On sharply vertex transitive 2-factorizations of the complete graph

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Abstract

We introduce the concept of a 2-starter in a group $G$ of odd order. We prove that any 2-factorization of the complete graph admitting $G$ as a sharply vertex transitive automorphism group is equivalent to a suitable 2-starter in $G$. Some classes of 2-starters are studied, with special attention given to those leading to solutions of some Oberwolfach or Hamilton–Waterloo problems.

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1. Introduction

Throughout the paper $K_v$ will denote the complete graph on $v$ vertices. By $V(K_v)$ and $E(K_v)$ we will, respectively, denote the vertex-set and the edge-set of $K_v$. Also, speaking of a cycle or, more generally, of a closed trail $A = (a_0, a_1, \ldots, a_{k-1})$, we mean the graph whose edges are $[a_i, a_{i+1}]$, $i = 0, 1, \ldots, k - 1$, where the subscripts are defined (mod $k$).

A cycle decomposition $\mathcal{D}$ of $K_v$ is a set of cycles whose edges partition $E(K_v)$ and it is obvious that its existence necessarily implies $v$ to be odd.

A 2-factor $F$ of $K_v$ is a set of cycles whose vertices partition $V(K_v)$. A 2-factorization of $K_v$ is a set $\mathcal{F}$ of 2-factors such that any edge of $K_v$ appears in exactly one member of $\mathcal{F}$.

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Hence the cycles appearing in some factor of \( \mathcal{F} \) form, altogether, a cycle decomposition of \( K_v \) that we will call the underlying cycle decomposition of \( \mathcal{F} \).

Observe that a 2-regular subgraph of \( K_v \) is a collection of disjoint cycles and it is a 2-factor if and only if its vertex-set coincides with \( V(K_v) \).

A 2-factorization of \( K_v \) whose cycles all have the same length \( k \) is also called a resolvable \( k \)-cycle decomposition of \( K_v \). In the case of \( k = 3 \) one also speaks of a Kirkman triple system of order \( v \) (KTS(\( v \)) for short).

Let \( G \) be an additive group of odd order \( v \), denote by \( K_G \) the complete graph with vertex-set \( V(K_G) = G \) and consider the regular action of \( G \) on \( V(K_G) \) defined by \( a \to a + g \), for any \((a, g) \in V(K_G) \times G \). A cycle decomposition \( D \) of \( K_G \) is regular under the action of \( G \) if we have \( C + g \in D \) for any \( C \in D \) and for any \( g \in G \).

Again, if \( \mathcal{F} \) is a 2-factorization of \( K_G \), we say that \( \mathcal{F} \) is regular under the action of \( G \), or simply that it is regular, or \( G \)-regular, if we have \( F + g \in \mathcal{F} \) for any \( F \in \mathcal{F} \) and any \( g \in G \).

In some recent papers (see \([6,3,13,4]\)) regular 1-factorizations of \( K_G \) were studied for several groups \( G \) of even order, despite the fact that the existence is not guaranteed for an arbitrary group \( G \). A regular 1-factorization of \( K_G \) has been proved to be equivalent to the concept of a starter in a group of even order which was introduced in \([6]\).

In this paper we present a similar method to construct regular 2-factorizations of a complete graph. More precisely, we will introduce the definition of a 2-starter in a group \( G \) of odd order and we will prove that to give a \( G \)-regular 2-factorization of \( K_G \) is equivalent to give a suitable 2-starter in \( G \).

We will also observe that a \( G \)-regular 2-factorization of \( K_G \) exists for any group \( G \) of odd order and we will lay emphasis on particular 2-factorizations, which we will call elementary 2-factorizations, proving some existence results.

We analyze when our constructions provide solutions to the Oberwolfach and to the Hamilton–Waterloo problem, or HW-problem for short. These problems relate to seating arrangements at a conference. The first one \([12]\), asks whether it is possible to seat \( v \) people (\( v \) odd) on \((v - 1)/2 \) days at \( s \) round tables at which there are \( c_1, \ldots, c_s \) seats (with \( c_1 + \cdots + c_s = v \), \( c_i \geq 3 \), \( 1 \leq i \leq s \)) in such a way that each person sits next to every other person exactly once. The HW-problem asks a similar question in case the conference is held for \( r \) days at Hamilton and \( s \) days at Waterloo, with \( r + s = (v - 1)/2 \), and where the round tables seat \( a_1, \ldots, a_t \) people at Hamilton and \( b_1, \ldots, b_u \) people at Waterloo (so \( a_1 + \cdots + a_t = b_1 + \cdots + b_u = v \)).

In terms of factorizations, the Oberwolfach problem asks for a 2-factorization of \( K_v \) in which each factor consists of cycles of length \( c_1, \ldots, c_s \). If \( c_1 = \cdots = c_s = c \) and \( r \) is the number of cycles in each factor, the Oberwolfach problem is also denoted by \( OP(c; r) \). It is known that \( OP(c; r) \) has a solution for all \( r \geq 1 \) and \( c \geq 3 \), \([2]\). The HW-problem asks for a 2-factorization of \( K_v \) in which each factor consists of cycles of length \( a_1, \ldots, a_t \) and \( s \) factors consist of cycles of length \( b_1, \ldots, b_u \). If \( a_1 = \cdots = a_t = m \) and \( b_1 = \cdots = b_u = n \), for some integers \( m \) and \( n \), we will denote the HW-problem by \( HWP(v; m, n; r, s) \). See also \([1]\) for a similar notation and for solutions to \( HWP(v; m, n; r, s) \) for particular values of \( m \) and \( n \).

