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Static spherically symmetric wormholes with isotropic pressure

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ABSTRACT

In this paper we study static spherically symmetric wormhole solutions sustained by matter sources with isotropic pressure. We show that such spherical wormholes do not exist in the framework of zero-tidal-force wormholes. On the other hand, it is shown that for the often used power-law shape function there are no spherically symmetric traversable wormholes sustained by sources with a linear equation of state $p = \omega\rho$ for the isotropic pressure, independently of the form of the redshift function $\phi(r)$. We consider a solution obtained by Tolman at 1939 for describing static spheres of isotropic fluids, and show that it also may describe wormhole spacetimes with a power-law redshift function, which leads to a polynomial shape function, generalizing a power-law shape function, and inducing a solid angle deficit.

1. Introduction

In the framework of Einstein General Relativity the study of spherically symmetric traversable wormhole spacetimes [1] has been mostly focused in matter sources with anisotropic pressures. Mainly this is due to the fact that in order to correctly describe a wormhole spacetime one needs a redshift function without horizons, or the redshift and the shape functions giving a desired asymptotic. In this way, the theoretical construction of wormhole geometries is usually performed by assuming a priori the form of the redshift and the shape functions, in order to have a desired metric. But Einstein's field equations for spherically symmetric spacetimes imply that the radial and lateral (or transverse) pressures are not equal. In such a way, by imposing restricted choices on the redshift and the shape functions we will obtain, in general, for the energy–momentum tensor that $T^2_2 \neq T^3_3$. This condition implies that a wormhole configuration is necessarily supported by an anisotropic matter source. However, this method has some limitations since we can obtain for the energy density, radial and lateral pressures algebraic expressions which are physically unreasonable.

One may also follow a more conventional method used for finding solutions in general relativity, by prescribing the matter content with specific equations of state for the radial and/or the tangential pressures, and then solve Einstein's field equations in order to find the redshift and shape functions. The often used equation of state is the linear barotropic equation of state $p_r = \omega\rho$, which relates the radial pressure with the energy density [2]. From the cosmological setting, such an equation of state is associated with phantom dark energy if $\omega < -1$, and also may sustain traversable static wormholes [3,4].

Anisotropic stress contributions to the gravitational field can arise from specific matter fields. A fluid source with anisotropic stresses supporting wormholes may be for example of electromagnetic nature: linear [5] and nonlinear Maxwell fields [6] have been considered in the literature. An electric field coupled to a scalar field is considered in Ref. [7].

A spatially varying cosmological constant also has been considered in the framework of static wormholes supported by anisotropic matter sources [8]. In Ref. [9] the source of the stress-energy tensor supporting the wormhole geometries consists of an anisotropic brown dwarf "star" which smoothly joins the vacuum and may possess an arbitrary cosmological constant, while in Ref. [10] anisotropic vacuum stress-energy of quantized fields has been proposed as source for static wormholes.

Such anisotropic scenarios are obtained also for regular static, spherically symmetric solutions describing wormholes supported by dark matter non-minimally coupled to dark energy in the form of a quintessence scalar field [11].

The main purpose of this paper is to present and discuss static spherically symmetric wormhole spacetimes supported by a single perfect fluid, i.e. a matter source with isotropic pressure. As far as we know, the only spherical wormhole solution discussed up to now is the not asymptotically flat wormhole with isotropic

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pressure considered in Ref. [4]. We will discuss this solution in more detail below in Sections 4 and 5. It must be noticed that the study of wormhole solutions sustained by a perfect fluid allows us to consider phantom wormholes sustained by inhomogeneous and isotropic phantom dark energy.

The paper is organized as follows. In Sec. 2 we write the Einstein equations for static spherically symmetric spacetimes. In Sec. 3 we analyze the possibility of having zero-tidal-force wormholes sustained by a matter source with isotropic pressure, while in Sec. 4 we analyze the possibility of having spherical wormholes sustained by a fluid with linear equation of state. In Sec. 5 we reobtain an analytical solution, previously obtained by Tolman, which describes static spheres of fluids with isotropic pressure, and we show that it may describe also a non-asymptotically flat wormhole geometry.

