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Generalized Cochrane sums and Cochrane–Hardy sums ☆

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Abstract

In this paper, the generalized Cochrane sums and Cochrane–Hardy sums are defined. The arithmetic properties of the generalized Cochrane sums are studied, and the Cochrane–Hardy sums are expressed in terms of the generalized Cochrane sums. Analogues of Subrahmanyam's identity and Knopp's theorem are given and proved. Finally, the hybrid mean value of generalized Cochrane sums, Cochrane–Hardy sums and Kloosterman sums is studied, and a few asymptotic formulae are obtained.

MSC: 11F20; 11B68; 11L05

Keywords: Cochrane sums; Hardy sums; Kloosterman sums

1. Introduction

For a positive integer q and an arbitrary integer h, the Cochrane sums are defined by

$$c(h,q)=\sum_{a=1}^{q}{}'\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{\bar{a}h}{q}\right)\right),$$

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where $\sum_a'$ denotes the summation over all a such that $(a,q)=1$,

$$((x)) = \begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

and $\bar{a}$ is defined by the equation $a\bar{a} \equiv 1 \bmod q$. The second author [17] gave the following upper bound estimate:

$$\left|c(h,q)\right| \ll \sqrt{q}\, d(q) \ln^2 q$$

and

$$\sum_{h=1}^{p-1} c^2(h,p) = \frac{5}{144}p^2 + O\left(p \exp\left(\frac{4\ln p}{\ln\ln p}\right)\right),$$

where $d(q)$ is the divisor function, and $\exp(y) = \mathrm{e}^y$.

In [14], the second author found that there are some relationships between $c(h,q)$ and Kloosterman sums

$$K(m,n;q) = \sum_{a=1}^{q}{}' \mathrm{e}\left(\frac{ma+n\bar{a}}{q}\right),$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$. For example, if q is a square-full number, then we have

$$\sum_{h=1}^{q}{}' c(h,q)K(h,1;q) = -\frac{1}{2\pi^2}q\phi(q) + O\left(q \exp\left(\frac{3\ln q}{\ln\ln q}\right)\right),$$

where $\phi(q)$ is the Euler function. For general integer $q \geqslant 3$, the second author [15] obtained the asymptotic formula

$$\sum_{h=1}^{q}{}' c(h,q)K(h,1;q) = -\frac{1}{2\pi^2}q\phi(q)\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{\frac{3}{2}+\epsilon}\right), \tag{1.1}$$

where ϵ be any fixed positive number. The authors [16] proved that the error term in (1.1) is $O(q^{1+\epsilon})$. Furthermore, Liu Hongyan [9] defined the generalized Cochrane sum as follows:

$$c(h,m,n,q) = \sum_{a=1}^{q}{}' \overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{\bar{a}h}{q}\right),$$

where

$$\overline{B}_n(x) = \begin{cases} B_n(x-[x]), & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

is the nth Bernoulli periodic function. For $m = n = 1$, $c(h, 1, 1, q) = c(h, q)$ is the classical Cochrane sums.

The Cochrane sums are analogous to the Dedekind sums:

$$s(h,q) = \sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

which is an important concept in number theory and has many applications (see [11]). In [2], Berndt studied the following Hardy sums:

$$S(h,q) = \sum_{a=1}^{q-1}(-1)^{a+1+[\frac{ha}{q}]}, \qquad s_1(h,q) = \sum_{a=1}^{q}(-1)^{[\frac{ha}{q}]}\left(\left(\frac{a}{q}\right)\right),$$

$$s_2(h,q) = \sum_{a=1}^{q}(-1)^{a}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right), \qquad s_3(h,q) = \sum_{a=1}^{q}(-1)^{a}\left(\left(\frac{ha}{q}\right)\right),$$

$$s_4(h,q) = \sum_{a=1}^{q-1}(-1)^{[\frac{ha}{q}]}, \qquad s_5(h,q) = \sum_{a=1}^{q}(-1)^{a+[\frac{ha}{q}]}\left(\left(\frac{a}{q}\right)\right),$$

which are related to the Dedekind sums, and obtained some arithmetic properties (see [3]). Sitaramachandrarao [12] and Pettet [10] used elementary methods to express the Hardy sums in terms of the Dedekind sums as follows:

$$\begin{aligned}
S(h,q) &= 8s(h,2q) + 8s(2h,q) - 20s(h,q), && \text{if } h+q \text{ is odd;}\\
s_1(h,q) &= 2s(h,q) - 4s(h,2q), && \text{if } h \text{ is even;}\\
s_2(h,q) &= -s(h,q) + 2s(2h,q), && \text{if } q \text{ is even;}\\
s_3(h,q) &= 2s(h,q) - 4s(2h,q), && \text{if } q \text{ is odd;}\\
s_4(h,q) &= -4s(h,q) + 8s(h,2q), && \text{if } h \text{ is odd;}\\
s_5(h,q) &= -10s(h,q) + 4s(2h,q) + 4s(h,2q), && \text{if } h+q \text{ is even.}
\end{aligned}$$

In [7,8], the authors gave some high power mean value formulae for Hardy sums.

