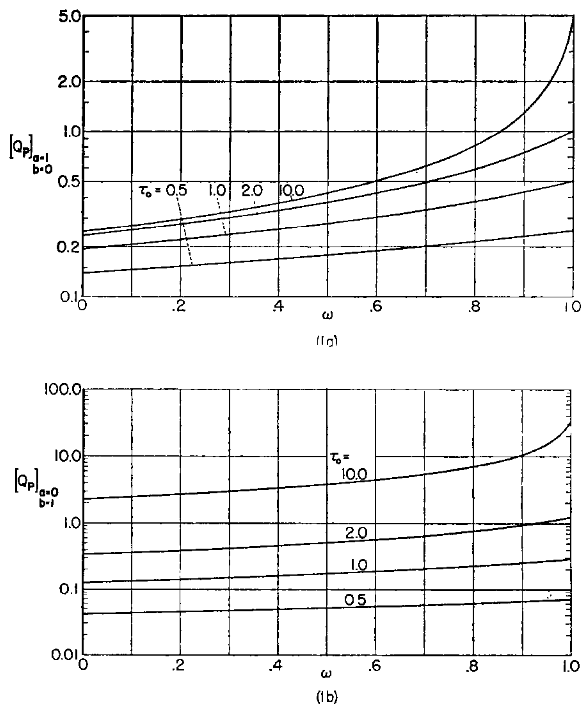

FIG. 1. Dimensionless flux loss from an slab with an internal source of energy release $P/\alpha A = a + b\tau$, showing dependence on the parameters τ_0 and ω. (a) Uniform energy release. (b) Linear energy release.

range $0 < \omega < 1$. The special flux predictions of Eqs. (12) and (29) may be evaluated by means of Sobouti's tables [4] of α_n, β_n, $n = 0, 1, 2$ and the tables of α_3, β_3 given in [2]. Figures 1 and 2 are graphical representations of these formulas.

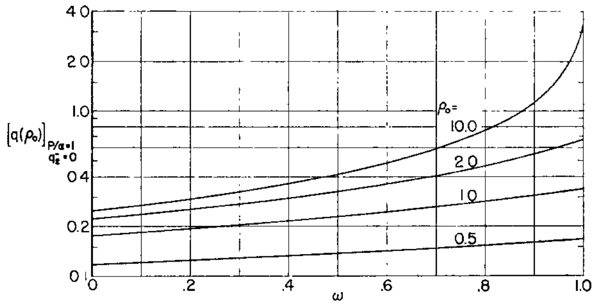

FIG. 2. Dimensionless flux loss from a sphere with uniform internal energy release, showing dependence on the parameters ρ_0 and ω.

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