2. Field equations for static spherically symmetric spacetimes

The spacetime ansatz for seeking static spherically symmetric solutions can be written in Schwarzschild coordinates as

$$ds^2 = e^{2\phi(r)}dt^2 - \frac{dr^2}{1-\frac{b(r)}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where $e^{\phi(r)}$ and $b(r)$ are arbitrary functions of the radial coordinate. In the case of wormholes these functions are referred to as redshift function and shape function respectively. The essential characteristics of a wormhole geometry are encoded in these functions, so in order to have a wormhole these two functions must satisfy some general constraints discussed by Morris and Thorne in Refs. [1,12].

By assuming that the matter content is described by a single imperfect fluid, from the metric (1) and the Einstein field equations $G_{\mu\nu} = -\kappa T_{\mu\nu}$ we obtain

$$\kappa\rho(r) = \frac{b'}{r^2}, \tag{2}$$

$$\kappa p_r(r) = 2\left(1-\frac{b}{r}\right)\frac{\phi'}{r} - \frac{b}{r^3}, \tag{3}$$

$$\kappa p_l(r) = \left(1-\frac{b}{r}\right)\times \left[\phi'' + \phi'^2 - \frac{b'r+b-2r}{2r(r-b)}\phi' - \frac{b'r-b}{2r^2(r-b)}\right], \tag{4}$$

where $\kappa = 8\pi G$, ρ is the energy density, and p_r and p_l are the radial and lateral pressures respectively. From the conservation equation $T^{\mu\nu}{}_{;\nu} = 0$ we obtain the hydrostatic equation for equilibrium of the matter sustaining the wormhole

$$p_r' = \frac{2(p_l - p_r)}{r} - (\rho + p_r)\phi'. \tag{5}$$

It becomes clear that the main condition for having a perfect fluid is given by

$$p_r = p_l. \tag{6}$$

This condition on the radial and lateral pressures allows us to get the following differential equation connecting functions $\phi(r)$ and $b(r)$:

$$\phi'' + \phi'^2 - \frac{b'r-3b+2r}{2r(r-b)}\phi' = \frac{b'r-3b}{2r^2(r-b)}. \tag{7}$$

We may consider Eq. (7) as a differential equation for one of these involved functions, by giving the remaining one. By supposing that the redshift function $\phi(r)$ is given, we obtain a first order differential equation for the shape function $b(r)$, whose general solution is given by

$$b(r) = \left(\int \frac{2r\left(r\phi'' + r\phi'^2 - \phi'\right)e^{\int \frac{2r^2\phi''+2r^2\phi'^2-3r\phi'-3}{r(1+r\phi')}dr}}{1+r\phi'}dr + C\right) \times e^{-\int \frac{2r^2\phi''+2r^2\phi'^2-3r\phi'-3}{r(1+r\phi')}dr}, \tag{8}$$

where C is an integration constant. Eqs. (7) and (8) have a general character, in the sense that they do not involve an equation of state for ρ and p. It must be remarked that for these static configurations, sustained by isotropic perfect fluids, the Einstein field equations are reduced to a set of three independent differential equations (2), (3) and (7) for four unknown functions, namely $\phi(r)$, $b(r)$, $\rho(r)$ and $p(r)$. Thus, to study solutions to these field equations, restricted choices of one of the unknown functions must be considered.

To obtain a realistic stellar model, one can start with an equation of state. Such input equations of state do not normally allow for closed form solutions. In arriving to exact solutions, one can solve the field equations by making an ad hoc assumption for one of the metric functions or for the energy density. Hence the equation of state can be computed from the resulting metric.

3. On zero-tidal-force wormholes with isotropic pressure

It is well known that a simple class of solutions corresponds to zero-tidal-force wormhole spacetimes, which are defined by the condition $\phi(r) = \phi_0 = const$ [1,12]. By putting $\phi(r) = const$ into Eq. (8) we obtain $b(r) = Cr^3$. By requiring that $b(r_0) = r_0$, the spacetime metric takes the form

$$ds^2 = dt^2 - \frac{dr^2}{1-\left(\frac{r}{r_0}\right)^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{9}$$

This metric represents a spacetime of constant curvature, for which the pressure and energy density are given by $p = -\rho/3 = -1/\kappa r_0^2$, and it is a particular case of the well-known static Einstein universe (for which we have $\kappa\rho = 3/r_0^2 - \Lambda$ and $\kappa p = -1/r_0^2 + \Lambda$, where Λ is the cosmological constant).