Naturally, one might consider whether the Hardy sums be generalized in the same way the Dedekind sums are generalized to the Cochrane sums? If yes, then what can be expected? These problems may be interesting.

First we define the Cochrane–Hardy sums as follows:

$$c_1(h,m,q) = {\sum_{a=1}^{q}}'(-1)^{[\frac{ha}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right), \quad m \equiv 1 \mod 2,$$

$$c_2(h,m,n,q) = {\sum_{a=1}^{q}}'(-1)^{a}\overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{h\bar{a}}{q}\right), \quad m \equiv n \mod 2,$$

$$c_3(h,n,q)=\sum_{a=1}^{q}{}'(-1)^a\overline{B}_n\left(\frac{h\bar{a}}{q}\right),\quad n\equiv 1\mod 2,$$

$$c_5(h,m,q)=\sum_{a=1}^{q}{}'(-1)^{a+[\frac{ha}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right),\quad m\equiv 1\mod 2.$$

We will study the arithmetic properties of $c(h,m,n,q)$ in Section 2, and give some interesting identities. In Section 3, the Cochrane–Hardy sums will be expressed in terms of the generalized Cochrane sums. Next we pose and prove generalized Subrahmanyam's identities and Knopp's theorems. Finally in Section 5, the hybrid mean value of generalized Cochrane sums, Cochrane–Hardy sums and Kloosterman sums is studied, and a few asymptotic formulae are obtained.

2. Arithmetic properties of the generalized Cochrane sums

From [1, Theorem 12.19] we know that

$$B_n(x)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\mathrm{e}(rx)}{r^n},\quad 0<x\leqslant 1. \tag{2.1}$$

Then for any real number x, one can easily deduce

$$\overline{B}_n(-x)=(-1)^n\overline{B}_n(x). \tag{2.2}$$

If x is not an integer, then by (2.1) we get

$$\sum_{j=0}^{k-1}\overline{B}_n\left(\frac{x+j}{k}\right)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{1}{r^n}\sum_{j=0}^{k-1}\mathrm{e}\left(\frac{rx+rj}{k}\right)=-\frac{kn!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0\\ k\mid r}}^{+\infty}\frac{\mathrm{e}(\frac{rx}{k})}{r^n}$$

$$=-\frac{n!}{k^{n-1}(2\pi i)^n}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{\mathrm{e}(sx)}{s^n}=\frac{1}{k^{n-1}}\overline{B}_n(x). \tag{2.3}$$

First we establish a connection between generalized Cochrane sums and Gauss sums:

Theorem 2.1. *Let $q\geqslant 3$ be an integer and $(h,q)=1$. Then we have*

$$c(h,m,n,q)=\frac{4m!n!}{(2\pi i)^{m+n}\phi(q)}\sum_{\substack{\chi\bmod q\\ \chi(-1)=(-1)^m=(-1)^n}}\overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(r,\chi)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(s,\chi)}{s^n}\right),$$

where $G(r,\chi)=\sum_{a=1}^{q}\chi(a)\mathrm{e}(\frac{ar}{q})$, a Gauss sum.

Proof. From the orthogonality relations for characters and (2.1) we can get

$$
\begin{aligned}
&c(h,m,n,q)\\
&=\sum_{a=1}^{q}{}' \overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{\bar{a}h}{q}\right)\\
&=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\left[\sum_{a=1}^{q}\chi(a)\overline{B}_m\left(\frac{a}{q}\right)\right]\left[\sum_{b=1}^{q}\chi(b)\overline{B}_n\left(\frac{bh}{q}\right)\right]\\
&=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(h)\left[\sum_{a=1}^{q}\chi(a)\overline{B}_m\left(\frac{a}{q}\right)\right]\left[\sum_{b=1}^{q}\chi(b)\overline{B}_n\left(\frac{b}{q}\right)\right]\\
&=\frac{m!n!}{(2\pi i)^{m+n}\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(h)\left[\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{1}{r^m}\sum_{a=1}^{q}\chi(a)\mathrm{e}\left(\frac{ra}{q}\right)\right]\left[\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{s^n}\sum_{b=1}^{q}\chi(b)\mathrm{e}\left(\frac{sb}{q}\right)\right]\\
&=\frac{m!n!}{(2\pi i)^{m+n}\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(h)\left(1+\frac{\chi(-1)}{(-1)^m}\right)\left(\sum_{r=1}^{+\infty}\frac{G(r,\chi)}{r^m}\right)\left(1+\frac{\chi(-1)}{(-1)^n}\right)\left(\sum_{s=1}^{+\infty}\frac{G(s,\chi)}{s^n}\right)\\
&=\frac{4m!n!}{(2\pi i)^{m+n}\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m=(-1)^n}}\overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(r,\chi)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(s,\chi)}{s^n}\right).
\end{aligned}
$$