Our Theorem 5.1 provides a cyclic solution to \( HWP(18n + 3; 3, 6n + 1; 3n, 6n + 1) \) for each positive integer \( n \).

Some solutions are also given as a consequence of Theorem 2.6.
2. Regular 2-factorizations and 2-starters in groups of odd order

In the rest of the paper when speaking of a group \( G \) we will always understand it is additive and of odd order. Also, if \( H \) is a subgroup of \( G \), then a system of distinct representatives for the left (respectively, right) cosets of \( H \) in \( G \) will be called a left transversal (respectively, a right transversal) for \( H \) in \( G \).

Given a \( k \)-cycle \( A = (a_0, a_1, \ldots, a_{k-1}) \) with vertices in \( G \), the stabilizer of \( A \) under the action of \( G \) is the subgroup \( G_A \) of \( G \) defined by \( G_A = \{ g \in G \mid A + g = A \} \). Generalizing what was observed in some previous papers (see, e.g., [7, Section 2]) in the case of \( G \) cyclic, we have the following proposition:

**Proposition 2.1.** Let \( A = (a_0, a_1, \ldots, a_{s-1}) \) be a st-cycle with vertices in \( G \) and let \( t \) be the order of \( G_A \). Then, there is an element \( g \in G \) of order \( t \) such that the following condition holds:

\[
    a_{i+s} - a_i = g \quad \forall i
\]

or, more explicitly:

\[
    A = (a_0, a_1, \ldots, a_{s-1}, a_0 + g, a_1 + g, \ldots, a_{s-1} + g, \ldots, a_0 + (t-1)g, a_1 + (t-1)g, \ldots, a_{s-1} + (t-1)g).
\]

Conversely, if \( g \) is an element of \( G \) of order \( t \), a sequence \( A = (a_0, a_1, \ldots, a_{s-1}) \) of \( st \) vertices of \( G \) satisfying (1) is a st-cycle with \( |G_A| = t \), if the following extra conditions are satisfied:

- \( s \) is the least divisor of \( st \) such that \( a_{i+s} - a_i \) does not depend on \( i \);
- \( a_0, a_1, \ldots, a_{s-1} \) lie in pairwise distinct left cosets of \( \langle g \rangle \) in \( G \).

Let \( A \) be a cycle as in Proposition 2.1. We define the list of partial differences of \( A \) to be the multiset

\[
    \partial A = \pm \{a_{i+1} - a_i \mid 0 \leq i < s\}
\]

and we set

\[
    \phi(A) = \{a_0, a_1, \ldots, a_{s-1}\}.
\]

If the stabilizer of \( A \) is trivial, then \( \partial A \) coincides with \( \Delta A \), the list of differences of \( A \) in the usual sense. In this case \( \phi(A) = V(A) \), the set of vertices of \( A \). More generally, if \( A = \{A_1, \ldots, A_n\} \) is a collection of cycles (in particular, a 2-regular graph) with vertices in \( G \), then we set \( \partial A = \partial A_1 \cup \cdots \cup \partial A_n \) and \( \phi(A) = \phi(A_1) \cup \cdots \cup \phi(A_n) \) (where in the union the elements have to be counted with their multiplicity).

The \( G \)-orbit of a cycle \( A \) is the set \( Orb_G(A) \) of all distinct cycles in the collection \( \{A + g \mid g \in G\} \). Its size (or length) is \( |G : G_A| \), the index of the stabilizer of \( A \) under \( G \) and \( Orb_G(A) = \{A + t \mid t \in T\} \) where \( T \) is a right transversal for \( G_A \) in \( G \).

**Proposition 2.2.** Let \( A = \{A_1, \ldots, A_n\} \) be a collection of cycles with vertices in \( G \). Then \( \mathcal{D} = \bigcup_{i=1}^n Orb_G(A_i) \) is a cycle decomposition of \( KG \) if and only if \( \partial A = G - \{0\} \).
Proof. For $i = 1, \ldots, n$, let $l_i$ be the length of $A_i$ and let $d_i$ be the order of the $G$-stabilizer of $A_i$. Assume $D$ is a cycle decomposition of $K_G$. The size of $Orb_G(A_i)$ is $v/d_i$ so that the number $|E(K_G)| = v(v - 1)/2$ of edges covered by $D$ may be also expressed as $v \sum_{i=1}^{n} \frac{l_i}{d_i}$.

It follows that $2 \sum_{i=1}^{n} \frac{l_i}{d_i} = v - 1$. Now note that the two sides of the last equality are the sizes of $\partial A$ and $G - \{0\}$, respectively. So, it is enough to show that any $x \in G - \{0\}$ appears at least once in $\partial A$. Given any non-zero element $x \in G$, we may claim by assumption that $[0, x]$ is an edge of $A_i + t$ for a suitable pair $(i, t) \in \{1, \ldots, n\} \times G$. It follows that $[0, x] = [a + t, b + t]$ where $A_i = (a, b, \ldots)$. This implies that $t = -a$ and hence $x = b + t = b - a \in \partial A_i \subset \partial A$.