The inverse of the radial metric component g_{rr}^{-1} vanishes at $r = r_0$, as we would expect for wormholes. However, for $r > r_0$ the radial metric component g_{rr} becomes negative, so the solution is valid only for $0 \le r \le r_0$. This implies that there are no zero-tidal-force wormhole solutions sustained everywhere by an isotropic perfect fluid. In other words, any zero-tidal-force wormhole must be filled by a single fluid with anisotropic pressure. Therefore, in order to generate spherically symmetric wormholes, sustained by a single matter source with isotropic pressure, we must consider spacetimes with $\phi(r) \neq const$.

Notice that this result does not mean that it is not possible to have a zero-tidal-force wormhole sustained by a perfect (ideal) fluid at spatial infinity. For an explicit example let us consider the spacetime

$$ds^2 = dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^\alpha - 1} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{10}$$

For $\alpha > 0$ the metric covers the range $r_0 \le r < \infty$ and describes a wormhole spacetime. The energy density, radial and lateral pressures are given by $\kappa\rho = -\frac{1+\alpha}{r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-2} + \frac{2}{r^2}$, $\kappa p_r =$

$\frac{1}{r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-2} - \frac{2}{r^2}$ and $\kappa p_l = \frac{\alpha}{2r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-2}$. Since $\phi(r) = 0$ this wormhole has anisotropic pressures as we would expect. The metric (10) includes non-asymptotically flat spacetimes. Note that for $\alpha = 2$ we have that $\rho = -3/r_0^2 + 2/r^2$, $p_r = 1/r_0^2 - 2/r^2$ and $p_l = 1/r_0^2$, obtaining at spatial infinity a spacetime with constant curvature and $p_r = p_l = -\rho/3$, which means that asymptotically we have a wormhole sustained by an ideal fluid (a string gas).

4. On spherical wormholes with isotropic pressure and linear equation of state

Since, for a vanishing redshift function, the only solution allowed by the isotropic pressure condition (6) is the spacetime (9), we shall now on take into consideration a non-vanishing redshift function in all considered wormhole solutions.

In this section, in order to follow with our study, we shall impose on the isotropic pressure the linear equation of state

$$p = \omega\rho, \tag{11}$$

where the state parameter ω is a constant. Thus from Eq. (5) we have

$$\omega\rho' = -(1+\omega)\rho\phi' \tag{12}$$

and then the energy density is given by

$$\rho(r) = Ce^{-\frac{1+\omega}{\omega}\phi(r)},$$

where C is an integration constant. If $\phi(r) = const$ we obtain that energy density is constant, in agreement with the result of the previous section.

From Eqs. (2), (7) and (12) the following master differential equation for the shape function $b(r)$ is obtained:

$$-2r^2\omega(1+\omega)(r-b)b'b''' + 4\omega\left(\omega+\frac{1}{2}\right)r^2(r-b)b''^2$$
$$+\omega r((1+\omega)rb' + (5\omega-3)b + 2r - 6\omega r)b'b''$$
$$-3b'^2\left(\left(\omega+\frac{1}{3}\right)(1+\omega)rb' - \left(1+\frac{5}{3}\omega^2+\frac{16}{3}\omega\right)b + \frac{8}{3}\omega r\right)$$
$$= 0. \tag{13}$$

In general it is hard to find analytical solutions to Eq. (13). Nevertheless, one can make some checks to prove the correctness of the above equation. Notice, for example, that Eq. (13) is fulfilled identically for $b(r) = Ar^3$. Thus from Eq. (12) we obtain that $\omega = -1$ or $\phi(r) = const$. For the latter case $p = -\rho/3 = -A$, while for $\omega = -1$ we have that $e^{\phi(r)} = 1 - Ar^2$ with $p = -\rho = -3A$.