This proves Theorem 2.1. □

Remarks. From Theorem 2.1 we know that, $c(h,m,n,q)=0$ if $m\not\equiv n \bmod 2$.

Next we prove the following identity, which will be used in Section 4.

Theorem 2.2. *Let $k>0$, $q>0$ and h be integers with $(q,k)=1$, then*

$$
c(kh,m,n,kq)=\sum_{d\mid k}\mu(d)\left(\frac{k}{d}\right)^{1-m}c(\bar{d}h,m,n,q),
$$

where $d\bar{d}\equiv 1 \bmod q$, and $\mu(d)$ is the Möbius function.

Proof. Let $u=ak+bq$. It is not hard to show

$$
\bar{u}=\bar{a}\bar{k}^2k+\bar{b}\bar{q}^2q,
$$

where $u\bar{u}\equiv 1 \bmod kq$, $k\bar{k}\equiv 1 \bmod q$, $q\bar{q}\equiv 1 \bmod k$, $a\bar{a}\equiv 1 \bmod q$, $b\bar{b}\equiv 1 \bmod k$. Therefore

$$
\begin{aligned}
&c(kh,m,n,kq)\\
&\quad=\sum_{u=1}^{kq}{}' \,\overline{B}_m\left(\frac{u}{kq}\right)\overline{B}_n\left(\frac{\bar{u}kh}{kq}\right)=\sum_{u=1}^{kq}{}' \,\overline{B}_m\left(\frac{u}{kq}\right)\overline{B}_n\left(\frac{\bar{u}h}{q}\right)\\
&\quad=\sum_{a=1}^{q}{}'\sum_{b=1}^{k}{}' \,\overline{B}_m\left(\frac{ak+bq}{kq}\right)\overline{B}_n\left(\frac{h(\bar{a}\bar{k}^2k+\bar{b}\bar{q}^2q)}{q}\right)=\sum_{a=1}^{q}{}'\sum_{b=1}^{k}{}' \,\overline{B}_m\left(\frac{a}{q}+\frac{b}{k}\right)\overline{B}_n\left(\frac{h\bar{a}\bar{k}^2k}{q}\right)\\
&\quad=\sum_{a=1}^{q}{}' \,\overline{B}_n\left(\frac{h\bar{a}\bar{k}}{q}\right)\sum_{b=1}^{k}{}' \,\overline{B}_m\left(\frac{a}{q}+\frac{b}{k}\right)=\sum_{d\mid k}\mu(d)\sum_{a=1}^{q}{}' \,\overline{B}_n\left(\frac{h\bar{a}\bar{k}}{q}\right)\sum_{b=0}^{\frac{k}{d}-1}\overline{B}_m\left(\frac{a}{q}+\frac{bd}{k}\right).
\end{aligned}
$$

Then from (2.3) we can have

$$
\begin{aligned}
c(kh,m,n,kq)&=\sum_{d\mid k}\mu(d)\sum_{a=1}^{q}{}' \,\overline{B}_n\left(\frac{h\bar{a}\bar{k}}{q}\right)\left(\frac{k}{d}\right)^{1-m}\overline{B}_m\left(\frac{ak}{qd}\right)\\
&=\sum_{d\mid k}\mu(d)\left(\frac{k}{d}\right)^{1-m}\sum_{a=1}^{q}{}' \,\overline{B}_n\left(\frac{h\bar{a}\bar{d}}{q}\right)\overline{B}_m\left(\frac{a}{q}\right)\\
&=\sum_{d\mid k}\mu(d)\left(\frac{k}{d}\right)^{1-m}c(\bar{d}h,m,n,q).
\end{aligned}
$$

This completes the proof of Theorem 2.2. □

3. Cochrane–Hardy sums in terms of the generalized Cochrane sums

We shall prove the following:

Theorem 3.1. *Let $q \geqslant 3$ be an integer and $(h,q)=1$. Then we have*

$$
\begin{aligned}
c_1(h,m,q)&=2c(h,m,1,q)-4c\left(\frac{h}{2},m,1,q\right), &&\text{if } h \text{ is even}; &(3.1)\\
c_2(h,m,n,q)&=-c(h,m,n,q), &&\text{if } q \text{ is even}; &(3.2)\\
c_3(h,n,q)&=2c(h,1,n,q)-4c(\bar{2}h,1,n,q), &&\text{if } q \text{ is odd}; &(3.3)\\
c_5(h,m,q)&=2c(h,m,1,q)-4c\left(\frac{h+q}{2},m,1,q\right), &&\text{if } h+q \text{ is even}, &(3.4)
\end{aligned}
$$

where $2\cdot\bar{2}\equiv 1 \bmod q$. Moreover, each one of

$$
\begin{cases}
c_1(h,m,q)\quad (h\ odd), & c_2(h,m,n,q)\quad (q\ odd),\\
c_3(h,n,q)\quad (q\ even), & c_5(h,m,q)\quad (h+q\ odd)
\end{cases}
\tag{3.5}
$$

is zero.

Proof. By [12, (5.8)] we know that

$$(-1)^{[x]} = 2((x)) - 4\left(\left(\frac{x}{2}\right)\right), \quad \text{if } x \text{ is not an integer.} \tag{3.6}$$

Then for any even number h,

$$\begin{aligned}
c_1(h,m,q) &= \sum_{a=1}^{q}{}' (-1)^{[\frac{ha}{q}]} \overline{B}_m\left(\frac{\bar{a}}{q}\right) \\
&= 2\sum_{a=1}^{q}{}' \left(\left(\frac{ha}{q}\right)\right)\overline{B}_m\left(\frac{\bar{a}}{q}\right) - 4\sum_{a=1}^{q}{}' \left(\left(\frac{ha}{2q}\right)\right)\overline{B}_m\left(\frac{\bar{a}}{q}\right) \\
&= 2c(h,m,1,q) - 4c\left(\frac{h}{2},m,1,q\right).
\end{aligned}$$

This proves (3.1).

If q is even, then $(h,k)=1$ only if h is odd. Therefore

$$\begin{aligned}
c_2(h,m,n,q) &= \sum_{a=1}^{q}{}' (-1)^a \overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{h\bar{a}}{q}\right) = -\sum_{a=1}^{q}{}' \overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{h\bar{a}}{q}\right) \\
&= -c(h,m,n,q).
\end{aligned}$$

This proves (3.2).

From the orthogonality relations for characters we can get

$$\begin{aligned}
c_3(h,n,q) &= \sum_{a=1}^{q}{}' (-1)^a \overline{B}_n\left(\frac{h\bar{a}}{q}\right) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left[\sum_{a=1}^{q} (-1)^a \chi(a)\right]\left[\sum_{b=1}^{q} \chi(b)\overline{B}_n\left(\frac{hb}{q}\right)\right] \\
&= \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left[\sum_{a=1}^{q} (-1)^a \chi(a)\right]\left[\sum_{b=1}^{q} \chi(b)\overline{B}_n\left(\frac{hb}{q}\right)\right],
\end{aligned}$$

where we used that $\sum_{b=1}^{q} \chi(b)\overline{B}_n(\frac{hb}{q}) = 0$ for even characters χ, if $n \equiv 1 \bmod 2$.

Noting that for odd characters (see [5])

$$\begin{aligned}
\sum_{a=1}^{q} (-1)^a \chi(a) &= 2\chi(2)\sum_{a=1}^{\frac{q-1}{2}} \chi(a) = \frac{2(1-2\chi(2))}{q}\sum_{a=1}^{q} a\chi(a) = 2\big(1-2\chi(2)\big)\sum_{a=1}^{q} \chi(a)\cdot\frac{a}{q} \\
&= 2\big(1-2\chi(2)\big)\sum_{a=1}^{q} \chi(a)\left(\left(\frac{a}{q}\right)\right) = 2\big(1-2\chi(2)\big)\sum_{a=1}^{q} \chi(a)\overline{B}_1\left(\frac{a}{q}\right),
\end{aligned}$$