Vice versa, assume that $\partial A = G - \{0\}$. So we have: $|\partial A| = 2 \sum_{i=1}^{n} \frac{l_i}{d_i} = v - 1$, and hence: $v \sum_{i=1}^{n} \frac{l_i}{d_i} = \frac{v(v - 1)}{2} = |E(K_G)|$. The left-hand side of this equality gives the number of edges covered by the cycles of $D$. So, to prove that each edge of $K_G$ is covered by the cycles of $D$ exactly once, it is sufficient to prove that this happens at least once. Let $[x, y]$ be an edge of $K_G$. By assumption, there is a suitable $i$ such that $A_i = (a, b, \ldots)$ with $a - b = x - y$. Then we have $[x, y] = [a, b] + (-b + y)$ and we may claim that $[x, y]$ is an edge of $A_i + (-b + y) \in Orb_G(A_i) \subset D$. \hfill $\Box$

In what follows we introduce a concept which makes it possible to describe algebraically any $G$-regular 2-factorization of $K_G$.

Definition 2.3. A 2-starter in $G$ is a collection $\Sigma = \{S_1, \ldots, S_n\}$ of 2-regular graphs with vertices in $G$ satisfying the following conditions:

- $\partial S_1 \cup \cdots \cup \partial S_n = G - \{0\}$;
- $\phi(S_i)$ is a left transversal for some subgroup $H_i$ of $G$ containing the stabilizers of all cycles of $S_i$, $i = 1, \ldots, n$.

Theorem 2.4. The existence of a $G$-regular 2-factorization of $K_G$ is equivalent to the existence of a 2-starter in $G$.

Proof. Suppose $\Sigma = \{S_1, \ldots, S_n\}$ is a 2-starter in $G$. By definition, for $i = 1, \ldots, n$, there is a suitable subgroup $H_i$ of $G$ such that $\phi(S_i)$ is a left transversal for $H_i$ in $G$. Set $F_i = \bigcup_{A \in S_i} Orb_{H_i}(A)$.

Given a cycle $C$, let $\ell(C)$ be its length. If $A \in S_i$, in $Orb_{H_i}(A)$ there are exactly $|H_i|/|G_A|$ cycles and each of them has length $\ell(A)$. So we have

$$\sum_{C \in F_i} \ell(C) = \sum_{A \in S_i} \ell(A)|H_i|/|G_A|. \quad (2)$$

On the other hand, since by assumption $\phi(S_i)$ is a left transversal for $H_i$ in $G$, we have

$$|\phi(S_i)| = \sum_{A \in S_i} \ell(A)/|G_A| = |G|/|H_i|. \quad (3)$$
From (2) and (3)) we get
\begin{equation}
\sum_{C \in F_i} \ell(C) = |G|.
\end{equation}

Now, observe that each \( g \in G \) is vertex of at least one cycle of \( F_i \). In fact, since \( \phi(S_i) \) is a left transversal for \( H_i \) in \( G \), we have \( g = x + h \) for a suitable pair \((x, h) \in \phi(S_i) \times H_i \). So, if \( A \) is the cycle of \( S_i \) such that \( x \in \phi(A) \), it is obvious that \( A + h \) is a cycle of \( F_i \) and that \( g \) is a vertex of it. This, together with (4), ensures that each \( g \in G \) is vertex of exactly one cycle of \( F_i \), i.e., \( F_i \) is a 2-factor of \( K_G \).

Consider the set of 2-factors \( \mathcal{F} = \text{Orb}_G(F_1) \cup \ldots \cup \text{Orb}_G(F_n) \). We prove that \( \mathcal{F} \) is a 2-factorization of \( K_G \) by proving that \( \overline{\mathcal{F}} \), the underlying set of cycles of \( \mathcal{F} \), is a cycle decomposition of \( K_G \). Let \( \mathcal{A} \) be the collection of cycles of \( \Sigma \) and observe that \( \text{Orb}_G(\mathcal{A}) = \overline{\mathcal{F}} \). In fact it is obvious that \( \overline{\mathcal{F}} \subset \text{Orb}_G(\mathcal{A}) \) and, vice versa, if \( B \in \text{Orb}_G(\mathcal{A}) \) we have \( B = A + h \) for some cycle \( A \in \mathcal{A} \) and for some \( g \in G \). Therefore \( A \in S_i \) for some \( S_i \in \Sigma \), and \( B \) is a cycle of \( \text{Orb}_G(F_i) \). By assumption, the set \( \Sigma \) is a 2-starter in \( G \) and then by Proposition 2.2, \( \overline{\mathcal{F}} \) is a cycle decomposition of \( K_G \). Obviously \( \mathcal{F} \) admits \( G \) as a sharply vertex transitive automorphism group.