On the other hand, the first term of Eq. (13) vanishes for $\omega = 0$ and $\omega = -1$. Thus for $\omega = 0$ we obtain $b(r) = A$, $e^{2\phi(r)} = 1 - A/r$, $\rho = p = 0$, i.e. the Schwarzschild solution. It is well known that the Schwarzschild solution may be interpreted as a non-traversable wormhole.

For $\omega = -1$ we obtain the Kottler solution, i.e. $b(r) = A + Br^3$, $e^{2\phi(r)} = 1 - A/r - Br^2$, $\rho = -p = 3B$ [13].

We turn next to often used power-law form of the shape function in wormhole spacetimes: $b(r) = A/r^n$. This choice ensures that for $r \to \infty$ and $n > -1$ the M-T constraint $b(r)/r \le 1$ is satisfied. Thus, by putting $b(r) = A/r^n$ into the master equation (13) we find that this equation is satisfied if $n = -3$ (for arbitrary ω) or $n = -5/3$ (and $\omega = -3$).

The obtained negative values of the n-parameter are less than -1. This implies that there are no spherically symmetric traversable wormholes characterized by a radial metric component given by $g_{rr}^{-1} = 1 - (r_0/r)^{n+1}$ (with $n > -1$), and sustained by isotropic pressure sources with a linear equation of state (11), independently of the form of the redshift function $\phi(r)$. The implications of this result tell us that we must consider more general forms of the shape function $b(r)$ and/or of the equation of state of the isotropic pressure $p(\rho)$.

Lastly, let us note that this conclusion is not in agreement with the result obtained in subsection III-B of the Ref. [4], where the author discusses the non-asymptotically flat wormhole given by

$$ds^2 = (r/r_0)^{2\omega\left(\frac{3-\alpha}{1+\omega}\right)}dt^2 - \frac{dr^2}{1-(r_0/r)^{1-\alpha}} -$$
$$r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14}$$

$$p = \omega\rho = -\frac{1}{8\pi r_0^2}\left(\frac{r_0}{r}\right)^{3-\alpha}, \tag{15}$$

where $\alpha = -1/\omega$ (see Eqs. (32) and (33) of Ref. [4]). Notice that the equation of state (15) has the linear form of Eq. (11) and the shape function just the form not allowed by the master Eq. (13), therefore this non-asymptotically flat solution is not consistent with Einstein equations (2)–(4). It becomes clear that this solution is defined by a redshift function of general form given by

$$e^{\phi(r)} = \left(\frac{r}{r_0}\right)^{\beta}, \tag{16}$$

with β a constant. In the following section we shall discuss the solution generated by field equations (2)–(4) with the restricted choice (16).

5. On spherical wormhole with $e^{\phi(r)} = \left(\frac{r}{r_0}\right)^{\beta}$

In order to show the correctness of the conclusion of the previous section, we must provide the static spherically symmetric solution with the specific choice of the power-law redshift function (16). In Ref. [14] Tolman provides explicit analytical solutions for static spheres of fluids with isotropic pressure. For our purpose, it is convenient to consider the solution V, obtained in Sec. 4 of Ref. [14], which justly takes into account the metric component g_{tt} having the form of Eq. (16). We shall re-obtain the Tolman solution by assuming that the redshift function is given by Eq. (16).

By putting Eq. (16) into Eq. (7) we find for the shape function

$$b(r) = \frac{\beta\,(\beta-2)}{\beta^2-2\,\beta-1}\,r - C\,r^{-\frac{(2\,\beta+1)(-3+\beta)}{1+\beta}}, \tag{17}$$

where C is a constant of integration. It becomes clear that this form of the shape function is more general than the discussed above power-law shape function $b(r) = A/r^n$, as we would expect. Thus, the metric, energy density and pressure are provided by