then we have

$$
\begin{aligned}
c_3(h,n,q) &= \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left[2\big(1-2\chi(2)\big)\sum_{a=1}^{q}\chi(a)\overline{B}_1\left(\frac{a}{q}\right)\right]\left[\sum_{b=1}^{q}\chi(b)\overline{B}_n\left(\frac{hb}{q}\right)\right] \\
&= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left[2\big(1-2\chi(2)\big)\sum_{a=1}^{q}\chi(a)\overline{B}_1\left(\frac{a}{q}\right)\right]\left[\sum_{b=1}^{q}\chi(b)\overline{B}_n\left(\frac{hb}{q}\right)\right] \\
&= 2\sum_{a=1}^{q}{}' \,\overline{B}_1\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{h\bar{a}}{q}\right) - 4\sum_{a=1}^{q}{}' \,\overline{B}_1\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{h\bar{2}\bar{a}}{q}\right) \\
&= 2c(h,1,n,q) - 4c(\bar{2}h,1,n,q).
\end{aligned}
$$

This proves (3.3).

From (3.6) we can get

$$
\begin{aligned}
c_5(h,m,q) &= \sum_{a=1}^{q}{}'(-1)^{a+[\frac{ha}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right) = \sum_{a=1}^{q}{}'(-1)^{[\frac{a(h+q)}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right) \\
&= \sum_{a=1}^{q}{}'\left[2\left(\left(\frac{a(h+q)}{q}\right)\right) - 4\left(\left(\frac{a(h+q)}{2q}\right)\right)\right]\overline{B}_m\left(\frac{\bar{a}}{q}\right) \\
&= 2c(h,m,1,q) - 4c\left(\frac{h+q}{2},m,1,q\right).
\end{aligned}
$$

This proves (3.4).

To prove (3.5), by (2.2) we have

$$
\begin{aligned}
c_1(h,m,q) &= \sum_{a=1}^{q}{}'(-1)^{[\frac{ha}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right) = \sum_{a=1}^{q}{}'(-1)^{[\frac{h(q-a)}{q}]}\overline{B}_m\left(\frac{\overline{q-a}}{q}\right) \\
&= \sum_{a=1}^{q}{}'(-1)^{h-[\frac{ha}{q}]-1}\overline{B}_m\left(-\frac{\bar{a}}{q}\right) = (-1)^{h+m-1}\sum_{a=1}^{q}{}'(-1)^{[\frac{ha}{q}]}\overline{B}_m\left(\frac{\bar{a}}{q}\right) \\
&= (-1)^h c_1(h,m,q).
\end{aligned}
$$

Then $c_1(h,m,q)=0$ if h is odd. The proofs of the remaining assertions in (3.5) are similar. □

4. Generalized Subrahmanyam's identities and Knopp's theorems

Subrahmanyam [13] proved the following identity:

Proposition 4.1. *For any positive integer d, we have*

$$\sum_{b \bmod d} s(h+bk, dk) = \sum_{c|d} \mu(c) s(hc, k) \sigma\left(\frac{d}{c}\right),$$

where $\sigma(n) = \sum_{d|n} d$.

By using the deep properties of $\log \eta(\tau)$, Knopp [6] got the following:

Proposition 4.2. *Let* $d > 0$, $k > 0$ *and* h *be integers, then*

$$\sum_{\substack{a\delta=d \\ \delta>0}} \sum_{b \bmod \delta} s(ah+bk, \delta k) = \sigma(d) s(h, k).$$

Proposition 4.1 in case $d = p$, is due to Dedekind [4]. Sitaramachandrarao [12] showed that Propositions 4.1 and 4.2 could be deduced from one another by elementary arguments.

In this section we shall pose and prove analogues of the Subrahmanyam identity and Knopp theorem for Cochrane sums. First we have

Theorem 4.1. *For any positive integers* d, k *with* $(d, k) = 1$, *we have*

$$d^{n-1} \sum_{b \bmod d} c(h+bk, m, n, dk) = \sum_{t|d} \mu(t) \left(\frac{d}{t}\right)^{1-m} c(\bar{t}h, m, n, k),$$

where $t\bar{t} \equiv 1 \bmod k$.

Proof. From (2.3) and Theorem 2.2 we can get

$$\begin{aligned}
&\sum_{b \bmod d} c(h+bk, m, n, dk) \\
&\quad= \sum_{b \bmod d} \sideset{}{'}\sum_{j=1}^{dk} \overline{B}_m\left(\frac{j}{dk}\right) \overline{B}_n\left(\frac{\bar{j}(h+bk)}{dk}\right) \\
&\quad= \sideset{}{'}\sum_{j=1}^{dk} \overline{B}_m\left(\frac{j}{dk}\right) \sum_{b \bmod d} \overline{B}_n\left(\frac{h\bar{j}}{dk} + \frac{b\bar{j}}{d}\right) = d^{1-n} \sideset{}{'}\sum_{j=1}^{dk} \overline{B}_m\left(\frac{j}{dk}\right) \overline{B}_n\left(\frac{h\bar{j}}{k}\right) \\
&\quad= d^{1-n} c(dh, m, n, dk) = d^{1-n} \sum_{t|d} \mu(t) \left(\frac{d}{t}\right)^{1-m} c(\bar{t}h, m, n, k).
\end{aligned}$$