Suppose now \( \mathcal{F} \) to be a \( G \)-regular 2-factorization of \( K_G \). Let \( \{F_1, \ldots, F_n\} \) be a complete system of representatives for the \( G \)-factor-orbits of \( \mathcal{F} \). For each \( i \), denote by \( H_i \) the stabilizer in \( G \) of \( F_i \) and let \( S_i \) be a complete system of representatives for the \( H_i \)-cycle-orbits that are contained in \( F_i \). Obviously, if \( A \) is a cycle of \( S_i \), then \( H_i \) contains \( G_A \) and hence the stabilizer of \( A \) in \( H_i \) coincides with \( G_A \). We prove that \( \Sigma:= \{S_1, \ldots, S_n\} \) is a 2-starter in \( G \). First of all observe that \( F_i = \bigcup_{A \in S_i} \text{Orb}_H(A) \) and \( \mathcal{F} = \text{Orb}_G(F_1) \cup \ldots \cup \text{Orb}_G(F_n) \). Therefore, if \( \mathcal{A} \) is the collection of cycles of \( \Sigma \), then \( \text{Orb}_G(\mathcal{A}) \) is the underlying cycle decomposition \( \overline{\mathcal{F}} \) of \( \mathcal{F} \). By assumption \( \overline{\mathcal{F}} \) is a cycle decomposition of \( K_G \) and by Proposition 2.2 we obtain \( \partial \mathcal{A} = \partial S_1 \cup \ldots \cup \partial S_n = G - \{0\} \). It remains to show that for each \( i \), \( \phi(S_i) \) is a left transversal for \( H_i \) in \( G \). First of all, \( \phi(S_i) \) has the right size since we have
\[ |G| = \sum_{A \in F_i} \ell(A) = \sum_{A \in S_i} \ell(A)|\text{Orb}_H(A)| = \sum_{A \in S_i} \ell(A)|H_i|/|G_A| \]
\[ \implies |\phi(S_i)| = \sum_{A \in S_i} \ell(A)/|G_A| = |G|/|H_i|. \]

Hence, it suffices to see that any \( g \in G \) may be expressed in the form \( g = x + h \) for some \((x, h) \in \phi(S_i) \times H_i \). Since \( F_i \) is a 2-factor of \( K_G \), each element \( g \in G \) is vertex of a cycle of \( F_i \). This implies the existence of a pair \((A, h) \in S_i \times H_i \) such that \( g \) is vertex of the cycle \( A + h \), say \( g = a + h \), \( a \in A \). On the other hand we also have \( a = x + h' \) with \( x \in \phi(A) \) and \( h' \in G_A \) (see (1) in Proposition 2.1). Therefore, recalling that \( H_i \) contains \( G_A \), we have \( g = x + h'' \) with \( x \in \phi(S_i) \) and \( h'' = h' + h \in H_i \). The assertion follows.

**Example 2.5.** Consider the following three cycles of respective lengths 14, 7, 7, and with vertices in \( G = Z_{21} \):
\[ A = (0, 7, 3, 10, 6, 13, 9, 16, 12, 19, 15, 1, 18, 4); \]
\[ B = (2, 5, 8, 11, 14, 17, 20); \]
\[ C = (0, 1, 3, 11, 16, 6, 12). \]
The stabilizer of $A$ and $B$ is $\langle 3 \rangle$, i.e., the subgroup of $G$ of order 7. Instead, $C$ has trivial stabilizer. Thus we have

$$\hat{\varepsilon}A = \{\pm 7, \pm 4\}; \quad \hat{\varepsilon}B = \{\pm 3\}; \quad \hat{\varepsilon}C = \{\pm 1, \pm 2, \pm 8, \pm 5, \pm 10, \pm 6, \pm 9\}.$$

$$\phi(A) = \{0, 7\}; \quad \phi(B) = \{2\}; \quad \phi(C) = \{0, 1, 3, 11, 16, 6, 12\}.$$

Now note that $\phi(A) \cup \phi(B) \equiv Z_3 \pmod{3}$ and that $\phi(C) \equiv Z_7 \pmod{7}$. So, setting $S = \{A, B\}$ and $T = \{C\}$, we see that $\Sigma = \{S, T\}$ is a 2-starter in $G$.

The base factors of the factorization $\mathcal{F}$ generated by $\Sigma$ are $F_1 = \{A, B\}$ and $F_2 = \{C, C + 7, C + 14\}$, and $\mathcal{F} = \text{Orb}_G(F_1) \cup \text{Orb}_G(F_2)$.

The existence question for cyclic resolvable $k$-cycle decompositions of $K_v$ has been solved in the case where $v = km$ with $k$ an odd prime and all prime factors of $m$ congruent to 1 (modulo $2k$) (see [10,9]). In view of our Theorem 2.4 these factorizations can be described in terms of 2-starters. For instance, the cyclic $\text{KTS}(3p)$ ($p$ a prime congruent to 1 (mod 6)) constructed by Gemma et al. [10] may be equivalently described as follows:

Let $G = Z_3 \oplus Z_p$, where $p = 6n + 1$ is a prime. Let $\rho$ be a primitive root (modulo $p$) and for $0 \leq i \leq n - 1$ consider the 3-cycles

$$A_i = ((0, \rho^j), (0, \rho^{2n+i}), (0, \rho^{4n+i})), \quad B_i = ((0, \rho^{3n+i}), (1, \rho^{5n+i}), (2, \rho^{n+i})), \quad C_i = ((0, \rho^j), (1, \rho^{2n+i}), (2, \rho^{4n+i})).$$

Let $S$ be the 2-regular graph whose cycles are $((0, 0), (1, 0), (2, 0))$, the $A_i$’s and the $B_i$’s. Then $\Sigma = \{S, C_0, C_1, \ldots, C_{n-1}\}$ is a 2-starter in $G$ giving rise to a cyclic $\text{KTS}(3p)$.

The question: For which groups $G$ does a $G$-regular 2-factorization of $K_G$ exist? naturally arises. Despite the fact that the analogous question for groups of even order and regular 1-factorizations does not seem easy to solve, [11,3,6,15,13], the answer to our question is quite simple if no additional restriction is made. In particular a $G$-regular 2-factorization in which each factor is fixed by $G$ exists as shown below.