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta}dt^2 - \frac{dr^2}{1-\frac{\beta\,(\beta-2)}{\beta^2-2\,\beta-1}+\tilde{C}\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}}$$
$$-r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{18}$$

$$\kappa\rho(r) = \tilde{C}\left(\frac{r}{r_0}\right)^{-2\,\frac{-2\,\beta-1+\beta^2}{1+\beta}}(2\,\beta+1)\,(\beta-3)\,(1+\beta)^{-1}\,r^{-2}$$
$$+\frac{(\beta-2)\,\beta}{\left(-2\,\beta-1+\beta^2\right)r^2}, \tag{19}$$

$$\kappa p(r) = (2\,\beta+1)\,\tilde{C}\left(\frac{r}{r_0}\right)^{-2\,\frac{-2\,\beta-1+\beta^2}{1+\beta}}r^{-2}$$
$$-\frac{\beta^2}{\left(-2\,\beta-1+\beta^2\right)r^2}, \tag{20}$$

respectively, where $\tilde{C} = C r_0^{-2(\beta^2-2\beta-1)/(1+\beta)}$. Note that the pressure is isotropic but not of barotropic type. We have such an equation of state only for $\tilde{C} = 0$ and $\beta = -1/2$, obtaining for the latter value $g_{tt} = r_0/r$, $g_{rr}^{-1} = \tilde{C} r_0/r - 4$, and $-p = \rho/5 = \frac{1}{\kappa r^2}$, i.e. $\omega = -1/5$. This solution exhibits a solid angle deficit, and does not describe a wormhole geometry.

By requiring the standard wormhole condition $b(r_0) = r_0$ on Eq. (17) we obtain for the metric (18) the following expression:

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - \frac{(1+2\beta-\beta^2)\,dr^2}{1-\left(\frac{r}{r_0}\right)^{\frac{2(1+2\beta-\beta^2)}{1+\beta}}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (21)$$

In order to the metric (21) describes a traversable wormhole the condition $\frac{1+2\beta-\beta^2}{1+\beta} < 0$ must be required, implying that the parameter β varies in the ranges $\beta > 1+\sqrt{2}$ or $-1 < \beta < 1-\sqrt{2}$. However, in this case $1+2\beta-\beta^2 < 0$, so the radial metric component g_{rr} becomes negative, and then the metric (21) does not describe a wormhole solution.

The metric component g_{rr} does not change its sign if we require $\frac{1+2\beta-\beta^2}{1+\beta} > 0$ and $1+2\beta-\beta^2 > 0$. This implies that the parameter β varies in the ranges $1-\sqrt{2} < \beta < 1+\sqrt{2}$. In this case the radial coordinate varies from zero to a maximum value $r_{max} = r_0 > 0$, hence the solution corresponds to a fluid sphere of radius r_0 with isotropic pressure.

Nevertheless, as we shall see in the following subsection, the considered Tolman solutions can describe wormhole geometries fulfilling the required conditions $g_{rr} > 0$ and $r \geq r_0$.

5.1. A truly wormhole geometry

In order to show that the metric (21) may correctly describe a Lorentzian spacetime for $r \geq r_0$ let us rewrite it in the following form:

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - \frac{(\beta^2-2\beta-1)\,dr^2}{\left(\frac{r}{r_0}\right)^{\frac{2(1+2\beta-\beta^2)}{1+\beta}} - 1} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (22)$$

It becomes clear that this metric describes a Lorentzian spacetime for $r \geq r_0$ if $\beta^2 - 2\beta - 1 > 0$ and $\frac{1+2\beta-\beta^2}{1+\beta} > 0$, which implies that the condition $\beta < -1$ must be required.

For the metric (22) the energy density and the isotropic pressure are given by

$$\kappa\rho = \frac{\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}(2\beta+1)(\beta-3)}{(1+\beta)(\beta^2-2\beta-1)r^2} + \frac{\beta(\beta-2)}{(\beta^2-2\beta-1)r^2}, \quad (23)$$

$$\kappa p = \frac{\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}(2\beta+1)}{(\beta^2-2\beta-1)r^2} - \frac{\beta^2}{(\beta^2-2\beta-1)r^2}. \quad (24)$$

It is interesting to note that this geometry describes a spacetime with a solid angle deficit (or excess). This can be seen directly by making the rescaling $\varrho^2 = (\beta^2-2\beta-1)r^2$. Then, the metric (22) becomes

$$ds^2 = \left(\frac{\varrho}{\varrho_0}\right)^{2\beta} dt^2 - \frac{d\varrho^2}{\left(\frac{\varrho}{\varrho_0}\right)^{\frac{2(1+2\beta-\beta^2)}{1+\beta}} - 1} - \frac{\varrho^2}{(\beta^2-2\beta-1)}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (25)$$