This proves Theorem 4.1. □

To extend Knopp's theorem to generalized Cochrane sums, we need the following:

Lemma 4.1. *Let* $f(h, d)$ *and* $g(h, d)$ *be two complex valued sequences defined for positive integers* h *and* d *with* $f(h+k, d) = f(h, d)$ *and* $g(h+k, d) = g(h, d)$, *then*

$$f(h, d) = \sum_{a\delta=d} g(\bar{a}h, \delta) \tag{4.1}$$

if and only if

$$g(h,d)=\sum_{a\delta=d}\mu(a)f(\bar{a}h,\delta), \tag{4.2}$$

where $a\bar{a}\equiv 1 \bmod k$.

Proof. Assume (4.2). Then

$$\begin{aligned}\sum_{a\delta=d}g(\bar{a}h,\delta)&=\sum_{a\delta=d}\sum_{ec=\delta}\mu(e)f(\bar{e}\cdot\bar{a}h,c)=\sum_{aec=d}\mu(e)f(\overline{ea}h,c)\\&=\sum_{bc=d}f(\bar{b}h,c)\sum_{ae=b}\mu(e)=f(h,d),\end{aligned}$$

which is (4.1). Similarly we can deduce (4.2) from (4.1). □

Now taking

$$f(h,d)=c(h,m,n,k)d^{1-m},\qquad g(h,d)=d^{n-1}\sum_{b \bmod d}c(h+bk,m,n,dk)$$

in Lemma 4.1, we have

Theorem 4.2. *Let d and k be positive integers with $(d,k)=1$, then*

$$\sum_{\substack{a\delta=d\\ \delta>0}}\delta^{n-1}\sum_{b \bmod \delta}c(\bar{a}h+bk,m,n,\delta k)=c(h,m,n,k)d^{1-m},$$

where $a\bar{a}\equiv 1 \bmod k$.

Using the same methods we can also get the following generalized Subrahmanyam's identities and Knopp's theorems for Cochrane–Hardy sums:

Theorem 4.3. *Let d and k be positive odd integers with $(d,k)=1$ and $(h,k)=1$, then*

$$\begin{cases}\displaystyle\sum_{b \bmod d}c_1(h+2bk,m,dk)=\sum_{t\mid d}\mu(t)\left(\frac{d}{t}\right)^{1-m}c_1(\bar{t}h,m,k),\\ \displaystyle\sum_{\substack{a\delta=d\\ \delta>0}}\sum_{b \bmod \delta}c_1(\bar{a}h+2bk,m,\delta k)=c_1(h,m,k)d^{1-m},\end{cases}\quad \text{if } h \text{ is even};$$

$$\begin{cases}\displaystyle d^{n-1}\sum_{b \bmod d}c_2(h+bk,m,n,dk)=\sum_{t\mid d}\mu(t)\left(\frac{d}{t}\right)^{1-m}c_2(\bar{t}h,m,n,k),\\ \displaystyle\sum_{\substack{a\delta=d\\ \delta>0}}\delta^{n-1}\sum_{b \bmod \delta}c_2(\bar{a}h+bk,m,n,\delta k)=c_2(h,m,n,k)d^{1-m},\end{cases}\quad \text{if } k \text{ is even};$$

$$\begin{cases} d^{n-1}\displaystyle\sum_{b \bmod d} c_3(h+bk,n,dk) = \sum_{t\mid d}\mu(t)c_3(\bar{t}h,n,k), \\ \displaystyle\sum_{\substack{a\delta=d\\ \delta>0}} \delta^{n-1}\sum_{b \bmod \delta} c_3(\bar{a}h+bk,n,\delta k) = c_3(h,n,k), \end{cases} \quad \text{if } k \text{ is odd};$$

$$\begin{cases} \displaystyle\sum_{b \bmod d} c_5(h+2bk,m,dk) = \sum_{t\mid d}\mu(t)\left(\frac{d}{t}\right)^{1-m} c_5(\bar{t}h,m,k), \\ \displaystyle\sum_{\substack{a\delta=d\\ \delta>0}} \sum_{b \bmod \delta} c_5(\bar{a}h+2bk,m,\delta k) = c_5(h,m,k)d^{1-m}, \end{cases} \quad \text{if } h \text{ and } k \text{ are odd},$$

where $t\bar{t} \equiv 1 \bmod k$ *and* $a\bar{a} \equiv 1 \bmod k$.