**Theorem 2.6.** For any group $G$ of odd order, a $G$-regular 2-factorization of $K_G$ exists.

**Proof.** Let $G = \{0\} = X \cup -X$. For any $x \in X$ denote by $t_x$ the order of $x$ in $G$ and by $S_x$ the cycle $(0, x, \ldots, (t_x - 1)x)$. Observe that $\hat{\varepsilon}S_x = \{\pm x\}$ and that $\phi(S_x) = \{0\}$ is a left transversal for $G$ in $G$. Therefore the set $\Sigma = \{S_x \mid x \in X\}$ is a 2-starter in $G$. $\square$

A $G$-regular 2-factorization of $K_G$ in which each factor is fixed by $G$, is necessarily obtained in this manner as stated in the Proposition below. We call this factorization the natural 2-factorization of $K_G$ and we denote it by $N(G)$.

**Proposition 2.7.** Let $\mathcal{F}$ be a $G$-regular 2-factorization of $K_G$ such that each factor $F \in \mathcal{F}$ is fixed by $G$. Then $\mathcal{F}$ is isomorphic to $N(G)$.

**Proof.** Let $F_1, \ldots, F_t$ be the 2-factors of $\mathcal{F}$ and let $\Sigma$ be the 2-starter in $G$ obtained by $\mathcal{F}$ as in proof of Theorem 2.4. As $G$ fixes each factor $F_i$, we have $|\Sigma| = |\mathcal{F}|$. Set $\Sigma = \{S_1, \ldots, S_t\}$.
As $G$ is transitive on $V(K_G)$, each $S_i$ is a single cycle and we have $F_i = \text{Orb}_G(S_i)$. Without loss of generality, suppose that the cycle $S_i$ contains the edge $[0, x], x \in G - \{0\}$, therefore $S_i + x \in F_i$ and $S_i + x = S_i$ as it contains the vertex $x$. We conclude that $S_i = (0, x, \ldots, (t_x - 1)x)$. □

Observe that the factors of $\mathcal{N}(G)$ are all possible 2-regular Cayley graphs of $G$.

As an easy consequence of the previous proposition, we have

**Proposition 2.8.** If $p$ is an odd prime, then, up to isomorphisms, the only regular 2-factorization of $K_p$ is $\mathcal{N}(Z_p)$.

**Proof.** Let $\mathcal{F}$ be a regular 2-factorization of $K_{Z_p}$. The length of each factor-orbit under the action of $Z_p$ divides $p$ so that each factor of $\mathcal{F}$ is fixed by $Z_p$. The assertion follows. □

Note that $\mathcal{N}(Z_p)$ ($p$ odd prime) is Hamiltonian, i.e., all its factors consist of a single $p$-cycle. It has been proved [8] that a cyclic Hamiltonian factorization of $K_v$ exists for all odd values of $v \geq 3$ with the only definite exceptions of $v = 15$ and $v = p^2$ with $p$ a prime and $x > 1$.

We finally point out that it has been recently proved [5] that the 2-factorizations of the complete graph admitting a 2-transitive automorphism group are, up to isomorphisms, the natural 2-factorizations associated with an elementary abelian group of odd order, i.e., those of the form $\mathcal{N}(Z_p^n)$ with $p$ an odd prime and $n$ a positive integer. It should be possible to derive the same result from the more general classification of doubly transitive colorings of complete graphs which was recently obtained by Sibley [16].

3. Natural 2-factorizations with at most two non-isomorphic factors

Each factor of a natural 2-factorization is uniform, i.e., with all its cycles of the same length. It is also obvious that the number of non-isomorphic 2-factors is equal to the number of distinct orders of the non-zero elements of $G$. So the natural 2-factorization of $K_G$ provides a solution to an Oberwolfach problem exactly when $G$ is a group of order $p^n$ ($p$ an odd prime) in which all the non-zero elements have order $p$. Apart from the elementary abelian $p$-group $Z_p^n$, non abelian groups with this property can be found for any $n \geq 3$. As an example with $n = 3$ take the group $P$ having the following defining relations:

$$P = \langle a, b, c \mid a^p = b^p = c^p = 1, c^{-1}bc = ba, c^{-1}ac = a, b^{-1}ab = a \rangle.$$ 

For $n > 3$, it suffices to consider the direct sum of $P$ and $Z_p^{n-3}$.

If just two possible orders $m, n$ are admissible for the non-zero elements of a finite group $G$, then the natural 2-factorization of $K_G$ gives solutions to $\text{HW} P(|G|; m, n; r, s)$ for suitable integers $r$ and $s$. For the reader’s convenience, we prove the following proposition:

**Proposition 3.1.** Let $G$ be a finite group in which the non-zero elements have either order $m$ or $n$, $m > n$. There are two possibilities:

(i) $|G| = p^r$, $p$ prime; $m = p^2$, $n = p$.
(ii) $|G| = pq^l$, $p$ and $q$ primes with $q \equiv 1 \pmod{p}$; $m = q$, $n = p$.
**Proof.** Obviously it is either \( m = p^2 \) and \( n = p \) with \( p \) a prime, or \( m = q \) and \( n = p \) with \( p \) and \( q \) distinct primes. If the first case occurs, then \( G \) is a \( p \)-group and (i) follows. Suppose that the second possibility holds. Then the group \( G \) has order \( p^i q^j \) for suitable integers \( i \) and \( j \). By induction on \( i + j \), we prove that \( i = 1 \) and \( q \equiv 1 \pmod{p} \).

If \( i + j = 2 \), then \( |G| = pq \) and \( G \) is the Frobenius group; this necessarily implies \( q \equiv 1 \pmod{p} \).