This new form of the metric (22) shows explicitly the presence of a solid angle deficit for $-\infty < \beta < 1-\sqrt{3}$ or $1+\sqrt{3} < \beta < \infty$, and a solid angle excess for $1-\sqrt{3} < \beta < 1-\sqrt{2}$ or $1+\sqrt{2} < \beta < 1+\sqrt{3}$. These topological defects vanish for $\beta = 1 \pm \sqrt{3}$, obtaining a non-flat asymptotic spacetime.

The geometrical properties and characteristics of these solutions can be explored through the embedding diagrams, which helps to visualize the shape and the size of slices $t = const$, $\theta = \frac{\pi}{2}$ of the metric (22) by using a standard embedding procedure in ordinary three dimensional Euclidean space. In general, in order to embed two dimensional slices $t = const$, $\theta = \frac{\pi}{2}$ of the generic metric (1) the equation

$$\frac{dz(r)}{dr} = \frac{1}{\sqrt{\frac{r}{b(r)} - 1}} \quad (26)$$

is used for the lift function $z(r)$ [1]. Thus, for slices $t = const$, $\theta = \frac{\pi}{2}$ of the metric (22) we obtain for the first derivative of the lift function

$$\frac{dz(r)}{dr} = \sqrt{\frac{\beta(\beta-2) - \left(\frac{r}{r_0}\right)^{\frac{2(1+2\beta-\beta^2)}{\beta+1}}}{\left(\frac{r}{r_0}\right)^{\frac{2(1+2\beta-\beta^2)}{\beta+1}} - 1}}. \quad (27)$$

This expression implies that at the throat $\frac{dz(r)}{dr} = \infty$, and it vanishes for $\beta = 1 \pm \sqrt{2} > -1$, so we have the standard behaviour of this derivative at the wormhole throat. On the other hand, we can see from Eq. (27) that the embedding of considered spacetimes in ordinary three dimensional Euclidean space has a finite size since it extends from r_0 up to a maximum radial value r_{max}. Effectively, the radial coefficient g_{rr} of the metric (22) implies that the denominator of the fraction is positive for any $r > r_0$, thus the numerator must be also positive. Then the condition $\beta(\beta-2)\left(\frac{r}{r_0}\right)^{\frac{2(\beta^2-2\beta-1)}{\beta+1}} \leq 1$ must be required, implying that $r_0 \leq r \leq r_{max}$, where

$$r_{max} = r_0(\beta(\beta-2))^{-\frac{\beta+1}{2(\beta^2-2\beta-1)}}. \quad (28)$$

Notice that for any value of the β-parameter we have that $\frac{dz(r)}{dr}|_{r_0} = \infty$ and $\frac{dz(r)}{dr}|_{r_{max}} = 0$. It can be shown that for $\beta = -1$ this $r_{max} = r_0$, as well as for $\beta \to -\infty$ we have that $r_{max} \to r_0$. The sphere r_{max} has a maximum value for $\beta = -4.745695219$ where it takes the value $r_{max} = 1.232835973\, r_0$. In Fig. 1 we show the behaviour of r_{max} for $\beta \leq -1$ and $r_0 = 1$.

In order to study the shape and the size of slices $t = const$, $\theta = \frac{\pi}{2}$ of the metric (22) we shall consider for the β-parameter values $\beta = -2, -3, -4.745695219, -10, -15, -30, -100 < -1$. These embeddings are shown in Fig. 2. For a full visualization of the surfaces the diagrams must be rotated about the vertical z-axis. We conclude from these diagrams that for $\beta < -1$ the metric (22) has typical wormhole shapes, i.e. presence of a global minimum at $r = r_0$, where the throat of the wormhole is located. The radial extension for all embeddings is finite as we would expect, and the height of the z-function increases with decreasing β-parameter.