5. Hybrid mean value of generalized Cochrane sums, Cochrane–Hardy sums and Kloosterman sums

First we need the following:

Lemma 5.1. *Let* q *and* r *be integers with* $q \geqslant 2$ *and* $(r,q)=1$, χ *be a Dirichlet character modulo* q. *Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d\mid (q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d) \quad \text{and}$$

$$J(q) = \sum_{d\mid q}\mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ *denotes the summation over all primitive characters modulo* q, *and* $J(q)$ *denotes the number of primitive characters modulo* q.

Proof. This is Lemma 4 of [16]. □

Lemma 5.2. *Let* $q=uv$, *where* $(u,v)=1$, u *be a square-full number or* $u=1$, v *be a square-free number. Then for positive integers* m, n *with* $m \equiv n \bmod 2$, *we have*

$$\text{(I)} \quad \Psi_1 :\equiv \sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n \phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=(-1)^m}} \overline{\chi}(d_1d_2)L(m,\overline{\chi})L(n,\overline{\chi})$$

$$= \frac{q}{2}\prod_{p\| q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{\epsilon}\right) \quad \text{and}$$

$$\text{(II)} \quad \Psi_2 :\equiv \sum_{d\mid v}\sum_{d_1\mid \frac{v}{d}}\sum_{d_2\mid \frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n \phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=(-1)^m}} \overline{\chi}(2d_1d_2)L(m,\overline{\chi})L(n,\overline{\chi})$$

$$\ll q^{\epsilon}.$$

Proof. We only prove (I), since similarly we can deduce (II). For the case $m = n = 1$, see [16, Lemma 5]. Now suppose that $n \geqslant m > 1$. Let $\sigma_\alpha(r) = \sum_{d|r} d^\alpha$, then we have

$$\Psi_1 = \sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n \phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sum_{r=1}^{+\infty} \frac{\sigma_{m-n}(r)}{r^m} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=(-1)^m}} \overline{\chi}(d_1d_2r).$$

For $(a,k)=1$, from Lemma 5.1 we have

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^m}} \chi(a) &= \frac{1}{2} \sideset{}{^*}\sum_{\chi \bmod q} \bigl(1+\chi(-1)(-1)^m\bigr)\chi(a) = \frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod q}\chi(a) + \frac{(-1)^m}{2}\sideset{}{^*}\sum_{\chi \bmod q}\chi(-a) \\ &= \frac{1}{2}\sum_{u|(q,a-1)} \mu\left(\frac{q}{u}\right)\phi(u) + \frac{(-1)^m}{2}\sum_{u|(q,a+1)}\mu\left(\frac{q}{u}\right)\phi(u).\end{aligned}$$

Then using the same methods of Lemma 5 in [16] we have

$$\begin{aligned}\Psi_1 &= \frac{1}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n \phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sum_{\substack{r=1 \\ (r,ud)=1}}^{+\infty} \frac{\sigma_{m-n}(r)}{r^m} \sum_{s|(ud,d_1d_2r-1)} \mu\left(\frac{ud}{s}\right)\phi(s) \\ &\quad + \frac{(-1)^m}{2}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n \phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sum_{\substack{r=1 \\ (r,ud)=1}}^{+\infty} \frac{\sigma_{m-n}(r)}{r^m} \sum_{s|(ud,d_1d_2r+1)} \mu\left(\frac{ud}{s}\right)\phi(s) \\ &= \frac{1}{2}\sum_{d|v} u^2d^2 \sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) \sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_2^{n-m}\phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sum_{\substack{r=1 \\ (r,ud)=1 \\ d_1d_2r\equiv 1 \bmod s}}^{+\infty} \frac{\sigma_{m-n}(r)}{(d_1d_2r)^m} \\ &\quad + \frac{(-1)^m}{2}\sum_{d|v} u^2d^2 \sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) \sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_2^{n-m}\phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sum_{\substack{r=1 \\ (r,ud)=1 \\ d_1d_2r\equiv -1 \bmod s}}^{+\infty} \frac{\sigma_{m-n}(r)}{(d_1d_2r)^m} \\ &= \frac{1}{2}\sum_{d|v}\frac{u^2d^2}{\phi^2(q)}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s) + O\bigl(q^\epsilon\bigr) = \frac{u^2}{2\phi^2(q)}\sum_{d|v} d^2 J(ud) + O\bigl(q^\epsilon\bigr) \\ &= \frac{q}{2}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\bigl(q^\epsilon\bigr).\end{aligned}$$