Let \( i + j > 2 \). The group \( G \) is solvable (by the theorem of Burnside) and it is non-abelian (otherwise it should contain elements of order \( pq \)) so that \( G' \) (the derived subgroup of \( G \)) is not trivial. The group \( G/G' \) is abelian and its elements have either order \( p \) or \( q \); this implies either \( |G'| = p^i q^j \) with \( s < j \), or \( |G'| = p^i q^j \) with \( r < i \).

Suppose \( |G'| = p^i q^j \) with \( s < j \). By induction, we have \( q \equiv 1 \pmod{p} \) and \( i = 1 \), that is \( |G| = pq^j \).

On the contrary, suppose \( |G'| = p^i q^j \) with \( r < i \). Then, by induction, we have \( q \equiv 1 \pmod{p} \) and \( |G'| = pq^j \). Let \( M \) be a Sylow \( q \)-subgroup of \( G' \). By Sylow’s theorem, the number of Sylow \( q \)-subgroups in \( G' \) is a divisor of \( p \) congruent to 1 \( \pmod{q} \). So, as \( p \not\equiv 1 \pmod{q} \), we necessarily have \( M \triangleleft G' \). Furthermore, it is also \( M \triangleleft G \) since each conjugate of \( M \) is still in \( G' \). For each \( x \in M \), let \( y \) be a non-zero element of \( C_G(x) \), the centralizer of \( x \) in \( G \). The order of \( y \) is the same as \( x \), otherwise \( xy \) should have order \( pq \). This implies \( y \in M \) and then \( G \) is a Frobenius group with kernel \( M \) (see [14, Chapter 12]). So, if \( H \) is a complement of \( G \), then its order is \( p^j \). On the other hand, by [14, 12.6.15, p. 356] \( H \) must be cyclic so that \( i = 1 \). Therefore \( |G| = pq^j = |G'| \) and this possibility does not occur. \( \square \)

Indeed a group of order \( pq^j \), with \( p, q \) odd primes and \( q \equiv 1 \pmod{p} \) exists for any \( j \): the semidirect product \( S = Z_p \cdot Z_q^j \) defined by

\[
(x, y) \cdot (x', y') = (x + x', \epsilon^{xy} y + y'),
\]

where \( \epsilon \) is a fixed primitive \( p \)th root of unity in the field of order \( q^j \). Observe that the natural 2-factorization of \( K_9 \) is a solution to \( HWP(pq^j; p, q; q^j(q - 1)/2, (q^j - 1)/2) \).

For each integer \( e \), \( 1 \leq e \leq (r - 1)/2 \), the group obtained as the direct sum of \( e \) copies of the cyclic group \( Z_2 \) together with \( r - 2e \) copies of the cyclic group \( Z_p \) provides a solution to \( HWP(p^r; p, p^2; (p^r - e - 1)/2, (p^r - p^r - e)/2) \).

As an example, the natural factorization of \( Z_9 \) gives rise to the following solution of \( HWP(9; 3, 9; 1, 3) \):

\[
\begin{align*}
(0, 3, 6) & \quad (1, 4, 7) & \quad (2, 5, 8) \\
(0, 1, 2, 3, 4, 5, 6, 7, 8) & \\
(0, 2, 4, 6, 8, 1, 3, 5, 7) & \\
(0, 4, 8, 3, 7, 2, 6, 1, 5) & 
\end{align*}
\]

**4. Elementary 2-starters**

Apart from the natural 2-factorization of \( K_G \), many other regular 2-factorizations may be found especially when the lattice of subgroups of \( G \) is quite rich. Here we want to fix our attention on regular 2-factorizations such that all cycles of the underlying cycle
decomposition have trivial stabilizer under the action of $G$. There are particular 2-starters, that we call elementary, giving rise to factorizations with this property.

We say that a 2-starter $\Sigma = \{S_1, \ldots, S_t\}$ in $G$ is elementary if each $S_i$ is a single cycle with trivial stabilizer. In this case we say that $\Sigma$ is of type $\{d_1, \ldots, d_t\}$ if the length of the cycle in $S_i$ is $d_i$, $i = 1, \ldots, t$.

In view of Theorem 5.1, we have the following proposition:

**Proposition 4.1.** Let $S_1, \ldots, S_t$ be cycles with vertices in $Z_v$ and respective lengths $d_1, \ldots, d_t$. Then $\Sigma = \{S_1, \ldots, S_t\}$ is an elementary 2-starter of type $\{d_1, \ldots, d_t\}$ in $Z_v$ if and only if the following conditions are satisfied:

(a) $d_h$ is a divisor of $v$ for $1 \leq h \leq t$;
(b) $\sum_{h=1}^{t} d_h = (v - 1)/2$;
(c) $\bigcup_{h=1}^{t} \Delta S_h = Z_v - \{0\}$;
(d) $V(S_h) = Z_{d_h} (\text{mod } d_h)$ for $1 \leq h \leq t$;
(e) $lcm(d_1, \ldots, d_t) = v$.

**Proof.** If $\Sigma$ is an elementary starter each cycle $S_i$ has trivial stabilizer. Hence the above relations easily follow observing that $|\varphi(S_i)|$ is the length of $S_i$ and $\partial S_i = \Delta S_i$. Condition (e) is a consequence of (c) and (d). Assume that $lcm(d_1, \ldots, d_t) = m < v$. In this case, by (d), the vertices of each $S_h$ are pairwise distinct (mod $m$) so that $m$ cannot appear in the list of differences $\bigcup_{h=1}^{t} \Delta S_h$ in contradiction with (c).