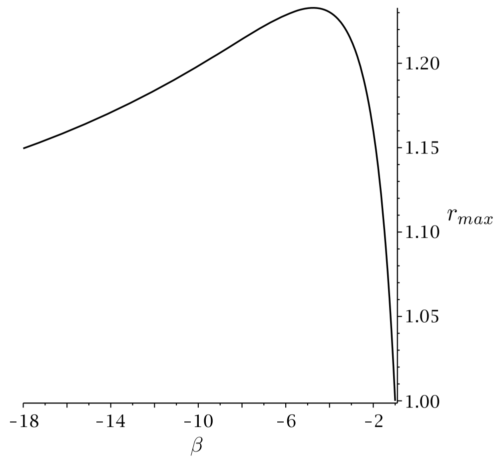

Fig. 1. Plot shows the behaviour of r_{max} of Eq. (28) for $\beta \leq -1$ and $r_0 = 1$. The maximum value is reached at $\beta = -4.745695219$ where $r_{max} = 1.232835973$. For $\beta \to -\infty$ we have $r_{max} \to 1$.

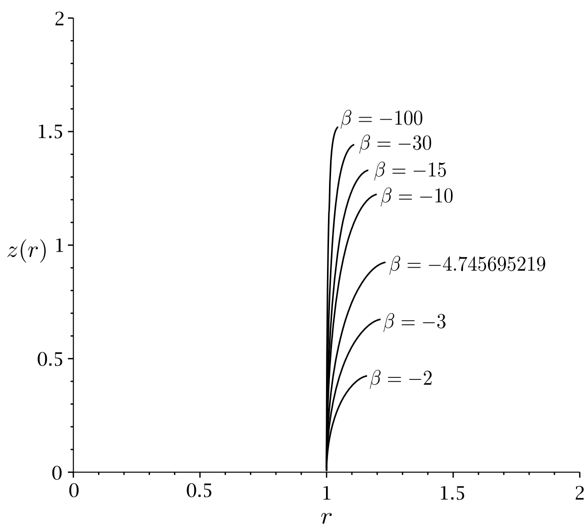

Fig. 2. Plots of the embedding function $z(r)$ for various values of the β-parameter are shown. The throat width has been set to $r_0 = 1$. The embedding of each slice $t = const$, $\theta = \frac{\pi}{2}$, $\beta = const$ of the wormhole (22) extends from the throat $r_0 = 1$ to $r_{max} > r_0$. The height of the z-function increases with decreasing β-parameter. It becomes clear that as $\beta \to -\infty$ the $r_{max} \to r_0 = 1$.

In Fig. 3 we show three-dimensional wormhole embedding diagrams for the values of the β-parameter $-2, -3, -4.745695219, -10$. It becomes clear that all embedding surfaces flare outward. This can be seen also from the fact that for the metric (22) the inverse of the embedding function $r(z)$ satisfies

$$d^2r/dz^2 = \frac{-r(\beta^2-2\beta-1)^2\left(\frac{r}{r_0}\right)^{2\beta}}{r_0^2(1+\beta)\left(\left(\frac{r}{r_0}\right)^{\frac{2\beta^2}{1+\beta}}(\beta^2-2\beta)-\left(\frac{r}{r_0}\right)^{\frac{2(2\beta+1)}{1+\beta}}\right)^2}, \tag{29}$$

implying that at the throat we have that

$$d^2r/dz^2|_{r_0} = -\frac{1}{r_0(1+\beta)}, \tag{30}$$

which is positive for any $\beta < -1$. Thus, the required flare-out condition for the wormhole throat is satisfied [1]. Note that Eq. (30) implies that for $\beta \to -\infty$ we have $d^2r/dz^2|_{r_0} \to 0$. By taking into account that we have also that $r_{max} \to r_0$ if $\beta \to -\infty$, then we may conclude that the shape of the wormhole embedding becomes a cylinder of radius r_0 for big negative values of the β-parameter.