This proves Lemma 5.2. $\square$

Now we can prove the following result:

Theorem 5.1. *For any positive integers $q \geqslant 3$, m and n with $m \equiv n \bmod 2$, we have*

$$\sum_{h=1}^{q}{}' c(h,m,n,q)K(h,1;q) = \frac{2m!n!\phi(q)q}{(2\pi i)^{m+n}} \prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{1+\epsilon}\right).$$

Proof. Let $q = uv$, where $(u,v)=1$, u be a square-full number or $u=1$, v be a square-free number. Using Theorem 2.1, (I) of Lemma 5.2 and the methods of [16] we have

$$\begin{aligned}
&\sum_{h=1}^{q}{}' c(h,m,n,q)K(h,1;q)\\
&\quad= \frac{4m!n!}{(2\pi i)^{m+n}\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m}} \left(\sum_{h=1}^{q}\overline{\chi}(h)K(h,1;q)\right)\left(\sum_{r=1}^{+\infty}\frac{G(r,\chi)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(s,\chi)}{s^n}\right)\\
&\quad= \frac{4m!n!}{(2\pi i)^{m+n}\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m}} \tau^2(\overline{\chi})\left(\sum_{r=1}^{+\infty}\frac{G(r,\chi)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(s,\chi)}{s^n}\right)\\
&\quad= \frac{4m!n!}{(2\pi i)^{m+n}\phi(q)} \sum_{d|v} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=(-1)^m}} \overline{\chi}^2\left(\frac{v}{d}\right)\mu^2\left(\frac{v}{d}\right)\tau^2(\overline{\chi})\\
&\qquad\times\left(\sum_{d_1|\frac{v}{d}}\frac{\phi(q)\chi(\frac{v}{dd_1})\mu(\frac{v}{dd_1})\tau(\chi)L(m,\overline{\chi})}{d_1^m\phi(\frac{q}{d_1})}\right)\left(\sum_{d_2|\frac{v}{d}}\frac{\phi(q)\chi(\frac{v}{dd_2})\mu(\frac{v}{dd_2})\tau(\chi)L(n,\overline{\chi})}{d_2^n\phi(\frac{q}{d_2})}\right)\\
&\quad= \frac{4m!n!\phi(q)}{(2\pi i)^{m+n}} \sum_{d|v}\sum_{d_1|\frac{v}{d}}\sum_{d_2|\frac{v}{d}} \frac{u^2d^2\mu(\frac{v}{dd_1})\mu(\frac{v}{dd_2})}{d_1^m d_2^n\phi(\frac{q}{d_1})\phi(\frac{q}{d_2})} \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=(-1)^m}} \overline{\chi}(d_1d_2)L(m,\overline{\chi})L(n,\overline{\chi})\\
&\quad= \frac{2m!n!\phi(q)q}{(2\pi i)^{m+n}} \prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{1+\epsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Noting that

$$c_1(2h,m,q) = c_3(2h,m,q) = c_5(2h-q,m,q), \quad \text{if } q \text{ is odd}.$$

Then using Theorem 3.1, (II) of Lemma 5.2 and Theorem 5.1 we also have the hybrid mean value formulae of Cochrane–Hardy sums and Kloosterman sums as follows:

Theorem 5.2. *Let $q \geqslant 3$ be an integer, then we have*

$$\sum_{h=1}^{q}{}' c_1(2h,m,q)K(h,1;q) = -\frac{8m!\phi(q)q}{(2\pi i)^{m+1}} \prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{1+\epsilon}\right), \quad \textit{if } q \textit{ is odd};$$

$$4\sum_{h=1}^{q}{}' c_2(h,m,1,q)K(h,1;q) = -\frac{8m!\phi(q)q}{(2\pi i)^{m+1}}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O\left(q^{1+\epsilon}\right), \quad \text{if } q \text{ is even};$$

$$\sum_{h=1}^{q}{}' c_3(2h,m,q)K(h,1;q) = -\frac{8m!\phi(q)q}{(2\pi i)^{m+1}}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O\left(q^{1+\epsilon}\right), \quad \text{if } q \text{ is odd};$$

$$\sum_{h=1}^{q}{}' c_5(2h-q,m,q)K(h,1;q) = -\frac{8m!\phi(q)q}{(2\pi i)^{m+1}}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O\left(q^{1+\epsilon}\right), \quad \text{if } q \text{ is odd}.$$

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