The converse is obvious by (c) and (d). $\square$

We give an example of elementary 2-starter.

**Example 4.2.** It is straightforward to check that the two cycles $(0, 4, 14)$ and $(0, 1, 9, 4, 6, 12, 3)$ form an elementary 2-starter of type $(3, 7)$ in $Z_{21}$. Here is, explicitly, the 2-factorization generated by it:

$$(0, 4, 14), (3, 7, 17), (6, 10, 20), (9, 13, 2), (12, 16, 5), (15, 19, 8), (18, 1, 11), (1, 5, 15), (4, 8, 18), (7, 11, 0), (10, 14, 3), (13, 17, 6), (16, 20, 9), (19, 2, 12), (2, 6, 16), (5, 9, 19), (8, 12, 1), (11, 15, 4), (14, 18, 7), (17, 0, 10), (20, 3, 13).$$

We conjecture that (a), (b) and (e) are sufficient conditions for the existence of an elementary 2-starter of type $\{d_1, \ldots, d_t\}$ in $Z_v$.

Recall that a partition of an integer $m$ is a multiset $P$ of positive integers whose sum is $m$. 

Let \( n \) be a positive integer and let an elementary 2-starter in \( \mathbb{Z}_F \) or any partition

**Conjecture.** For any partition \( P = \{d_1, \ldots, d_t\} \) of \((v - 1)/2\) into proper divisors of \( v \) with \( \text{lcm}(d_1, \ldots, d_t) = v \), there exists an elementary 2-starter of type \( P \) in \( \mathbb{Z}_v \).

In the following section, we give a construction for an elementary 2-starter in \( \mathbb{Z}_v \) for each \( v = 18n + 3 \) and we prove the above conjecture to be true when \( v = 3p \) with \( p \) an odd prime.

**5. Construction for an elementary 2-starter of type \( \{3, 3, \ldots, 3, 6n + 1\} \) in \( \mathbb{Z}_{18n+3} \)**

**Theorem 5.1.** Let \( n \) be a positive integer and let \( v = 18n + 3 \). An elementary 2-starter of type \( \{3, 3, \ldots, 3, 6n + 1\} \) exists in \( \mathbb{Z}_v \).

**Proof.** For the case \( n = 1 \) see Example 4.2. The assertion is also true for \( n = 2, 3 \). In fact an elementary 2-starter in \( \mathbb{Z}_{39} \) is

\[
\{(0, 13, 14), (0, 5, 28), (0, 2, 37, 29, 9, 38, 17, 10, 27, 18, 21, 6, 33)\}
\]

and an elementary 2-starter in \( \mathbb{Z}_{57} \) is

\[
\{(0, 19, 20), (0, 11, 40), (0, 22, 26), (0, 2, 18, 41, 55, 42, 35, 43, 53, 1, 33, 6, 51, 45, 12, 27, 30, 9, 48)\}
\]

From now on we assume \( n \geq 4 \) and we identify \( \mathbb{Z}_v \) with the direct sum \( G = \mathbb{Z}_3 \oplus \mathbb{Z}_m \), \( m = 6n + 1 \). By Proposition 4.1 we have to find a set \( \{T_0, T_1, \ldots, T_{n-1}, C\} \) of \( n + 1 \) cycles with vertices in \( G \) satisfying the following conditions:

(i) For \( i = 0, 1, \ldots, n - 1 \), the projection of \( V(T_i) \) on \( \mathbb{Z}_3 \) is bijective;  
(ii) the projection of \( V(C) \) on \( \mathbb{Z}_m \) is bijective; 
(iii) \( \Delta T_0 \cup \cdots \cup \Delta T_{n-1} \cup \Delta C = G - \{0\} \).

Let \((z_1, \ldots, z_{n-1})\) be a Skolem sequence or a hooked Skolem sequence of order \( n - 1 \) (see, e.g., [15]). So we have

\[
\bigcup_{i=1}^{n-1} \{z_i, z_i + i\} = \{1, 2, \ldots, 2n - 1\} - \{z\}
\]

with \( z = 2n - 2 \) or \( 2n - 1 \) according to whether we have \( n \equiv 1, 2 \) or \( n \equiv 0, 3 \) (mod 4), respectively.

Set \( T_0 = ((0, 0), (1, 0), (2, n + 2)) \) and \( T_i = ((0, 0), (1, i), (2, -z_i - n - 2)) \) for \( i = 1, \ldots, n - 1 \). We have \( \Delta T_0 = \pm(\{1\} \times (0, n + 2, -(n + 2))) \) and \( \Delta T_i = \pm(\{i\} \times (i, z_i + n + 2, -(z_i + i + n + 2))) \). By (5), the \( n - 1 \) pairs \( \{z_i + n + 2, z_i + i + n + 2\} \) cover the set \( \{n + 3, n + 4, \ldots, 3n - 1, 3n\} \) or the set \( \{n + 3, n + 4, \ldots, 3n - 1, 3n + 1\} \), according to whether we have \( n \equiv 1, 2 \) or \( n \equiv 0, 3 \) (mod 4), respectively.  