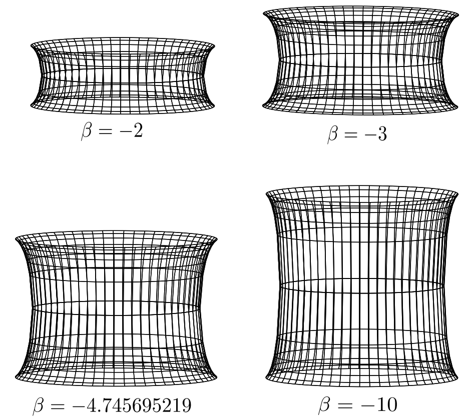

Fig. 3. The figure shows three dimensional wormhole embedding diagrams for $\beta = -2, -3, -4.745695219, -10$. The heights of the diagrams increase with decreasing β-parameter. The throat width is the same for all diagrams.

In conclusion, the metric (22) describes a wormhole geometry with isotropic pressure which extends from $r = r_0$ to $r = \infty$. Since these wormholes are not asymptotically flat, and the embedding in ordinary three dimensional Euclidean space extends from r_0 to r_{max}, we may match them, as an interior spacetime, to an exterior vacuum spacetime at the finite junction surface $r = r_{max}$.

From Eqs. (23) and (24) we conclude that at the throat we have for the energy density that $\rho(r_0) < 0$ if $-3 < \beta < -1$ and $\rho(r_0) \geq 0$ for $\beta \leq -3$, while for the pressure we obtain $p(r_0) = -1/r_0^2 < 0$. On the other hand we have that

$$\rho + p = \frac{2\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}(2\beta+1)(\beta-1)}{(1+\beta)(\beta^2-2\beta-1)r^2} - \frac{2\beta}{(\beta^2-2\beta-1)r^2}, \tag{31}$$

then at the throat $\rho + p = \frac{2}{(1+\beta)r_0^2}$ is fulfilled. Thus for $\beta < -1$ we have that always $\rho + p < 0$, which allows us to conclude that the energy conditions are not satisfied at the wormhole throat.

Since we are interested in studying static wormhole configurations, for which we must require $\beta < -1$, we conclude that we have wormholes only exhibiting a solid angle deficit, for which we obtain that $0 < \frac{1}{\beta^2-2\beta-1} < 1/2$. The polar and azimuthal angles are restricted to $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, respectively, therefore $0 \leq \tilde{\theta} \leq \pi/\sqrt{2}$ and $0 \leq \tilde{\phi} < \sqrt{2}\pi$, where $\tilde{\theta} = \theta/\sqrt{\beta^2-2\beta-1}$ and $\tilde{\phi} = \phi/\sqrt{\beta^2-2\beta-1}$.

6. Conclusions

The study of spherically symmetric traversable wormholes in General Relativity has been mostly focused in sources with anisotropic pressures. In this work we present and discuss static spherical wormhole spacetimes supported by a single perfect fluid, for which the condition $p_r = p_l$ must be required for radial and lateral pressures.

We show that it is not possible to sustain a zero-tidal-force wormhole by a perfect fluid, thus a single fluid threading a zero-tidal-force wormhole must be necessarily anisotropic. This implies that if we want to generate spherically symmetric wormholes, sustained by a single matter source with isotropic pressure, we must consider spacetimes with $\phi(r) \neq const$.

Also we discuss the possibility of having isotropic fluids with a linear equations of state. In particular, we show that a wormhole with a power-law shape function cannot be supported by an ideal fluid with a linear equation of state. Therefore we consider more general forms for the shape function and the equation of state of the isotropic pressure.

In this manner, we generate and discuss the general solution for a non-asymptotically flat family of static wormholes characterized by a redshift function given by Eq. (16). The obtained wormhole solutions exhibit always a solid angle deficit and do not satisfy energy conditions. The embeddings of these spacetimes in ordinary three dimensional Euclidean space have a finite size since they extend from r_0 up to a maximum radial value r_{max} given by Eq. (28). However, notice that the metric (22), or equivalently the metric (25), is well behaved for $r \geq r_0$, including the sphere r_{max}. For $r \geq r_{max}$ wormhole slices cannot be embedded in an ordinary Euclidean space. Instead, a space with indefinite metric must be used. It is interesting to note that one may match such a non-asymptotically flat wormhole, as an interior spacetime, to an exterior vacuum spacetime at the finite junction surface $r = r_{max}$.

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