Then, observing that \( 3n + 1 = -3n \) (mod \( m \)), we may say that for each \( i \in \{0, 1, \ldots, 3n\} \) \(- (n, n + 1, n + 2) \) exactly one of the two pairs \((1, i)\) and \((-1, i)\) appears in the list \( \Delta T = \bigcup_{i=0}^{n-1} \Delta T_i \).
Note that we may write $Z_m - \{0\} = \{ \pm hn \mid 1 \leq h \leq 3n \}$ since we obviously have $\gcd(m, n) = 1$. Also note that $\pm 5n \equiv \mp (n + 1) \pmod{m}$ and that $\pm 11n \equiv \mp (n + 2) \pmod{m}$.

In view of this, the above paragraph may be reformulated by saying that for any $h \in \{1, 2, \ldots, 3n\} - \{1, 5, 11\}$ exactly one of the pairs $(1, hn)$ and $(-1, hn)$ appears in $\Delta T$. So, we may define a map $f : \{hn \mid 1 \leq h \leq 3n, h \neq 1, 5, 11\} \to \{-1, 1\}$ in such a way that $f(hn)$ is the unique element of $\{1, -1\}$ such that $(f(hn), hn) \in \Delta T$.

Let $(y_0, y_1, \ldots, y_{m-1})$ be the permutation on $Z_m$ defined by

$$y_i = \begin{cases} n(-1)^i \left\lfloor \frac{i+1}{2} \right\rfloor & \text{for } 11 \leq i < 3n + \beta, \\ n(-1)^i \left\lceil \frac{i+1}{2} \right\rceil & \text{for } 3n + \beta \leq i \leq 6n, \end{cases}$$

where $\beta = 0$ or 1 according to whether $n$ is odd or even, respectively.

Set $\theta = -f(3n) - \sum_{i=5}^{3n} f(y_i - y_{i-1})$ and fix $\psi \in Z_3 - \{0, \theta\}$.

Consider the $m$-cycle $C = (c_0, c_1, \ldots, c_{m-1})$ with vertices in $G$ and with $c_i = (x_i, y_i)$, the $x_i$'s being defined by the following rules:

$$x_0 = 0; \quad x_1 = \psi; \quad x_2 = \theta; \quad x_3 = \theta - f(3n); \quad x_4 = -\psi - \theta - f(3n);$$

$$x_i = x_{i-1} - f(y_i - y_{i-1}) \quad \text{for } 5 \leq i \leq 3n;$$

$$x_i = 0 \quad \text{for } 3n + 1 \leq i \leq 6n.$$

We are going to show that any element of $G - \{0\}$ appears in $\Delta T \cup \Delta C$. First of all, observe that $\{\pm (y_{i+1} - y_i) \mid i = 3n + 1, \ldots, 6n\} = \{\pm hn \mid 1 \leq h \leq 3n\}$. Then, considering that the $x_i$'s are all equal to 0 for $i \geq 3n + 1$, we have

$$\{\pm (0, hn) \mid 1 \leq h \leq 3n\} = \{\pm (c_{i+1} - c_i) \mid i = 3n + 1, \ldots, 6n\}.$$

Given $h \in \{1, 2, \ldots, 3n\} - \{1, 5, 11\}$, we have $(f(hn), hn) \in \Delta T$ and

$$(-f(hn), hn) = \begin{cases} c_4 - c_5 & \text{if } h = 2; \\ c_3 - c_2 & \text{if } h = 3; \\ c_5 - c_6 & \text{if } h = 4; \\ c_9 - c_{10} & \text{if } h = 6; \\ c_7 - c_6 & \text{if } h = 7; \\ c_9 - c_8 & \text{if } h = 8; \\ c_{11} - c_{10} & \text{if } h = 9; \\ c_7 - c_8 & \text{if } h = 10; \\ (-1)^h(c_h - c_{h-1}) & \text{if } 12 \leq h \leq 3n. \end{cases}$$

Now observe that $c_1 - c_0 = (\psi, n)$ and $c_{3n} - c_{3n+1} = (x_{3n} - x_{3n+1}, y_{3n} - y_{3n+1}) = (x_{3n}, y_{3n} - y_{3n+1})$. With the use of the iterating formula giving $x_i$ for $5 \leq i \leq 3n$, we get

$$x_{3n} = x_4 - \sum_{i=5}^{3n} f(y_i - y_{i-1}) = x_4 + \theta + f(3n) = -\psi.$$

Also, checking that $y_{3n} - y_{3n+1} = n$, one obtains:

$$\{(1, n), (-1, n)\} = \{c_1 - c_0, c_{3n} - c_{3n+1}\}.$$
Moreover, since $\psi \neq \emptyset$, we obtain:
$$\{(1, 5n), (-1, 5n)\} = \{c_1 - c_2, c_4 - c_3\};$$

Finally
$$(1, 11n), (-1, 11n) \in \Delta T_0.$$

Now note that $\Delta T \cup \Delta C$ has size $6n + 2m = |G - \{0\}|$ and hence, by the pigeon-hole principle, we may claim that $\Delta T \cup \Delta C$ covers exactly once $G - \{0\}$. The assertion follows. $\square$

**Remark.** The above Theorem 5.1 proves our conjecture to be true for $v = 3p$, $p$ an odd prime. In fact, it is immediate to check that in this case $P$ is a partition of $(v - 1)/2$ into proper divisors of $v$ if and only if $p = 6n + 1$ and $P = \{3, \ldots, 3, p\}$. The theorem also provides a solution to $HWP(18n + 3; 3, 6n + 1; 3n, 6n + 1)$.

**References**


